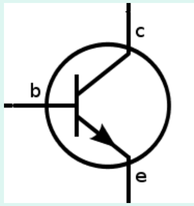
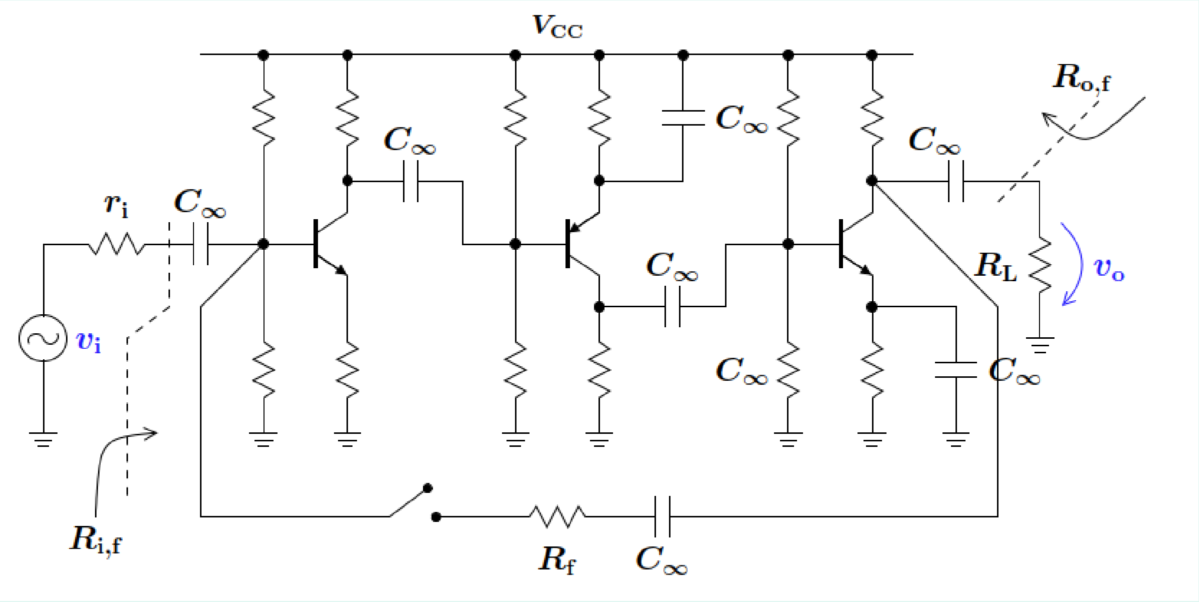
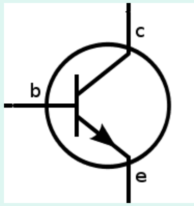


