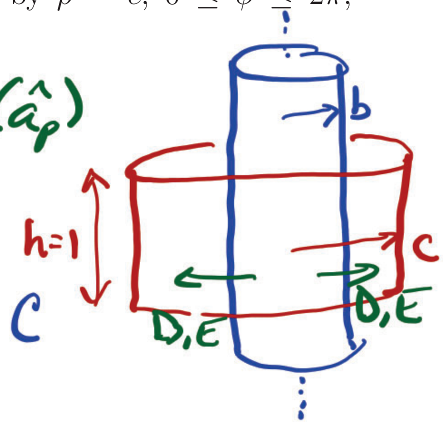
Part I - ELEC 211 questions

Point values specified for each question. Part marks may be awarded. For all questions, you don't need to do long hand calculations, but simplify your answers where possible.

1. $\boxed{\text{10 marks}}$ Electric Flux & Flux Density

   (a) (2 points) A charge distribution in free space given by $\rho_v = N\ C/m^3$ (where N is a constant) exists in the region defined by $0 \le \rho \le b$, $-\infty \le z \le \infty$, and $0 \le \phi \le 2\pi$. Find the expression for the total electric flux, Ψ, through the surface defined by $\rho = c$, $0 \le \phi \le 2\pi$, $0 \le z \le 1$, assuming $b < c$.

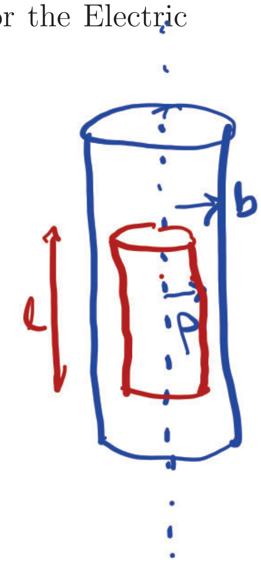


Field is radially outward ($\hat{a}_\rho$)

$$\oiint D \cdot dS = Q_{encl} = \Psi$$

$$\Psi = Q_{encl} = N\pi\rho^2 h = N\pi b^2 (1)\ C$$

   (b) (3 points) A charge distribution in free space given by $\rho_v = N\rho^2\ C/m^3$ (where N is a constant) exists in the region defined by $0 \le \rho \le b$, $-\infty \le z \le \infty$, and $0 \le \phi \le 2\pi$. Find the expression for the Electric Flux Density, $\vec{D}$, in the region $\rho \le b$.



Field is radially outward ($\hat{a}_\rho$)

$$\oiint \bar{D} \cdot d\bar{S} = Q_{enclosed}$$

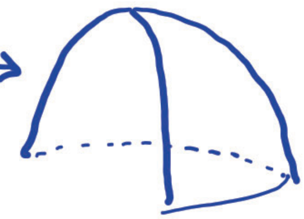
$$D(2\pi\rho l) = \int_0^l \int_0^{2\pi} \int_0^{\rho} N\rho'^2\, \rho' d\rho' d\phi dz$$

$$D(\cancel{2\pi}\cancel{\rho}\cancel{l}) = N(\cancel{2\pi})(\cancel{l})\frac{\rho^{\cancel{4}\,3}}{4}$$

$$\bar{D} = \frac{N}{4}\rho^3\ \hat{a}_\rho\ C/m^2$$

(c) (3 points) If, in spherical coordinates, an Electric Flux Density is defined as $\vec{D} = 2(r-1)^3\hat{a}_r\ C/m^2$ over the region $2 \le r \le 5$ m, $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$, what is the total electric flux, Ψ, through the surface defined by $\underbrace{r = 5 \text{ m}, 0 \le \theta \le \pi/2, 0 \le \phi \le \pi}$?

