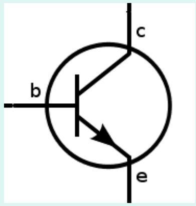
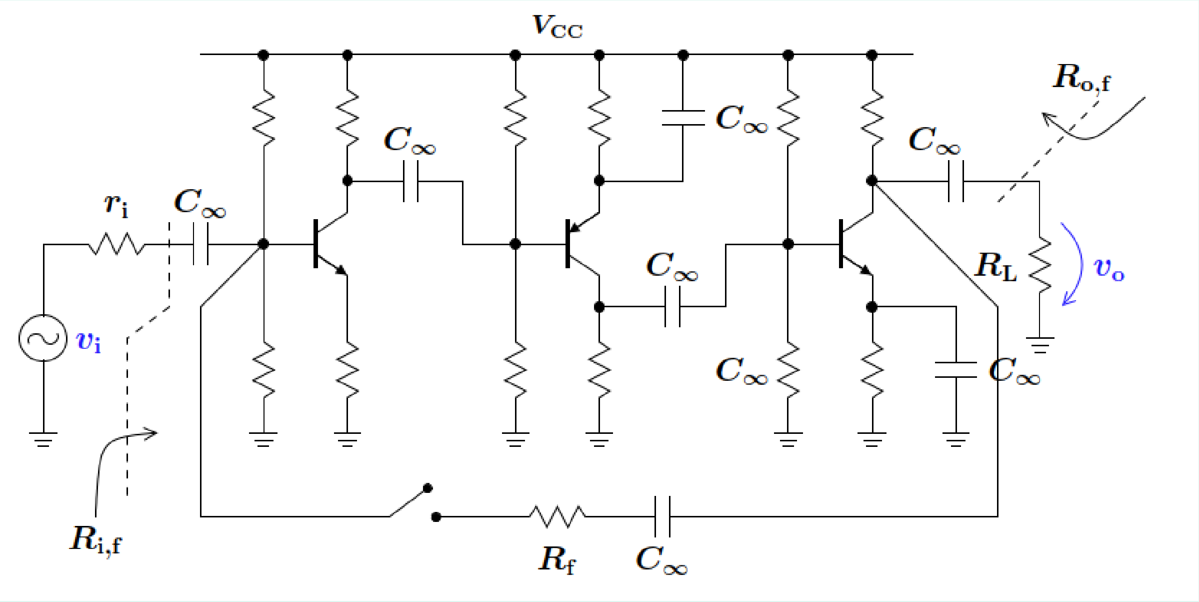
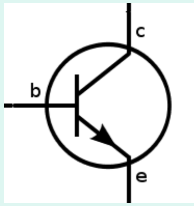


ELEC 301 - LTI systems

L06 - Sep 16

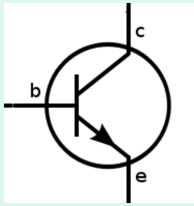
Instructor: Edmond Cretu





Last time

- General analysis approaches for LTI systems
- From time domain analysis to Fourier Transform
- From Fourier Transform to Laplace transform
- LT properties and usage



The unilateral Laplace transform

$$x(t) \overset{L_u}{\longleftrightarrow} X(s)$$

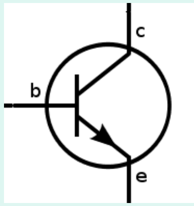
The unilateral Laplace transform: $X(s) = L_u\{x(t)\} = \int_{0-}^{\infty} x(t)e^{-st}dt$

The inverse unilateral transform: $x(t)u(t) = L_u^{-1}\{X(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$

$$y(t) = h(t) * x(t) \overset{LT}{\longleftrightarrow} Y(s) = H(s)X(s)$$

- Main applications:

- solving linear differential equations with constant coefficients
- solving electrical circuits (transfer functions) by mapping them into s-domain



Laplace Transform as "Operational calculus"

