Q3 Phase and Amplitude response

Note: The original question was phrased incorrectly

Phase lag: $\measuredangle H(j\omega) < 0$ for $\omega > 0$

① $$|H_1(j\omega)|^2 = \left| \frac{1 + j\omega/10}{1 + 10j\omega} \right|^2 = \frac{1 + \frac{\omega^2}{100}}{1 + 100\omega^2} \leq 1 \ \forall \omega$$

$\Rightarrow$ no amplification

$$H_1(j\omega) = \frac{(1 + j\omega/10)(1 - 10j\omega)}{1 + 100\omega^2} = \frac{1 + \omega^2 + j\left(\frac{\omega}{10} - 10\omega\right)}{1 + 100\omega^2}$$

$$\measuredangle H_1(j\omega) = \tan^{-1}\left(\frac{\frac{\omega}{10} - 10\omega}{1 + \omega^2}\right) < 0 \ \text{for } \omega > 0$$

$\Rightarrow$ phase lag for $\omega > 0$

② $$H_2(j\omega) = \frac{1}{H_1(j\omega)}$$

$$= \frac{1}{|H(j\omega)|} e^{-j \measuredangle H_1(j\omega)}$$

$\Rightarrow$ amplification for all ω

but no phase lag

③ $$|H_3(j\omega)| = \frac{|(1 + j\omega/10)(1 + j\omega/10)|}{|(1 + 10j\omega)(1 + 10j\omega)(1 + 10j\omega)|} = \frac{|1 + j\omega/10|^2}{|1 + 10j\omega|^3} < 1 \ \forall \omega$$

$\Rightarrow$ no amplification

④ $$|H_4(j\omega)|^2 = \frac{1+\omega^2/100}{|-100\omega^2 + 10j\omega + 1|^2} = \frac{1+\omega^2/100}{(1-100\omega^2)^2 + 100\omega^2}$$

$$= \frac{1+\omega^2/100}{1+10^4\omega^4 - 100\omega^2} > 1 \text{ for } |\omega| < \frac{\sqrt{100+\frac{1}{100}}}{100}$$

amplification

numerically evaluate $\measuredangle H_4(j\omega)$

$\Rightarrow$ $\measuredangle H_4(j\omega) < 0$ for $\omega > 0$

$\Rightarrow$ phase lag

⑤ $$|H_5(j\omega)| = \frac{1+100\omega^2}{\left(1-\frac{\omega^2}{100}\right)^2 + \frac{1}{25}\omega^2} \geq 1 \quad \forall \omega$$

$\Rightarrow$ amplification

numerically evaluate $\measuredangle H_5(j\omega)$

$\Rightarrow$ $\measuredangle H_4(j\omega) < 0$ for $\omega > 9.9$ rad

$\Rightarrow$ phase lag

Q4 CT filters and phase response

1 a $H(j\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise} \end{cases}$

1 mark

overall system is LTI with

$$Y(j\omega) = X(j\omega) - X(j\omega) H(j\omega)$$
$$= X(j\omega) \underbrace{(1 - H(j\omega))}_{\text{overall frequency response } Z(j\omega)}$$

$$Z(j\omega) = \begin{cases} 0 & |\omega| \geq \omega_c \\ 1 & |\omega| < \omega_c \end{cases}$$

$\Rightarrow$ ideal lowpass filter //

1 b $Z(j\omega) = 1 - |H(j\omega)| e^{jf(\omega)}$

1 mark

$$= \begin{cases} 1 - e^{jf(\omega)} & |\omega| \geq \omega_c \\ 1 & |\omega| < \omega_c \end{cases}$$

$\Rightarrow$ $|Z(j\omega)|$ generally nonzero for $|\omega| > \omega_c$

can be > 1 for some ω

$\Rightarrow$ not an (ideal) lowpass filter anymore //

(2) $Y_1(j\omega) = H_1(j\omega)\, X(j\omega)$

1mark $Y_2(j\omega) = H_2(j\omega)\, X(j\omega) = H_1(j\omega)\, e^{j\omega\alpha}\, X(j\omega)$

$= Y_1(j\omega)\, e^{j\omega\alpha}$

$\Rightarrow \quad y_2(t) = y_1(t+\alpha)$

$\Rightarrow$ system 2 shifts the output compared to system 1 //

($\alpha < 0$: shift = delay

$\alpha > 0$: shift = advance)

