Q2 System properties

six variants

① $y(t) = x(t)\,x(t-1)$

- [ ] Linear
- [x] Causal
- [ ] Memoryless
- [x] Time-invariant
- [x] Stable
- [ ] Invertible

Linear: $x_1(t) = a\,x(t) \longrightarrow y_1(t) = a\,x(t) \cdot a\,x(t-1)$

$$\neq a\,y(t)$$

Causal: y at time t_0 depends only on $x(t)$ for $t \leq t_0$ ✓

Memoryless: y at time t_0 depends on $x(t)$ for $t \neq t_0$

Time invariant: $x_1(t) \longrightarrow y_1(t) = x_1(t)\,x_1(t-1)$

$$x_2(t) = x_1(t-t_0) \longrightarrow y_2(t) = x_2(t)\,x_2(t-1)$$
$$= x_1(t-t_0)\,x_1(t-t_0-1)$$
$$= y_1(t-t_0) \checkmark$$

Stable: $\underbrace{|x(t)| \leq B}_{\text{finite}} \;\forall t \Rightarrow |y(t)| \leq B^2 \;\forall t$

$\Rightarrow$ y has a finite bound ✓

Invertible: counterexample:

$$x_1(t) = 1, \quad x_2(t) = -1$$
$$y_1(t) = 1, \quad y_2(t) = 1 = y_1(t)$$

$\Rightarrow$ different inputs lead to the same output

② $y(t) = at\big(x(t) + x(t-1)\big) \qquad a \neq 0$

- [x] Linear
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- [ ] Time-invariant
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- [ ] Invertible

Linear: If $x_1(t) \longrightarrow y_1(t) = at\big(x_1(t) + x_1(t-1)\big)$

$x_2(t) \longrightarrow y_2(t) = at\big(x_2(t) + x_2(t-1)\big)$

then $z(t) = bx_1(t) + cx_2(t) \rightarrow at\Big(\underbrace{bx_1(t) + cx_2(t)}_{z(t)} + \underbrace{bx_1(t-1) + cx_2(t-1)}_{z(t-1)}\Big)$

$$= b\,at\big(x_1(t) + x_1(t-1)\big) + c\,at\big(x_2(t) + x_2(t-1)\big)$$

$$= b\,y_1(t) + c\,y_2(t) \quad ✓$$

Causal: y at time t_0 depends only on $x(t)$ for $t \leq t_0$ ✓

Memoryless: y at time t_0 depends on $x(t)$ for $t \neq t_0$

Time-invariant: $x_1(t) = x(t-t_0) \longrightarrow y_1(t) = at\big(x_1(t) + x_1(t-1)\big)$

$$= at\big(x(t-t_0) + x(t-t_0-1)\big)$$

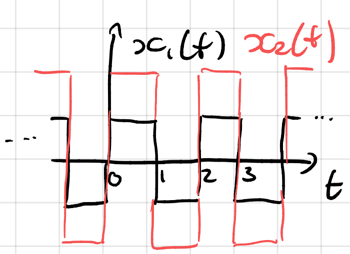
$$\neq y(t-t_0)$$

$$= a(t-t_0)\big(x(t-t_0) + x(t-t_0-1)\big)$$

stable: $|x(t)| \leq B \;\; \forall t \Rightarrow |y(t)| \leq |at|\, 2B$

upper bound grows without bound as t increases

Invertible: Counterexample: periodic signals



$y_1(t) = 0 \;\; \forall t$

$y_2(t) = 0 \;\; \forall t.$

Note: this is a somewhat tricky case

③ $$y(t) = \begin{cases} x(t) & t<0 \\ 1 & t \geq 0 \end{cases}$$

- [ ] Linear
- [x] Causal
- [x] Memoryless
- [ ] Time-invariant
- [x] Stable
- [ ] Invertible

Linear: if $x(t) = 0 \ \forall t$ then $y(t) \neq 0$ for $t \geq 0$

$\Rightarrow$ homogeneity not satisfied

Causal: y at time t_0 depends only on $x(t)$ for $t \leq t_0$ ✓

Memoryless: y at time t_0 depends only on $x(t)$ for $t = t_0$ ✓

(which includes not even depending on $x(t_0)$)