The surface is $\frac{1}{4}$ of a full spherical shell of radius 5.



$$\Psi = \iint D \cdot dS = \int_0^{\pi}\int_0^{\pi/2} 2(r-1)^3 \cancel{\hat{a}_r} \cdot \cancel{\hat{a}_r}\, r^2 \sin\theta\, d\theta\, d\phi$$

where $r = 5$

$$\Psi = \int_0^{\pi}\int_0^{\pi/2} 2(4)^3(5)^2 \sin\theta\, d\theta\, d\phi$$

$$\Psi = 2\pi(4)^3(5)^2\left(\underbrace{-\cos\theta}_{1}\right)\Big|_0^{\pi/2}$$

$$= (50)(64)\pi\ C$$

(d) (2 points) For the Electric Flux Density defined in part (c), what is the volume charge density, ρ_v, at radius $r = 4$ m?

$$\nabla \cdot \bar{D} = \rho_v$$

$$\nabla \cdot \bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right)$$ since we only have an $\hat{a}_r$ term

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(2(r-1)^3\right)\right)$$

$$= \frac{2}{r^2}\frac{\partial}{\partial r}\left(r^2(r-1)^3\right)$$

$$= \frac{2}{r^2}\left[2r(r-1)^3 + r^2(3)(r-1)^2\right]$$

cont.

This blank page is for your solution to ***Question 1*** *if you need more space.*

d) continued.

$$\nabla \cdot \bar{D} = \frac{4}{r}(r-1)^3 + 6(r-1)^2 = \rho_v \ C/m^3$$

No need to simplify further. Can sub in $r=4$. to get

$$\rho_v = \nabla \cdot \bar{D} = \frac{4}{4}(3)^3 + 6(3)^2$$

$$= 3^3 + 6(3)^2$$

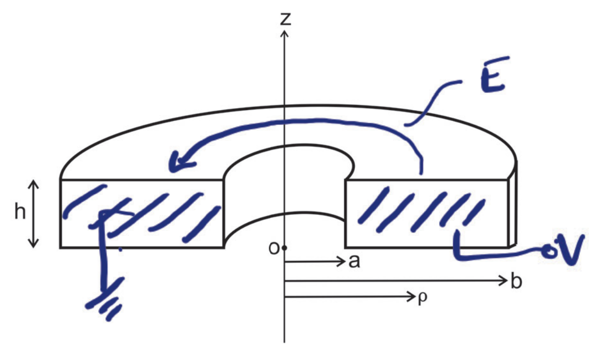
$$= 27 + 6(9)$$

$$= 81 \ C/m^3$$

2. $\boxed{\text{15 marks}}$ Conductors, Resistance, & Capacitance

NOTE: this description applies to parts (a) to (c). Parts (d) and (e) are independent of each other and of parts (a) to (c).

A metal conductor in the shape of a thick half-circle as shown in the figure has conductivity $\sigma = 1$ x 10^6 S/m and dimensions $a = 2$ cm, $b = 4$ cm, and $h = 1$ cm. The two rectangular faces are aligned with the z-axis (i.e., if the conductor was a complete circle, the z-axis would be at it's centre). If a potential difference is applied between the two rectangular faces, the resultant Electric Field Intensity is $\vec{E} = (0.1/\rho)\ \hat{a}_\phi$ V/m.



(a) (2 points) What potential difference, V, exists between the two rectangular faces of this structure?

$$V = -\int \bar{E}\cdot d\bar{\rho} = -\int_{\pi}^{0} \frac{0.1}{\cancel{\rho}}\, \cancel{\hat{a}_\phi}\cdot\cancel{\hat{a}_\phi}\, \cancel{\rho}\, d\phi$$

$$= 0.1\pi \quad V$$

(b) (4 points) What is the total current flowing in the structure?

Use $\bar{J} = \sigma \bar{E}$ and $I = \iint \bar{J} \cdot d\bar{S}$

$$\bar{J} = (1 \times 10^6)\left(\frac{0.1}{\rho}\right) \hat{a}_\phi \ A/m^2$$

$$I = \int_0^{0.01} \int_{0.02}^{0.04} (1 \times 10^6)\left(\frac{0.1}{\rho}\right) \cancel{\hat{a}_\phi} \cdot \cancel{\hat{a}_\phi} \, d\rho \, dz$$

$$= (1 \times 10^6)(0.1) \ln\left(\frac{.04}{.02}\right)(.01)$$

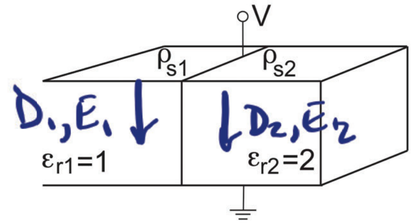
$$= 1 \times 10^3 \ln(2) \ A.$$

(c) (1 points) What is the resistance between the two rectangular faces?