ELEC 301 - LTI systems

L05 - Sep 15

Instructor: Edmond Cretu



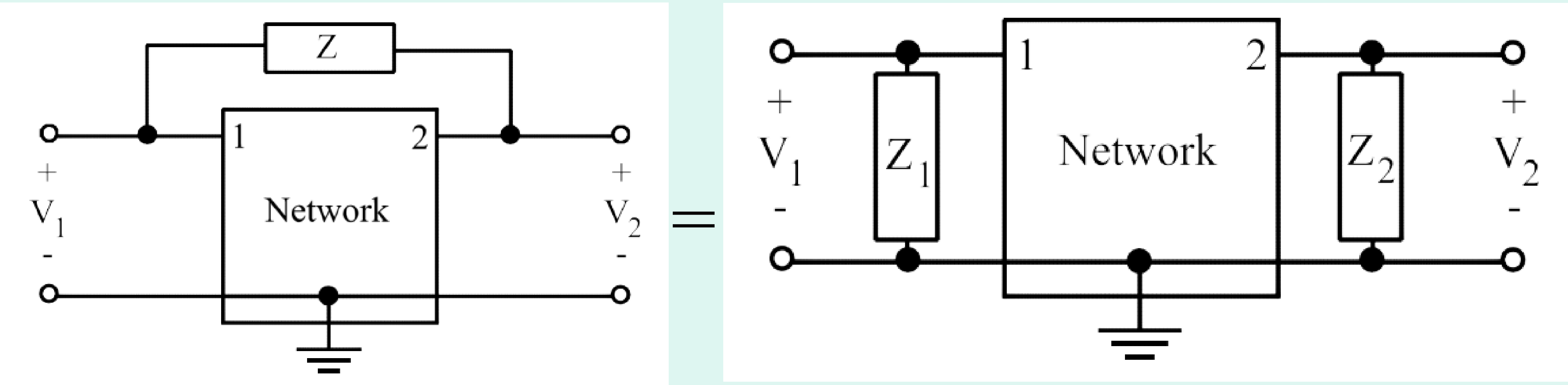


Last time

- BJT vs MOSFET (current-controlled vs voltage-controlled)
- Systems aspects: design, **analysis**, optimization, synthesis, approximation
- Linear (and time-invariant) systems - linearity, superposition, Thevenin/Norton reduction, Millman and Miller theorems

Miller's theorem - decoupling feedback

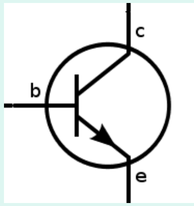
- Replace the Z feedback with two impedances Z1 and Z2



$$V_2 = kV_1$$

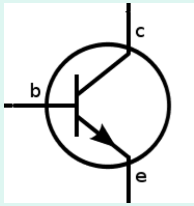
$$V_2 = kV_1$$

$$Z_1 = Z\frac{1}{1-k} \qquad Z_2 = Z\frac{k}{k-1}$$



Today

- Review of analysis methods for LTI systems
  - time-domain (unit impulse response + convolution product)
  - Spectral domain - Fourier transform
  - (Extended) spectral domain - Laplace transform, operational calculus
  - transfer function h(t) < = > H(s)
- Next: Frequency response - Bode plots, LP-BP-HP transfer functions



General analysis principle

- Decompose the input signal into elementary components (a complete base set of signals)
- Characterize the system response to the elementary signals
- Use superposition and time invariance to reconstruct the output from the responses to elementary components

System modeling and analysis in the time domain

LTI systems are completely characterized by their unit impulse response

Main principle: “divide et impera” – break a complex problem into simpler one: **decompose signals using an elementary signals set (basis)**, and exploit linearity and time invariance for LTI systems

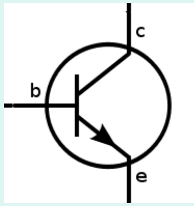
Time-domain analysis of LTI systems

The response of a given LTI system to unit impulse is enough in order to know everything about the system behavior

Principle: decompose signals in terms of Dirac/unit impulses

$$x(t)=\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)\,d\tau,\; x[n]=\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

$$H\{\delta(t)\}=h(t)\Rightarrow H\{x(t)\}=h*x(t)=\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$



LTI description

Time-domain description – based on representing signals as linear combination of shifted impulses
Easy signal decomposition, but the unit impulse changes its shape in propagating through a system (unit impulse response)

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \longrightarrow \boxed{H\{\ \}} \longrightarrow y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

A more general perspective: because of linearity, any linear decomposition of the input signal in terms of a set of basic signals will give an output as a linear combination of the system's responses to the basic signals considered

$$x[n] = \sum_{-\infty}^{\infty} a_k f_k[n] \longrightarrow \boxed{H\{\ \}} \longrightarrow y[n] = \sum_{-\infty}^{\infty} a_k H\{f_k[n]\}$$

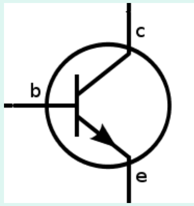
Frequency-domain analysis of LTI systems

Principle: decompose the input signal in terms of harmonic exponentials [Why?]

$$Fourier\ transform: x(t) = \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega \xleftrightarrow{F} X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$h(t) \xleftrightarrow{F} H(\omega) = transfer\ function = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt$$

$$x(t) \xrightarrow{H} y(t) = h * x(t) \Leftrightarrow X(\omega) = F(x(t)) \xrightarrow{H} Y(\omega) = H(\omega) X(\omega)$$



The Fourier Transform

Used to represent a continuous-time nonperiodic signal as a superposition of complex sinusoids

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega \xleftrightarrow{FT} X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