Q5 DTFT System Cascade

① • $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})}$

Table 5.2 $\quad X(e^{j\omega}) = \dfrac{1}{1-\frac{1}{2}e^{-j\omega}}$

$$y[n] = (n+1)\left(\tfrac{1}{2}\right)^n u[n] - 3\left(\tfrac{1}{2}\right)^n u[n]$$

Table 5.2 $\quad Y(e^{j\omega}) = \dfrac{1}{(1-\frac{1}{2}e^{-j\omega})^2} - 3\dfrac{1}{1-\frac{1}{2}e^{-j\omega}}$

$$= \frac{1-3+\frac{3}{2}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})^2}$$

$$= \frac{-2+\frac{3}{2}e^{-j\omega}}{(1-\frac{1}{2}e^{-j\omega})^2}$$

0.5 mark

$$\Rightarrow H(e^{j\omega}) = \frac{-2+\frac{3}{2}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}} \quad //$$

• $h[n]$: $\quad H(e^{j\omega}) = -2\dfrac{1}{1-\frac{1}{2}e^{-j\omega}} + \dfrac{3}{2}\dfrac{e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$

Table 5.1, 5.2 $\Rightarrow h[n] = -2\left(\tfrac{1}{2}\right)^n u[n] + \tfrac{3}{2}\left(\tfrac{1}{2}\right)^{n-1} u[n-1] \quad //$

0.5 mark

$$= -2\left(\tfrac{1}{2}\right)^n u[n] + 3\left(\tfrac{1}{2}\right)^n u[n-1]$$

$$= -2\,\delta[n] + \left(\tfrac{1}{2}\right)^n u[n-1]$$

2. $h[n] = 0 \quad \forall n<0 \Rightarrow$ causal //

0.5

$$\sum_{n=-\infty}^{\infty} |h[n]| = 2 + \sum_{n=1}^{\infty}\left(\tfrac{1}{2}\right)^n = 1 + \sum_{n=0}^{\infty}\left(\tfrac{1}{2}\right)^n$$

$$= 1 + \frac{1}{1-\frac{1}{2}} = 3$$

0.5

$\Rightarrow$ stable //

3. $$H^{inv}(e^{j\omega}) = \frac{1}{H(e^{j\omega})} = \frac{1-\frac{1}{2}e^{-j\omega}}{-2+\frac{3}{2}e^{-j\omega}}$$

$$= \frac{-\frac{1}{2}+\frac{1}{4}e^{-j\omega}}{1-\frac{3}{4}e^{-j\omega}}$$

$$= -\frac{1}{2}\frac{1}{1-\frac{3}{4}e^{-j\omega}} + \frac{1}{4}\frac{e^{-j\omega}}{1-\frac{3}{4}e^{-j\omega}}$$

0.5 mark

Table 5.1
5.2 $\Rightarrow h^{inv}[n] = -\frac{1}{2}\left(\frac{3}{4}\right)^n u[n] + \frac{1}{4}\left(\frac{3}{4}\right)^{n-1} u[n-1]$

$$= -\frac{1}{2}\left(\frac{3}{4}\right)^n u[n] + \frac{1}{3}\left(\frac{3}{4}\right)^n u[n-1]$$

$$= -\frac{1}{2}\delta[n] - \frac{1}{6}\left(\frac{3}{4}\right)^n u[n-1]$$

$h[n] = 0$ for $n<0 \Rightarrow$ causal 0.25

$\sum_n |h[n]| < \infty \Rightarrow$ stable 0.25 marks

4

- $$H(e^{j\omega}) = \frac{-2 + \frac{3}{2} e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$\left(-2 + \frac{3}{2} e^{-j\omega}\right) X(j\omega) = \left(1 - \frac{1}{2} e^{-j\omega}\right) Y(e^{j\omega})$$

0.5 mark $$-2\, x[n] + \frac{3}{2} x[n-1] = y[n] - \frac{1}{2} y[n-1]$$

- inverse system: change roles of x and y

0.5 marks $$-2\, y[n] + \frac{3}{2} y[n-1] = x[n] - \frac{1}{2} x[n-1]$$