Time-invariant: $x_1(t) = x(t-t_0) \rightarrow y_1(t) = \begin{cases} x_1(t) & t<0 \\ 1 & t \geq 0 \end{cases}$

$$= \begin{cases} x(t-t_0) & t<0 \\ 1 & t \geq 0 \end{cases}$$

$$\neq y(t-t_0)$$

$$= \begin{cases} x(t-t_0) & t-t_0<0 \\ 1 & t-t_0 \geq 0 \end{cases}$$

Stable: if $|x(t)| \leq B \ \forall t$ then $|y(t)| \leq \max(B, 1) \ \forall t$ ✓

Invertible: Counter example:

$$\left.\begin{array}{l} x_1(t) = x_2(t) \text{ for } t<0 \\ x_1(t) \neq x_2(t) \text{ for } t \geq 0 \end{array}\right\} y_1(t) = y_2(t)$$

④ $y[n] = a\,x^2[-n] \qquad a \neq 0$

- [ ] Linear
- [ ] Causal
- [ ] Memoryless
- [ ] Time-invariant
- [x] Stable
- [ ] Invertible

Linear: $x_1[n] + x_2[n] \longrightarrow a\left(x_1[-n] + x_2[-n]\right)^2$

$$= \underbrace{a\,x_1^2[-n]}_{y_1[n]} + \underbrace{a\,x_2^2[-n]}_{y_2[n]} + 2a\,x_1[-n]\,x_2[-n]$$

$$\neq y_1[n] + y_2[n]$$

Causal: Counter example: $y[-5]$ depends on $x[5]$

Memoryless: Counter example: $y[-5]$ depends on $x[5]$

Time invariant: $x_1[n] = x[n-n_0] \longrightarrow y_1[n] = a\,x_1^2[-n]$

$$= a\,x^2[-n-n_0]$$

$$y[n-n_0] = a\,x^2[-(n-n_0)] = a\,x^2[-n+n_0] \neq y_1[n]$$

stable: if $|x[n]| \leq B \;\forall$ then $|y[n]| \leq a B^2 \;\forall n$ ✓

invertible: $x_1[n] = -x_2[n] \longrightarrow y_1[n] = y_2[n]$

⑤ $y[n] = x[n+1] + \sum_{k=-\infty}^{n} x[k]$

- [x] Linear
- [ ] Causal
- [ ] Memoryless
- [x] Time-invariant
- [ ] Stable
- [x] Invertible

Note: $y[n] = \sum_{k=-\infty}^{n+1} x[k]$

Linear: If $x_1[n] \longrightarrow y_1[n] = \sum_{k=-\infty}^{n+1} x_1[k]$

$$x_2[n] \longrightarrow y_2[n] = \sum_{k=-\infty}^{n+1} x_2[k]$$

then $a x_1[n] + b x_2[n] \longrightarrow \sum_{k=-\infty}^{n+1} (a x[k] + b x_2[k])$

$$= a \sum_{k=-\infty}^{n+1} x_1[k] + b \sum_{=-\infty}^{n+1} x_2[k]$$

$$= a y_1[n] + b y_2[n] \checkmark$$

Causal: $y[n]$ depends on $x[n+1]$

Memoryless: not causal $\rightarrow$ not memoryless

Time invariant: $x_1[n] = x[n-n_0] \longrightarrow y_1[n] = \sum_{k=-\infty}^{n+1} x_1[k]$

$$= \sum_{k=-\infty}^{n+1} x[k-n_0]$$

$$= \sum_{k=-\infty}^{n-n_0+1} x[k]$$

$$= y[n-n_0] \checkmark$$

Stable: counter example

$x[n] = B \ \forall n$ then $y[n] = \sum_{k=-\infty}^{n+1} B \rightarrow \infty$

Invertible: $y[n-1] - y[n-2] = x[n]$

$\Rightarrow$ inverse system: $w[n] \longrightarrow z[n] = w[n-1] - w[n-2]$

(6) $$y[n] = \begin{cases} x[n] - x[n-1] & n < 0 \\ x[-n] & n \geq 0 \end{cases}$$