The inverse Fourier transform

$$x(t) = F^{-1}\{X(j\omega)\}$$

The (direct) Fourier transform

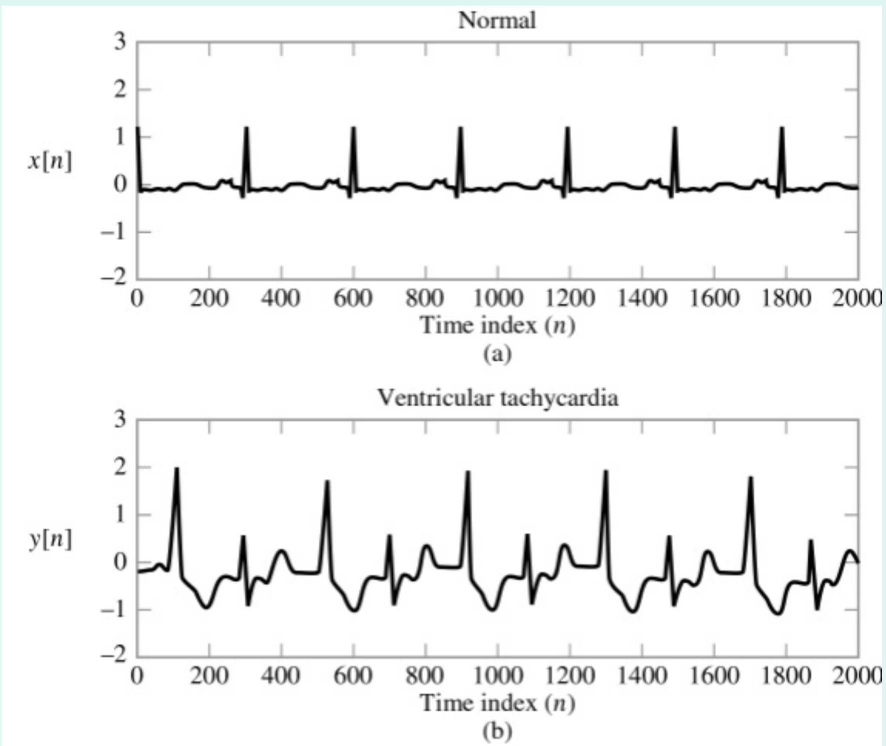
$$X(j\omega) = F\{x(t)\}$$

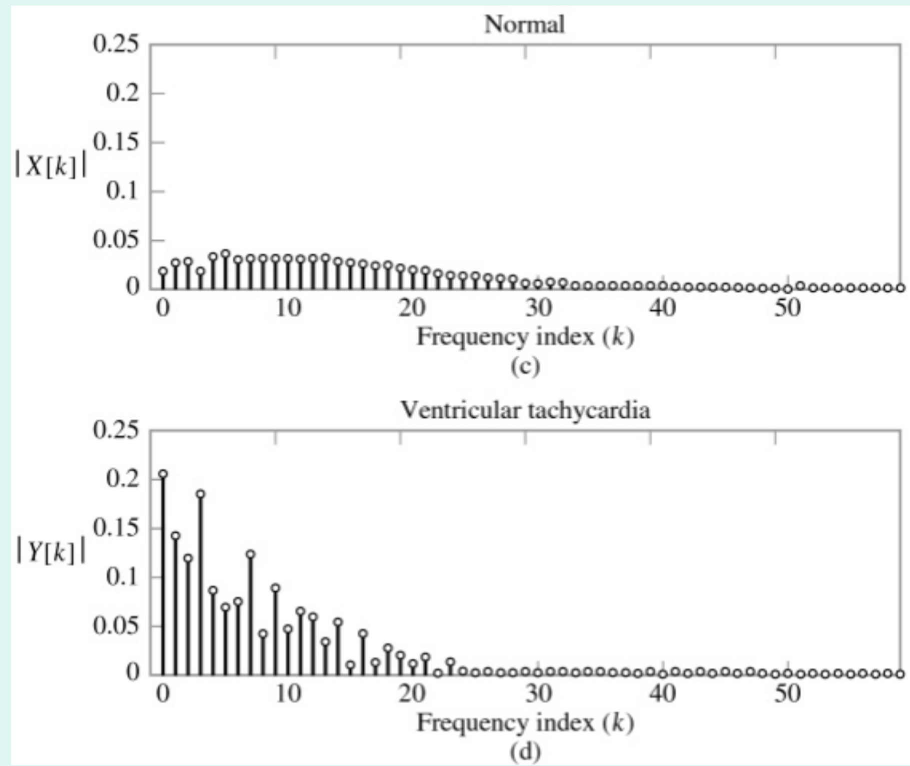
We say that x(t) and X(jω) are a **Fourier transform pair**