$$R = \frac{V}{I} = \frac{0.1\pi}{1 \times 10^3 \ln(2)} \ \Omega$$

$$= \frac{1 \times 10^{-4}\pi}{\ln(2)} \ \Omega$$

(d) (3 points) A parallel plate capacitor is half filled with air and half with a dielectric material as shown. Circle the correct relationship for each quantity in the table provided.



Quantity			
ρ_s	$\boxed{\rho_{s1} < \rho_{s2}}$	$\rho_{s1} > \rho_{s2}$	$\rho_{s1} = \rho_{s2}$
D	$\boxed{\vec{D}_1 < \vec{D}_2}$	$\vec{D}_1 > \vec{D}_2$	$\vec{D}_1 = \vec{D}_2$
E	$\vec{E}_1 < \vec{E}_2$	$\vec{E}_1 > \vec{E}_2$	$\boxed{\vec{E}_1 = \vec{E}_2}$

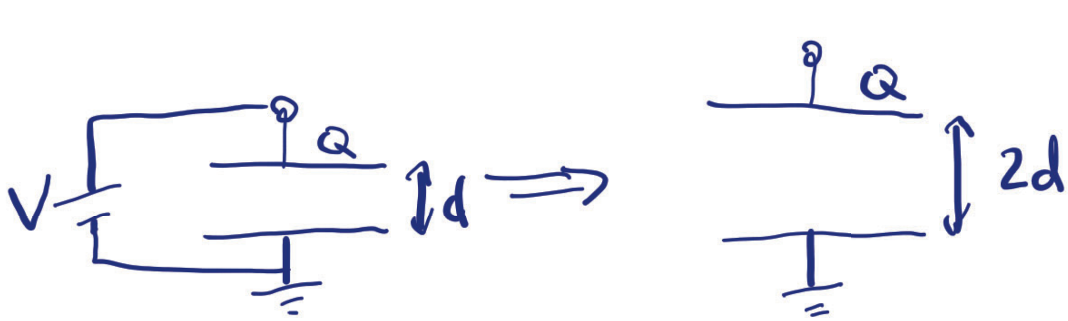
E_{TAN} = continuous at dielectric/air boundary

$D = \varepsilon_r \varepsilon_0 E \quad \therefore \; D_2 > D_1$

$D_N = \rho_s$ at conductor boundaries

(e) (5 points) An air-filled parallel plate capacitor has plate surface area S, plate separation d, and a potential, V, supplied by a battery connected between the two plates. If the battery is disconnected and then the separation between the plates is increased to 2d <u>all without</u> discharging the capacitor, how will each of the following values change? Place a check mark in the appropriate column for each quantity.

Quantity	Up	Down	Unchanged
E			✓
C		✓	
V	✓		
ρ_s			✓
D			✓



· Q, ρ_s unchanged

· $D_N = \rho_s$ ∴ unchanged

$E = \dfrac{D}{\varepsilon_0}$ ∴ unchanged

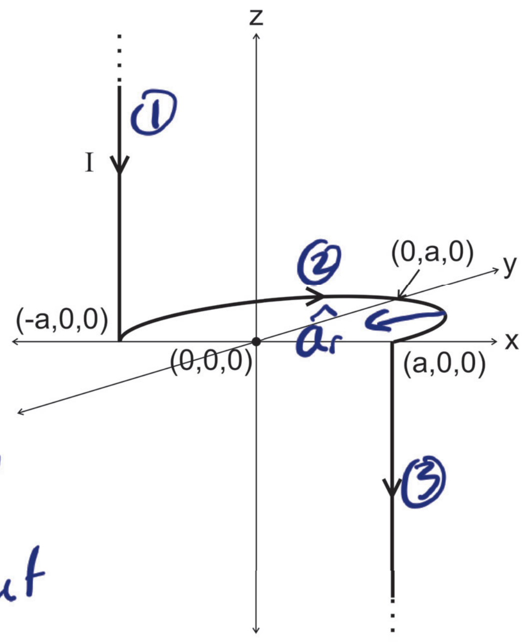
$C = \dfrac{\varepsilon_0 S}{d} \rightarrow \dfrac{\varepsilon_0 S}{2d} \; \therefore \downarrow$