- [x] Linear
- [x] Causal
- [ ] Memoryless
- [ ] Time-invariant
- [x] Stable
- [ ] Invertible

Linear: if $x_1[n] \longrightarrow y_1[n] = \begin{cases} x_1[n] - x_1[n-1] & n<0 \\ x_1[-n] & n \geq 0 \end{cases}$

$x_2[n] \longrightarrow y_2[n] = \begin{cases} x_2[n] - x_2[n-1] & n<0 \\ x_2[-n] & n \geq 0 \end{cases}$

then $z[n] = a x_1[n] + b x_2[n] \longrightarrow w[n] = \begin{cases} z[n] - z[n-1] & n<0 \\ z[-n] & n \geq 0 \end{cases}$

$$= \begin{cases} a x_1[n] + b x_2[n] - a x_1[n-1] - b x_2[n-1] & n<0 \\ a x_1[-n] + b x_2[-n] & n \geq 0 \end{cases}$$

$$= a \begin{cases} x_1[n] - x_1[n-1] & n<0 \\ x_1[-n] & n \geq 0 \end{cases} + b \begin{cases} x_2[n] - x_2[n-1] & n<0 \\ x_2[-n] & n \geq 0 \end{cases}$$

$$= a\, y_1[n] + b\, y_2[n] \quad \checkmark$$

Causal: $n<0$: $y[n] = x[n] - x[n-1]$
$n \geq 0$: $y[n] = x[-n]$
$y[n]$ only depends on past or current input ✓

memoryless: $y[n]$ depends on past input

Time invariant: $x_1[n] = x[n-n_0] \longrightarrow y_1[n] = \begin{cases} x_1[n] - x_1[n-1] & n<0 \\ x_1[-n] & n \geq 0 \end{cases}$

$$= \begin{cases} x[n-n_0] - x[n-n_0-1] & n<0 \\ x[-n-n_0] & n \geq 0 \end{cases}$$

$$\neq y[n-n_0]$$

$$= \begin{cases} x[n-n_0] - x[n-n_0-1] & n-n_0<0 \\ x[-n+n_0] & n-n_0 \geq 0 \end{cases}$$

stable: if $|x[n]| \le B \ \forall n$ then $|y[n]| \le 2B \ \forall n$ ✓

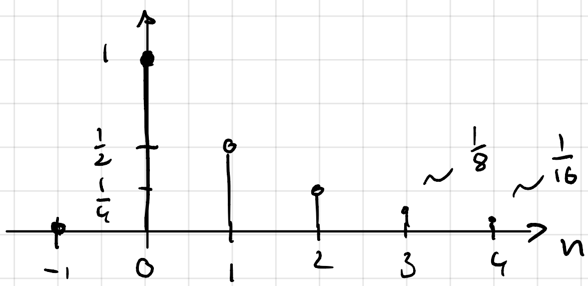
invertible: $x_1[n] = 1 \ \forall n<0$, $x_2[n] = 2 \ \forall n<0$ $\Big\}$ $y_1[n] = y_2[n] = 0 \quad n<0$

Q3: A series of convolutions

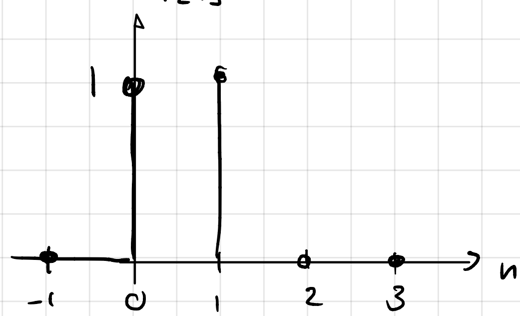
ⓐ impulse response $h[n] = u[n] - u[n-2] = \delta[n] + \delta[n-1]$