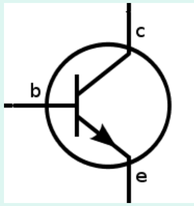
The transform X(jω) describes the signal as a function of frequency ω and it is called the **frequency-domain representation** of x(t)

Exm: electrocardiograms

EGG waveform analysis – normal heart beat vs. ventricular tachycardia – first 60 coefficients







Key points:

The importance of exponential and harmonic signals:

- For LTI systems – such signals propagate without changing their shape
- In nature – many natural sensors operate based on the harmonic decomposition of the signals (e.g. ear)
- Periodic signals – attention to the periodicity of DT signals
- You need a minimum 2nd order differential eqn. to generate an oscillatory solution!

Interpretation of the frequency response

Sinusoidal steady-state response:

$$H(\omega) = |H(\omega)| e^{j\arg\{H(\omega)\}}$$ - polar form

|H(ω)| is the **magnitude response** of the system – it shows the scaling of the input

arg{H(ω)} is the **phase response** of the system – it shows the (frequency-dependent) delay in propagating a sinusoid through the system

The frequency response is reduced to two real-valued functions

LTI systems analysis

Time vs. frequency – 2nd look

$$\delta(t) \longrightarrow \boxed{\text{LTI S}} \longrightarrow h(t)$$

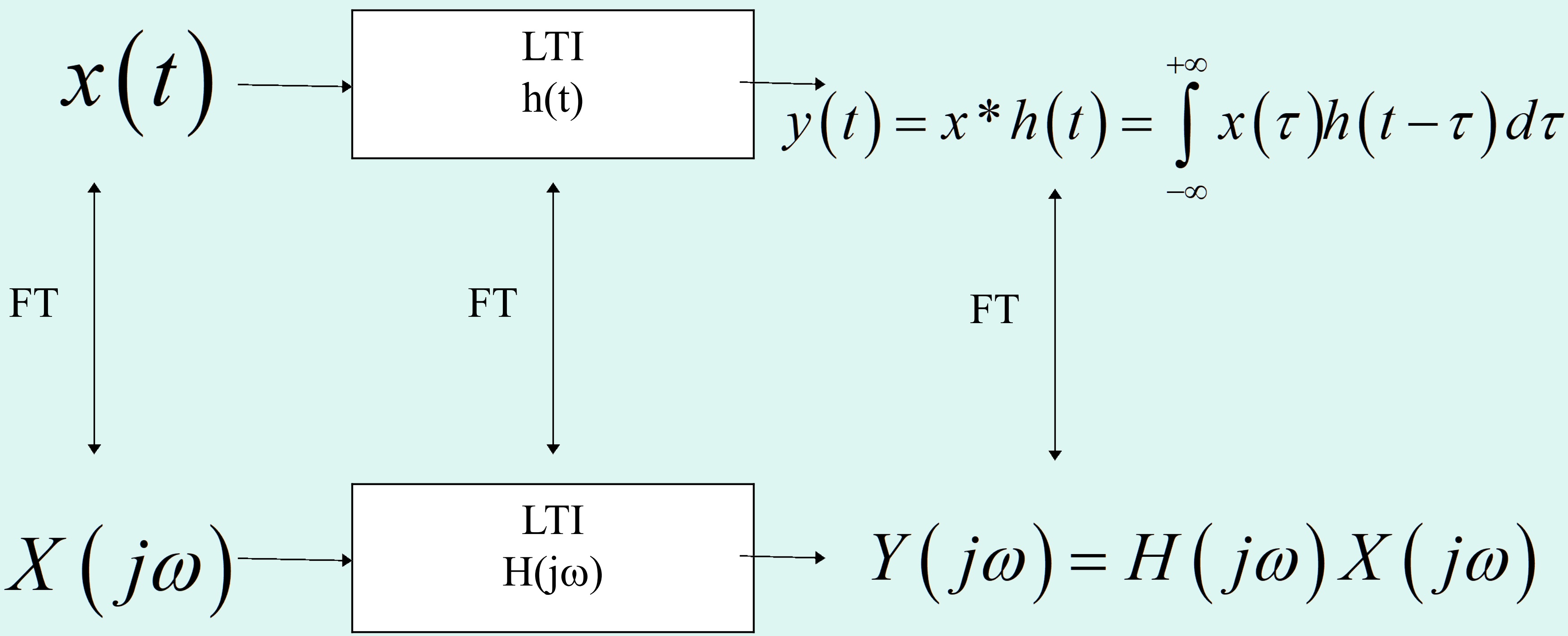
$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)\,d\tau \longrightarrow \boxed{\text{LTI S}} \longrightarrow y(t) = x * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau$$