- Technique for solving linear systems (e.g. electrical circuits)
- First developed on a large scale by Oliver Heaviside (UK) as a collection of rules (**operational calculus**)

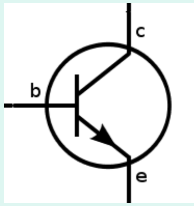
"Shall I refuse my dinner because I do not fully understand the process of digestion?" – O. Heaviside

Linear differential equations with initial conditions - linear circuits

- Is perhaps the main application of the unilateral Laplace transform
- Main used properties:
  - - differential operators transformed into algebraic operations in the complex domain
  - - initial conditions explicitly incorporated into the solution (differentiation property)

$$\frac{d}{dt}x(t) \xleftrightarrow{L_u} sX(s) - x(0^-)$$

$$\frac{d^2}{dt^2}x(t) \xleftrightarrow{L_u} s^2X(s) - sx(0^-) - \frac{dx}{dt}(0^-)$$



Linear differential equations

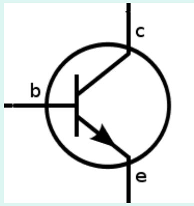
- Physical LTI systems reduced to differential equations (implicit dynamics description of input-output relation)
- Example:

$$a_2 \frac{d^2 y}{dt^2}(t) + a_1 \frac{dy}{dt}(t) + a_0 y(t) = b_1 \frac{dx}{dt}(t) + b_0 x(t)$$

- Mapping to Laplace domain:

$$a_2 \left(s^2 Y(s) - s y(0^-) - \frac{dy}{dt}(0^-) \right) + a_1 \left(sY(s) - y(0^-) \right) + a_0 Y(s) = b_1 \left(sX(s) - x(0^-) \right) + b_0 X(s)$$

$$Y(s) = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} X(s) - \frac{b_1}{a_2 s^2 + a_1 s + a_0} x(0^-) + \frac{a_2 s + a_1}{a_2 s^2 + a_1 s + a_0} y(0^-) + \frac{a_2}{a_2 s^2 + a_1 s + a_0} y'(0^-)$$



Initial- and final-value theorems

We can compute the initial value x(0+) and the final value x(+∞) of x(t) directly from X(s)

The initial-value theorem:
Restriction: it does not apply to rational functions

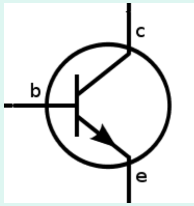
$$x(0^+) = \lim_{s\to\infty} sX(s)$$

$$X(s) = \frac{Z(s)}{P(s)}, \text{ with } grade\{Z(s)\} \geq grade\{P(s)\}$$

Interpretation: small time behavior is dominated by high frequencies (poles far from the Re{s} axis)

$$x(\infty) = \lim_{s\to 0} sX(s)$$

The final-value theorem:
Restriction: it applies only if all the poles of X(s) are in the left half of s-plane, with at most a single pole at s=0
Interpretation: the long time behavior is dominated by low frequencies or poles close to or on the Re{s} axis



Example

$$x(t) = e^{-\alpha t}\cos(\omega_0 t)u(t) \overset{L_u}{\longleftrightarrow} X(s) = L[x(t)] = \frac{s+\alpha}{(s+\alpha)^2 + {\omega_0}^2}$$

First alternative: computation in time domain

$$x(0^+) = \lim_{t\to 0} x(t) = \lim_{t\to 0} e^{-\alpha t}\cos(\omega_0 t)u(t) = 1$$

$$x(\infty) = \lim_{t\to\infty} x(t) = 0$$

Computation using Laplace initial- and final-value theorems

$$x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{s(s+\alpha)}{(s+\alpha)^2 + \omega_0^2} = 1$$

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{s(s+\alpha)}{(s+\alpha)^2 + \omega_0^2} = 0$$

Linear differential equations (2)