$$\Rightarrow \sum_{n=-\infty}^{\infty} |h[n]| = 2 \Rightarrow \text{stable}$$

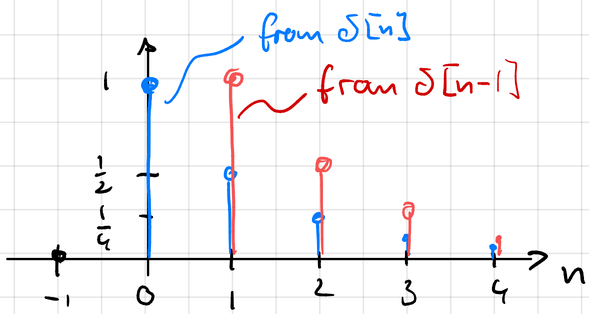
ⓑ $x[n] = \left(\frac{1}{2}\right)^n u[n]$



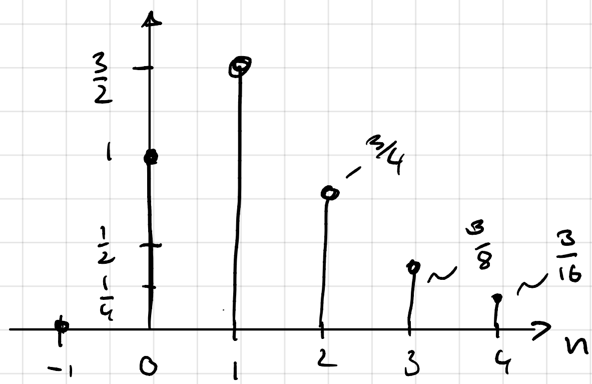
$h[n]$



$x[n] * h[h]$



$\Rightarrow$



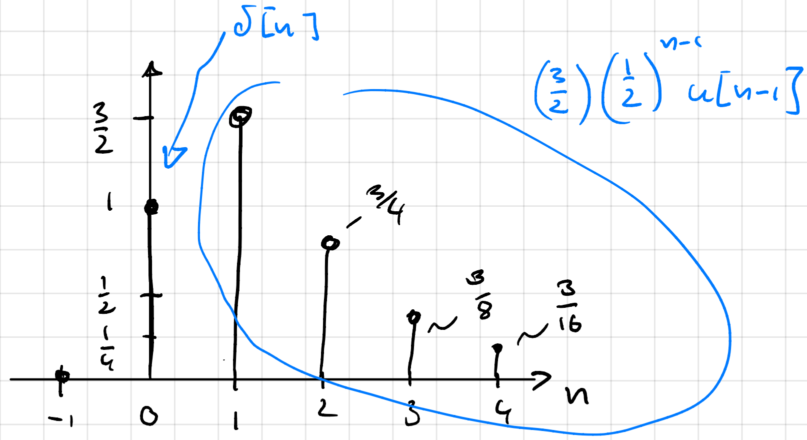
ⓒ $x[n] * h[n] = x[n] + x[n-1]$

$$= \underbrace{\left(\tfrac{1}{2}\right)^n u[n]} + \left(\tfrac{1}{2}\right)^{n-1} u[n-1]$$

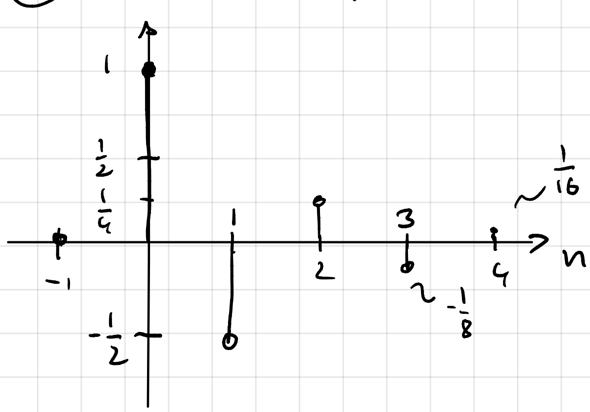
$$= \delta[n] + \left(\tfrac{1}{2}\right)^n u[n-1] + \left(\tfrac{1}{2}\right)^{n-1} u[n-1]$$