$V = E \cdot 2d \; \therefore \uparrow$

3. $\boxed{\text{15 marks}}$ Magnetic Fields, Magnetic Flux & Flux Densities

NOTE: Part (a) is independent of parts (b) through (d).

(a) (5 points) An infinite current filament carrying a current, I, is bent into the shape shown in the figure. The section lying in the $x-y$ plane is a semicircle of radius a. Find the expression for the Magnetic Field Intensity, $\vec{H}$, at the origin.



Consider 3 segments

Seg. ① field in $-\hat{a}_y$ direction

Seg. ③ field in $+\hat{a}_y$ direction

H_1 & H_3 are equal and opposite ∴ cancel out

Seg ② has field in $-\hat{a}_z$ direction

use Biot Savart : $H = \int_0^{\pi} \frac{I dL \times \hat{a}_r}{4\pi r^2}$

where $dL = a\,d\phi\,\hat{a}_\phi$ and $\hat{a}_r = -\hat{a}_\rho$

in cylindrical coords.

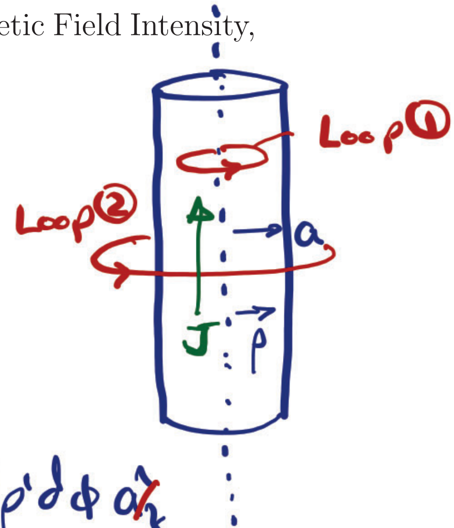
$$\therefore H = \int_{\pi}^{0} \frac{I a\,d\phi\,\hat{a}_\phi \times (-\hat{a}_\rho)}{4\pi a^2} = \frac{I \cancel{a} (-\cancel{\pi})\,\hat{a}_z}{4\cancel{\pi} a^{\cancel{2}}}$$

$$\therefore \bar{H} = \frac{I}{4a}(-\hat{a}_z)\ A/m$$

(b) (5 points) A long cylindrical conductor of radius a is coincident with the $\hat{a}_z$ axis and has a current density defined by

$$\vec{J} = \frac{A}{a}\rho \ \hat{a}_z \ A/m^2$$

where A is a constant. Find expressions for the Magnetic Field Intensity, $\vec{H}$, for all ρ.



Two regions, $\rho \leq a$ & $\rho > a$

for $\rho \leq a$ (Loop 1)

$$\oint \bar{H} \cdot d\bar{L} = \iint \bar{J} \cdot d\bar{S}$$

$$H(2\pi\rho) = \int_0^{2\pi}\int_0^{\rho} \frac{A}{a} \rho' \cancel{\hat{a}_z} \cdot \rho' d\rho' d\phi \, \cancel{\hat{a}_z}$$

$$H(\cancel{2}\pi\cancel{\rho}) = \cancel{2}\pi \frac{A}{a} \frac{\rho^{\cancel{3}\,2}}{3}$$

$$\therefore \bar{H} = \frac{A}{3a}\rho^2 \, \hat{a}_\phi \quad A/m$$

Same integral but $\int_0^{2\pi}\int_0^{a} \bar{J} \cdot d\bar{S}$.