- Clear separation between the effects of initial conditions and the input signal
- The effect of ICs, for a stable system, attenuates exponentially in time, disappearing in the steady-state solution

$$Y(s)=\frac{b_1 s+b_0}{a_2 s^2+a_1 s+a_0}X(s)\underbrace{-\frac{b_1}{a_2 s^2+a_1 s+a_0}x(0^-)+\frac{a_2 s+a_1}{a_2 s^2+a_1 s+a_0}y(0^-)+\frac{a_2}{a_2 s^2+a_1 s+a_0}y'(0^-)}_{ICs}$$

The **transfer function**:

$$H(s)=\left.\frac{Y(s)}{X(s)}\right|_{IC=0}=\frac{b_1 s+b_0}{a_2 s^2+a_1 s+a_0}$$

The effect of the ICs in the steady-state solution, for a stable system (no pole in the rhp):

$$\lim_{t\to\infty}\left.y(t)\right|_{x(t)=0}=\lim_{s\to 0}s\left(-\frac{b_1}{a_2 s^2+a_1 s+a_0}x(0^-)+\frac{a_2 s+a_1}{a_2 s^2+a_1 s+a_0}y(0^-)+\frac{a_2}{a_2 s^2+a_1 s+a_0}y'(0^-)\right)=0$$

Poles and zero

LT links the transfer function description with the implicit description as dynamical system

The transfer function for LTI systems characterized by differential equations is in the form of rational function:

$$H(s) = H_0 \frac{(s - z_1)(s - z_2)\ldots(s - z_M)}{(s - p_1)(s - p_2)\ldots(s - p_N)}$$

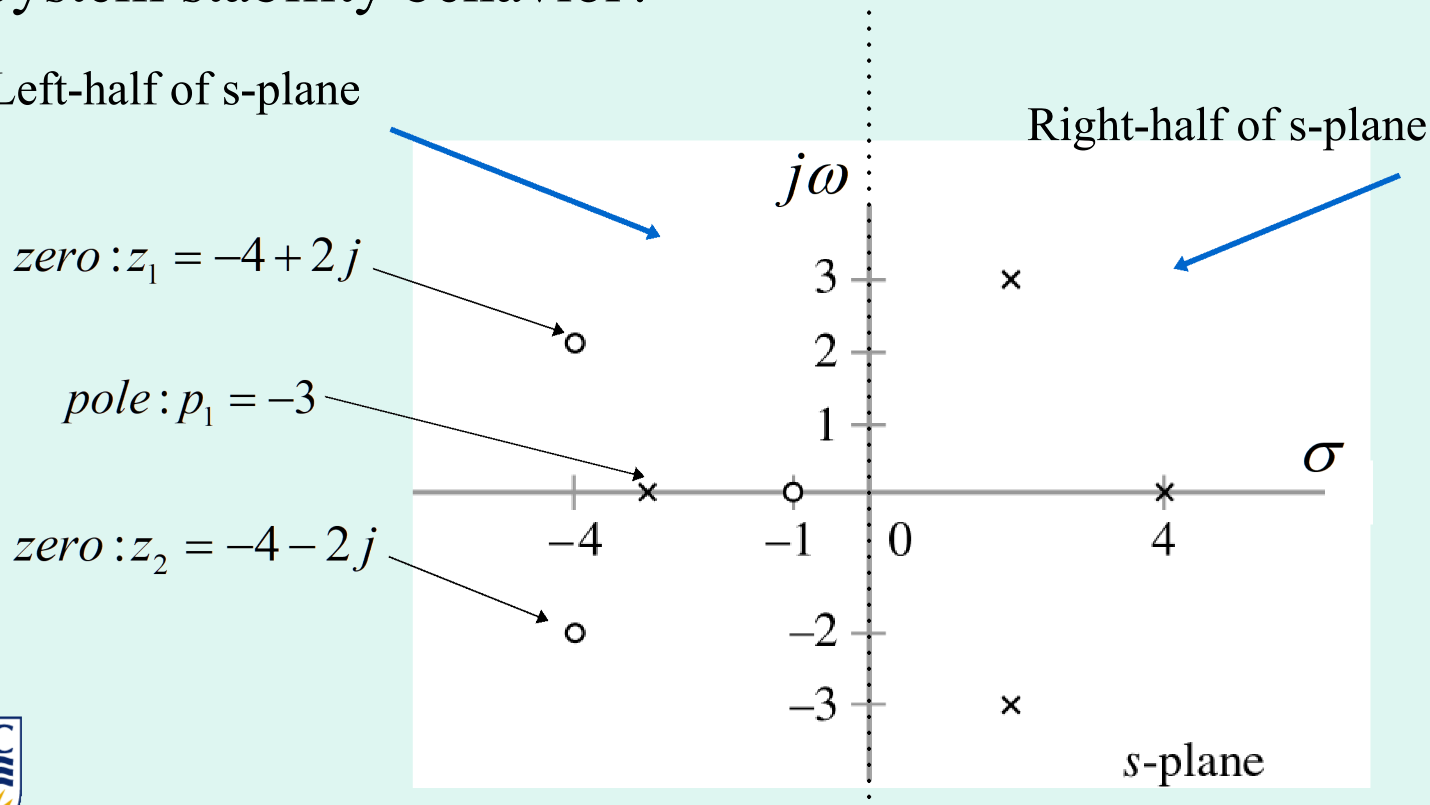
The roots of the numerator polynomial=the **zeros** of H(s) (marked 0), $H(z_i)=0$