$$= \delta[n] + \tfrac{3}{2}\left(\tfrac{1}{2}\right)^{n-1} u[n-1]$$

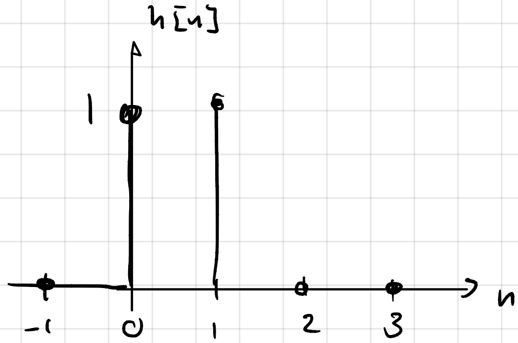
$\Rightarrow$



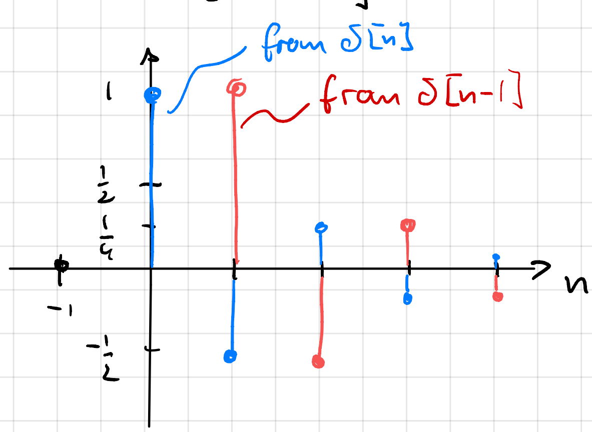
(d) $x[n] = \left(-\frac{1}{2}\right)^n u[n]$



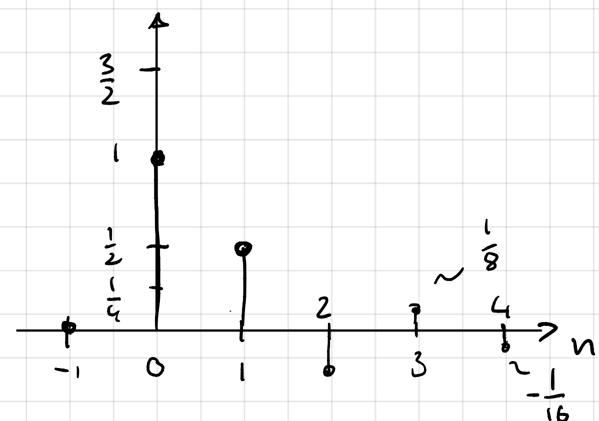
$h[n]$



$x[n] * h[n]$



$\Rightarrow$



$$x[n] * h[n] = x[n] + x[n-1]$$

$$= \underbrace{\left(-\tfrac{1}{2}\right)^n u[n]} + \left(-\tfrac{1}{2}\right)^{n-1} u[n-1]$$

$$= \overbrace{\delta[n] + \left(-\tfrac{1}{2}\right)^n u[n-1]} + \left(-\tfrac{1}{2}\right)^{n-1} u[n-1]$$

$$= \delta[n] + \left(-\tfrac{1}{2} + 1\right)\left(-\tfrac{1}{2}\right)^{n-1} u[n-1]$$

$$= \delta[n] + \tfrac{1}{2}\left(-\tfrac{1}{2}\right)^{n-1} u[n-1]$$

plot as above

Q4: Series concatenation

(a) S1: If $x_1[n] \longrightarrow y_1[n] = x_1[n] + \frac{1}{2}x_1[n-1]$

$x_2[n] \longrightarrow y_2[n] = x_2[n] + \frac{1}{2}x_2[n-1]$

then $z[n] = a x_1[n] + b x_2[n] \longrightarrow w[n] = z[n] + \frac{1}{2} z[n-1]$

$$= a x_1[n] + b x_2[n] + \frac{1}{2} a x_1[n-1] + \frac{1}{2} b x_2[n]$$

$$= a\left(x_1[n] + \frac{1}{2}x_1[n-1]\right) + b\left(x_2[n] + \frac{1}{2} x_2[n-1]\right)$$

$$= a y_1[n] + b y_2[n] \checkmark$$

S2: If $x_1[n] \longrightarrow y_1[n] = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k x_1[n-k]$