for $\rho > a$ (loop 2)

$$H(\cancel{2}\pi\rho) = \frac{\cancel{2}\pi A a^2}{3}$$

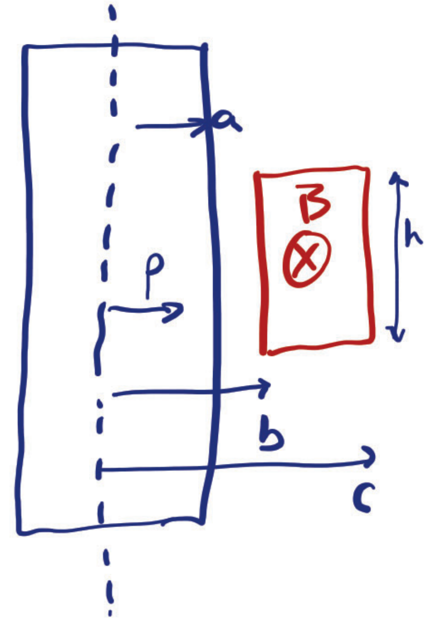
$$\therefore \bar{H} = \frac{Aa^2}{3\rho} \, \hat{a}_\phi \quad A/m.$$

(c) (3 points) For the conductor in part (b), find the total Magnetic Flux, Φ, through the surface defined by $b \leq \rho \leq c$, $0 \leq z \leq h$, and $\phi = 0$ (assume $a < b < c$ and $h > 0$).

Ok to either leave μ as is or write $\mu = \mu_0$.

$$\phi = \iint B \cdot dS = \int_0^h \int_b^c \frac{\mu_0 A a^2}{3\rho} \hat{a}_\phi \cdot \hat{a}_\phi \, d\rho \, dh$$

$$= \frac{\mu_0 A a^2 h}{3} \ln\left(\frac{c}{b}\right) \text{ Wb.}$$



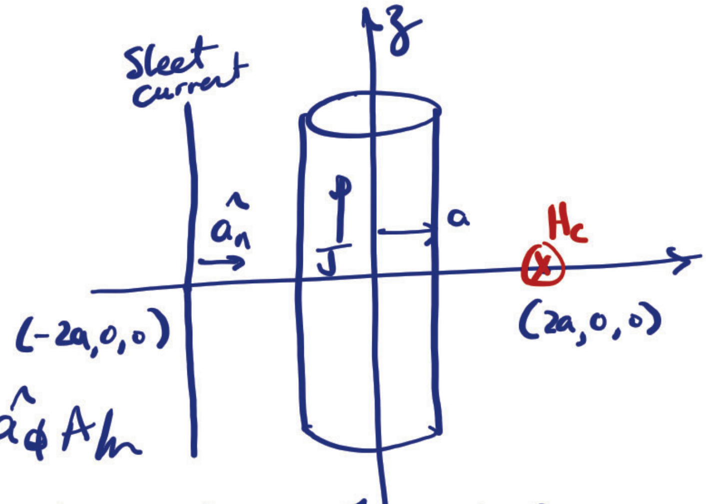
(d) (2 points) Assume that $\phi = 0$ corresponds with the x-axis of a Cartesian coordinate system. Consider the point $P(2a, 0, 0)$ in Cartesian coordinates. Now assume that in addition to the conductor in part (b) there is an infinite sheet current located in the $x = -2a$ plane. Determine the magnitude and direction of the sheet current density, $\vec{K}$, needed to achieve $\vec{H} = 0$ A/m at the point P.

H_c = H of conductor

Want $H_K = -H_c$

$$\bar{H}_K = \frac{1}{2} \bar{K} \times \hat{a}_n$$

$$H_c(2a,0,0) = \frac{A a^2}{3(2a)} = \frac{Aa}{6} \hat{a}_\phi \text{ A/m}$$



where $\hat{a}_\phi = \hat{a}_y$ at $(2a,0,0)$

cont.

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3d) Set $H_k = \frac{1}{2}\bar{K} \times \hat{a}_n$

$$|H_c| = |H_K| \Rightarrow \frac{Aa}{6} = \frac{1}{2}K$$

$\therefore |K| = \dfrac{Aa}{3}$ and

direction must be $(-\hat{a}_z)$ to get an H direction of $-\hat{a}_y$

$(= -\hat{a}_\varphi @ (2a,0,0))$