The roots of the denominator polynomial=the **poles** of H(s) (marked X), $H(p_j)\rightarrow\infty$

The location of poles and zeros in the s-plane uniquely specifies H(s) up to a constant gain factor

The s-plane. Poles and zeros

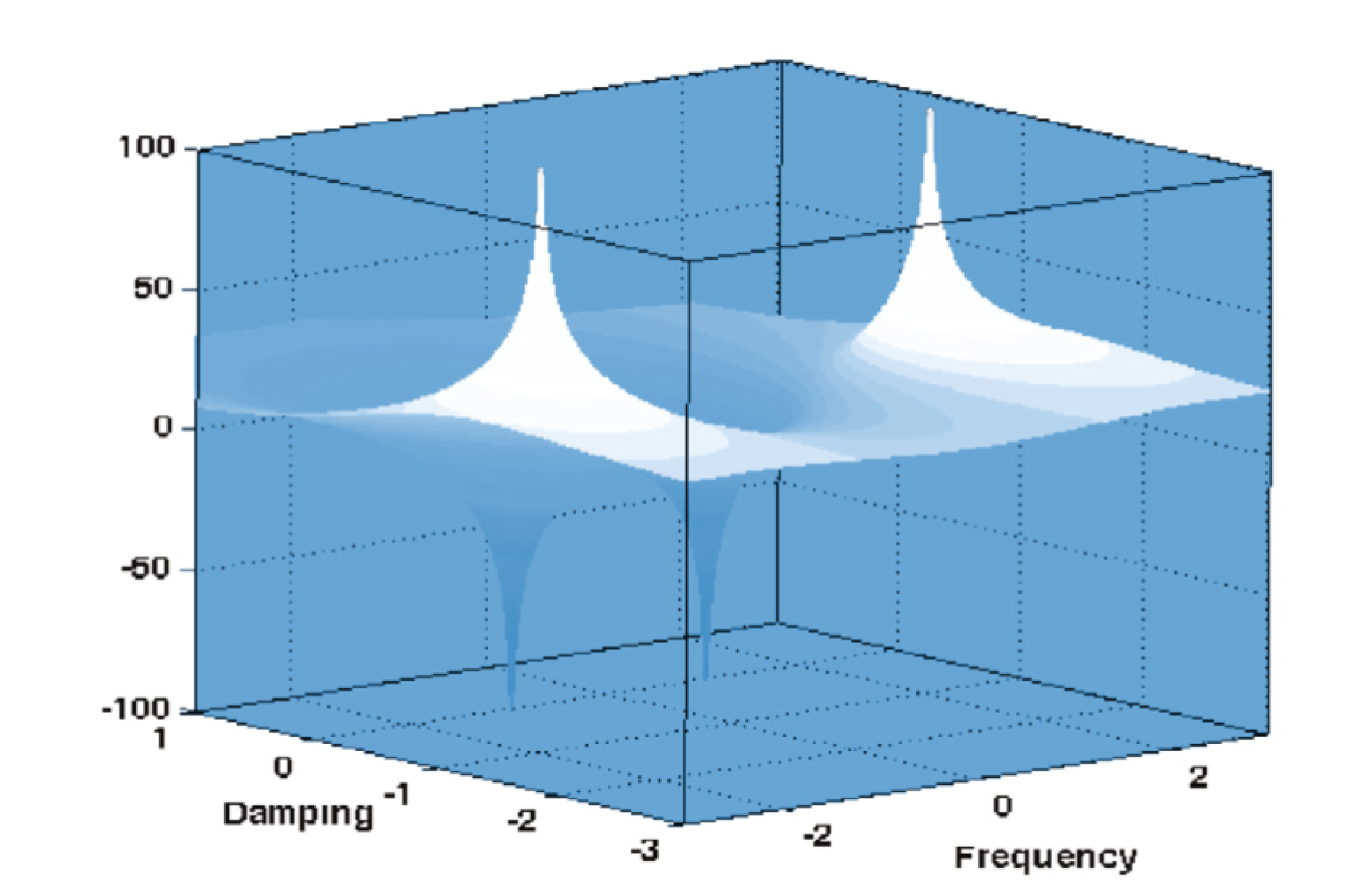
LT introduces an exponential damping factor σ, besides the sinusoidal frequency ω => complex s-plane
FT operates only on the imaginary axis => if x(t) is absolutely integrable, then we may obtain FT from LT by setting σ=0
System stability behavior?