$x_2[n] \longrightarrow y_2[n] = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k x_2[n-k]$

then $z[n] = a x_1[n] + b x_2[n]$

$$\longrightarrow w[n] = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k z[n-k] = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k \left(a x_1[n-k] + b x_2[n-k]\right)$$

$$= a \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k x_1[n-k] + b \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k x_2[n-k]$$

$$= a y_1[n] + b y_2[n] \checkmark$$

(b) $h[n] = h_1[n] * h_2[n]$

$h_1[n] = \delta[n] + \frac{1}{2}\delta[n-1], \quad h_2[n] = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k \delta[n-k]$

$$h[n] = h_2[n] + \frac{1}{2} h_2[n-1]$$

$$= \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k \delta[n-k] + \frac{1}{2}\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k \delta[n-1-k]$$

$$\frac{1}{2}\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1}\delta[n-k]$$

$$= -\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k}\delta[n-k]$$

$$= -\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^{k}\delta[n-k] + \delta[n]$$

$$= \delta[n]$$

ⓒ S1 is invertible because S2 is its inverse system (*) but see below

$$\underbrace{x[n] \longrightarrow \boxed{S1} \longrightarrow \boxed{S2}}_{h[n]=\delta[n]} \longrightarrow y[n]$$

$$h[n] = \delta[n] \Rightarrow y[n] = x[n] * \delta[n] = x[n] \quad ✓$$

ⓓ

$$h[n] = h_1[n] * h_2[n] \underset{\uparrow}{=} h_2[n] * h_1[n]$$

conv. is commutative

$$\Rightarrow \underbrace{x[n] \longrightarrow \boxed{S2} \longrightarrow \boxed{S1}}_{h[n]=\delta[n]} \longrightarrow y[n]$$

$$h[n] = \delta[n] \Rightarrow y[n] = x[n]$$

S2 is invertible because S1 is its inverse system. ✓

(*) a special case that also got full marks:

Let $x[n] = \left(-\frac{1}{2}\right)^n$ be input to S1: Then $y[n] = \left(-\frac{1}{2}\right)^n + \frac{1}{2}\left(-\frac{1}{2}\right)^{n-1} = 0$

$\Rightarrow$ $y[n] = 0 \ \forall n$ is obtained for $x[n] = \left(-\frac{1}{2}\right)^n$ and for $x[n] = 0 \ \forall n$

$\Rightarrow$ the system is not invertible

What happened? If we apply this signal to $\boxed{S1} \rightarrow \boxed{S2}$ then we get

$$y[n] = \sum_{k=0}^{\infty} \left(-\tfrac{1}{2}\right)^k x[n-k] + \frac{1}{2} \sum_{k=0}^{\infty} \left(-\tfrac{1}{2}\right)^k x[n-1-k]$$

$$= \sum_{k=0}^{\infty} \left(-\tfrac{1}{2}\right)^k \left(-\tfrac{1}{2}\right)^{n-k} + \frac{1}{2} \sum_{k=0}^{\infty} \left(-\tfrac{1}{2}\right)^k \left(-\tfrac{1}{2}\right)^{n-1-k}$$

$$= \left(-\tfrac{1}{2}\right)^n \underbrace{\sum_{k=0}^{\infty} 1} - \left(-\tfrac{1}{2}\right)^n \underbrace{\sum_{k=1}^{\infty} 1}$$

difference of sums is 1 but they do not converge !

Note that $x[n] = \left(-\tfrac{1}{2}\right)^n$ is not bounded though.