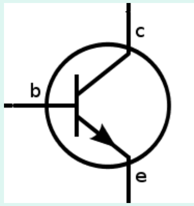
$$e^{j\omega t} \longrightarrow \boxed{\text{LTI S}} \longrightarrow H(j\omega)e^{j\omega t}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega \longrightarrow \boxed{\text{LTI S}} \longrightarrow y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)H(j\omega)e^{j\omega t}\,d\omega$$

Input-output perspective

LTI system





(Unilateral) Laplace Transform

Laplace Transform as extension of Fourier Transform

- (Unilateral) Laplace Transform – extend the range of possible signals (useful for stability analysis, transient analysis)
- Unlike Fourier Transform, Laplace Transform does not preserve the energy of the signal in the spectral domain ⇨ the numerical inversion of Laplace Transform is difficult
- LT - used as “operational calculus” directly mapping circuits (+ their initial conditions) from time domain into the s-domain ($d/dt \rightarrow s$, $\int dt \rightarrow 1/s$)

The unilateral Laplace transform

In normal analysis procedures, we are interested with what happens with the evolution of a system only after a certain moment of time t_0, and store all the past history in only a couple of values ("initial conditions")

we assume an arbitrary time origin ($t_0=0$), and all the input signals as causal (zero for $t<0$)

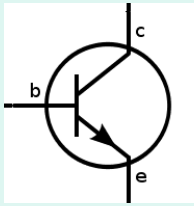
The causality assumption removes the ambiguity existent in bilateral LT (so we do need to consider the ROC)

Main applications: the analysis of linear electrical circuits, or systems described by differential equations with initial conditions

$$x(t) \overset{L_u}{\longleftrightarrow} X(s)$$

The unilateral Laplace transform: $X(s) = L_u\{x(t)\} = \int_{0-}^{\infty} x(t)e^{-st}dt$

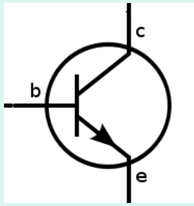
The inverse unilateral transform: $x(t)u(t) = L_u^{-1}\{X(s)\} = \dfrac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$



Remarks

$$X(s)=\int_{0^-}^{\infty} x(t)e^{-st}\,dt$$

- The lower 0^- limit means that we do include in the integral discontinuities and impulses that occur at t=0 (e.g. Dirac impulse) the unilateral and bilateral Laplace transforms are equivalent for causal signals (zero for t<0)
- The inverse LT is generally computed not by using contour integration in the complex plane, but by combining properties with partial-fraction expansion
- **Convergence**: for a signal x(t) to have a Laplace transform, it is sufficient to satisfy $\int_{0-}^{\infty} |x(t)|e^{-\sigma t}\,dt < \infty$ for some real positive $\sigma>0$



Fourier vs. Laplace transform

Definitions:

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}\,d\omega \xleftrightarrow{FT} X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}\,dt$$

$$x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}\,ds \xleftrightarrow{LT} X(s)=\int_{0-}^{+\infty}x(t)e^{-st}\,ds,$$

$$s=\sigma+j\omega, with\,\sigma>\sigma_{\min}$$

- **Energy preservation**: only FT preserves the energy (orthogonal transform)⇨ easier to numerically compute both the direct and inverse FT (good numerical stability)
- Class of signals: LT allowed for an extended class of (causal) signals
- **Linearity**: both FT and LT are linear operators
- Transformations on calculus operators (d/dt, integral): mapping to algebraic operations (jω* or 1/(jω))
- Initial conditions: explicitly kept in LT (transient +steady-state solutions)

• LT properties

Linearity: $ax(t)+by(t)\overset{L_u}{\longleftrightarrow}aX(s)+bX(s)$

Convolution (causal signals): $x(t)*y(t)\overset{L_u}{\longleftrightarrow}X(s)Y(s)$

Scaling: $x(at)\overset{L_u}{\longleftrightarrow}\dfrac{1}{a}X\left(\dfrac{s}{a}\right)$, for $a>0$

Time shift: $\mathrm{x}(t-\tau)\overset{L}{\longleftrightarrow}e^{-s\tau}X(s)$, for $\tau>0$

s-plane shift: $e^{s_0 t}x(t)\overset{L_u}{\longleftrightarrow}X(s-s_0)$

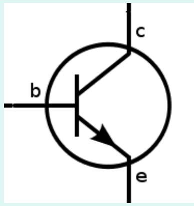
s-domain differentiation: $-tx(t)\overset{L_u}{\longleftrightarrow}\dfrac{d}{ds}X(s)$

Time differentiation: $\dfrac{d}{dt}x(t)\overset{L_u}{\longleftrightarrow}sX(s)-x(0^-)$

$$\frac{d^2x}{dt^2}(t)\overset{L_u}{\longleftrightarrow}s^2X(s)-sx(0^-)-\frac{dx}{dt}(0^-)$$

Time integration:

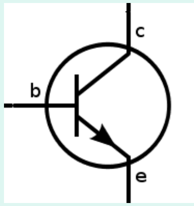
$$\int_{-\infty}^{t}x(\tau)d\tau\overset{L_u}{\longleftrightarrow}\frac{X(s)}{s}+\frac{1}{s}x^{(-1)}(0^-),\ \text{where } x^{(-1)}(0^-)=\int_{-\infty}^{0^-}x(\tau)d\tau$$



Laplace transform pairs

Table 2.3 Important Laplace Transform Pairs

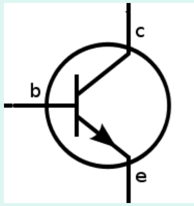
$f(t)$	$F(s)$
Step function, $u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$f^{(k)}(t) = \frac{d^k f(t)}{dt^k}$	$s^k F(s) - s^{k-1} f(0^-) - s^{k-2} f'(0^-) - \ldots - f^{(k-1)}(0^-)$



Laplace transform pairs (2)

Table 2.3 Important Laplace Transform Pairs

$f(t)$	$F(s)$
$\int_{-\infty}^{t} f(t)\,dt$	$\frac{F(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} f(t)\,dt$
Impulse function $\delta(t)$	1
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{1}{\omega}[(\alpha-a)^2+\omega^2]^{1/2}e^{-at}\sin(\omega t+\phi)$, $\phi = \tan^{-1}\frac{\omega}{\alpha-a}$	$\frac{s+\alpha}{(s+a)^2+\omega^2}$
$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin \omega_n\sqrt{1-\zeta^2}t,\ \zeta < 1$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$



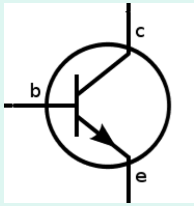
Transfer function

- In the case of a general dynamical system description (implicit characterization through a system of differential eqns.):

$$\frac{d^N y}{dt^N} + a_{N-1}\frac{d^{N-1} y}{dt^{N-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y = b_M\frac{d^M x}{dt^M} + b_{M-1}\frac{d^{M-1} x}{dt^{M-1}} + .. + b_0 x$$

- The input-output **transfer function** is obtained when **all ICs are set to zero**:

$$H(s) = \left.\frac{Y(s)}{X(s)}\right|_{IC=0} = \frac{b_M s^M + b_{M-1}s^{M-1} + \ldots + b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + .. + a_1 s + a_0}$$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