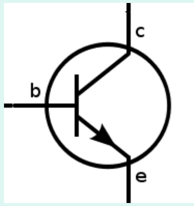


Example

Log(|H(s)|) plot

$$H(s) = \frac{2(s^2+1)}{s^2+2s+5} = \frac{2(s+j)(s-j)}{(s+1-2j)(s+1+2j)}$$





Some common unilateral Laplace transforms

$$\delta(t) \xleftrightarrow{L_u} 1$$

$$u(t) \xleftrightarrow{L_u} \frac{1}{s}$$

$$t^n u(t) \xleftrightarrow{L_u} \frac{n!}{s^{n+1}}$$

$$e^{at} u(t) \xleftrightarrow{L_u} \frac{1}{s-a}$$

$$\cos(\omega t) u(t) \xleftrightarrow{L_u} \frac{s}{s^2+\omega^2}$$

$$\sin(\omega t) u(t) \xleftrightarrow{L_u} \frac{\omega}{s^2+\omega^2}$$

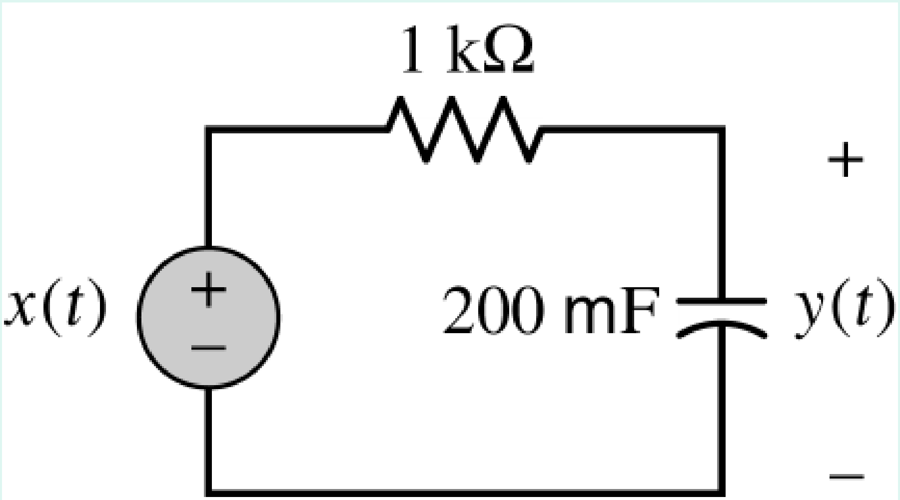
$$e^{at} \cos(\omega t) u(t) \xleftrightarrow{L_u} \frac{s-a}{(s-a)^2+\omega^2}$$

$$e^{at} \sin(\omega t) u(t) \xleftrightarrow{L_u} \frac{\omega}{(s-a)^2+\omega^2}$$

Solving circuits using Laplace transform

- Convert the circuit from time domain to Laplace s-domain (ICs on capacitors and inductors will become equivalent voltage/current sources)
- Solve the circuit to find the required output Y(s) like you would do for DC networks (incremental circuit reduction techniques, mesh or nodal analysis)
- Once Y(s) is determined, compute y(t) through inverse Laplace transform (usually through partial fraction expansion)

LT in energy flow modeling: RC circuit analysis



$$x(t) = \frac{3}{5} e^{-2t} V u(t)$$

$$y(0^-) = -2V$$

Alternative (2) approach - ODE in time domain from circuit analysis

$$x(t) = R\,i(t) + y(t) = RC\frac{dy}{dt}(t) + y(t)$$

$$RC\frac{dy}{dt}(t) + y(t) = x(t) \xrightarrow{LT} RC\left(sY(s) - y(0^-)\right) + Y(s) = X(s)$$

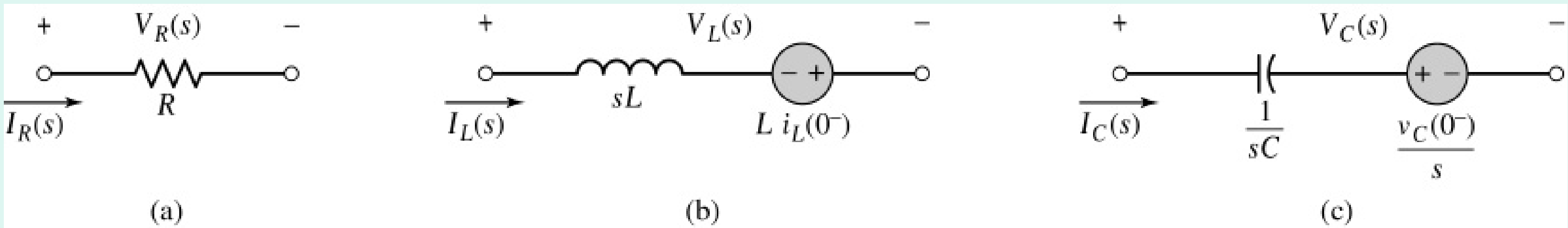
$$Y(s) = \frac{1}{1+sRC}X(s) + \frac{RC}{1+sRC}y(0^-) = \frac{1}{1+200s}\frac{3}{5}\frac{1}{s+2}V + \frac{200}{1+200s}(-2)V$$

$$Y(s) = \frac{3}{1000}\frac{1V}{(s+2)(s+0.005)} - \frac{2V}{s+0.005}$$

Partial fraction decomposition + ILT

Mapping circuits in the s-domain

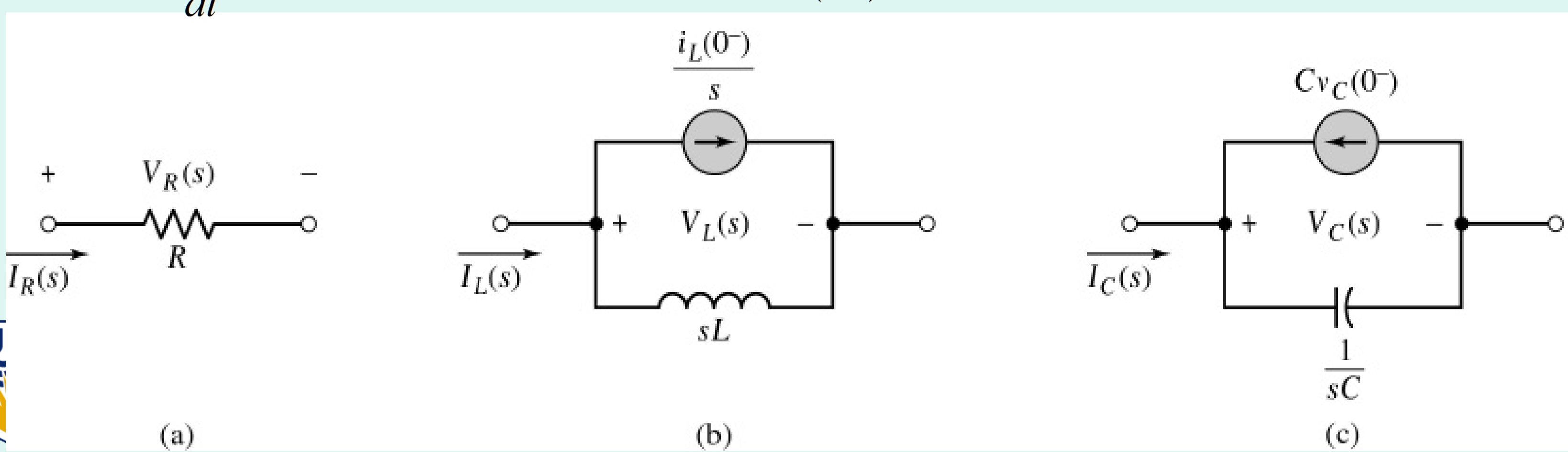
Linear circuits with R,L,C + sources elements can be directly mapped into the complex frequency space (Alternative (3)



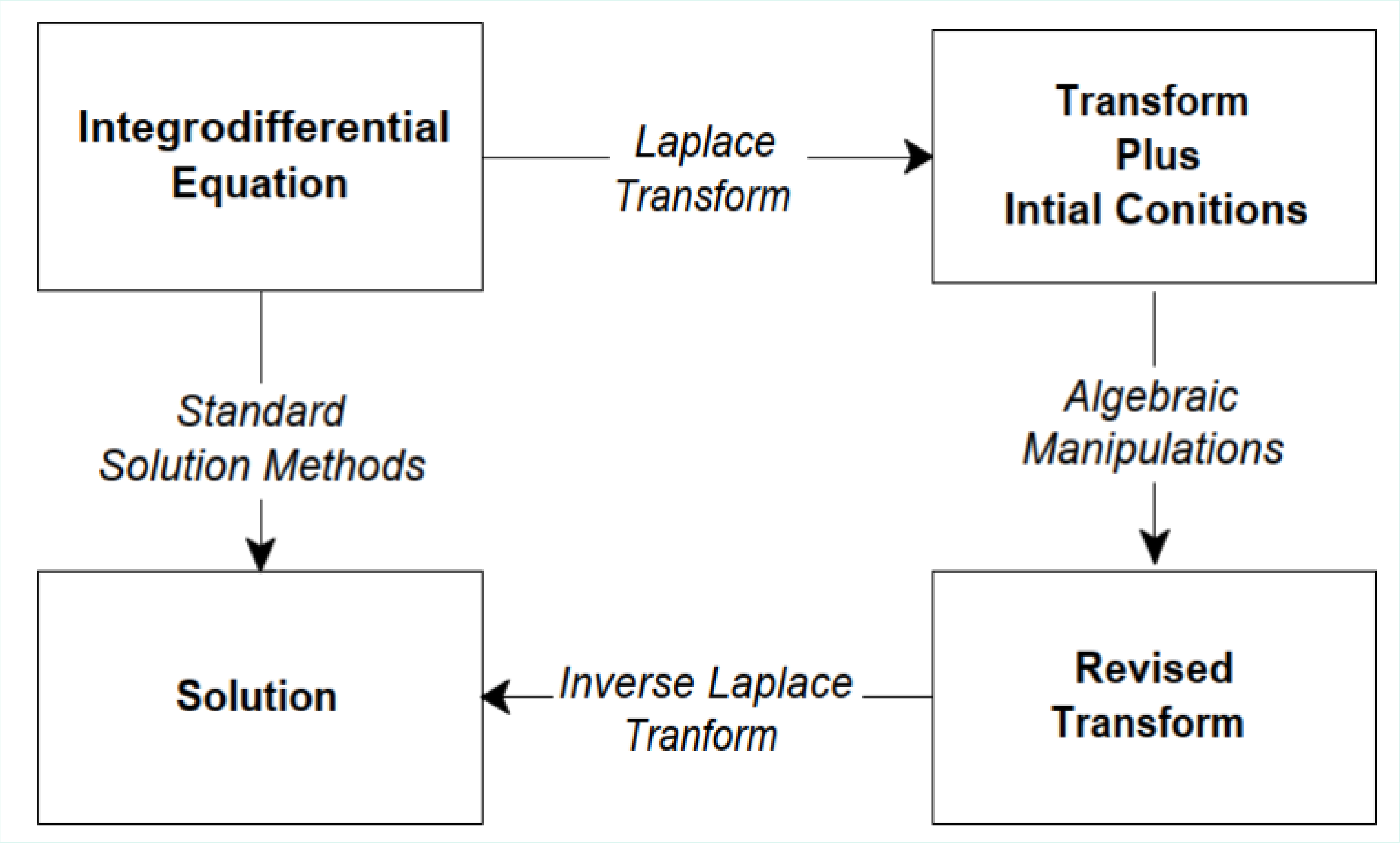
$$v_R(t) = Ri_R(t) \overset{L_u}{\longleftrightarrow} V_R(s) = RI_R(s)$$

$$v_L(t) = L\frac{di_L}{dt}(t) \overset{L_u}{\longleftrightarrow} V_L(s) = sLI_L(s) - Li_L(0^-)$$

$$i_C(t) = C\frac{dv_C}{dt}(t) \overset{L_u}{\longleftrightarrow} I_c(s) = sCV_C(s) - Cv_C(0^-)$$



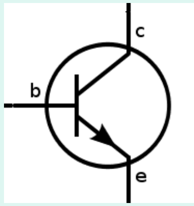
The use of Laplace transform



Frequency response - Bode plots

- **Steady-state response** of a system to a sinusoidal input test signal (no ICs)
- Frequency response is the transfer function H(s) when s=jω
- Bode plots - graphically display the log- magnitude and phase of H(s) vs. log(ω)
- REM: Bode plots are **graphical approximation techniques** in the spectral domain
- System bandwidth concept

$$H(s) = H_0 \frac{(s - z_1)(s - z_2)\dots(s - z_M)}{(s - p_1)(s - p_2)\dots(s - p_N)}$$

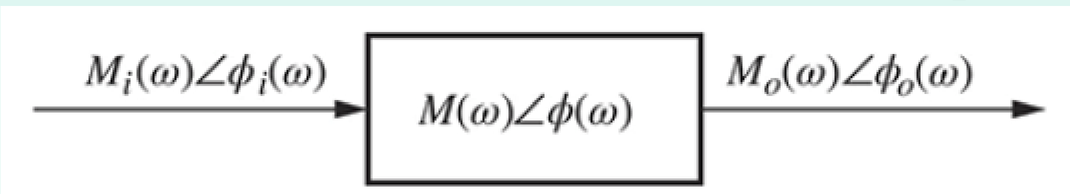


Frequency response methods

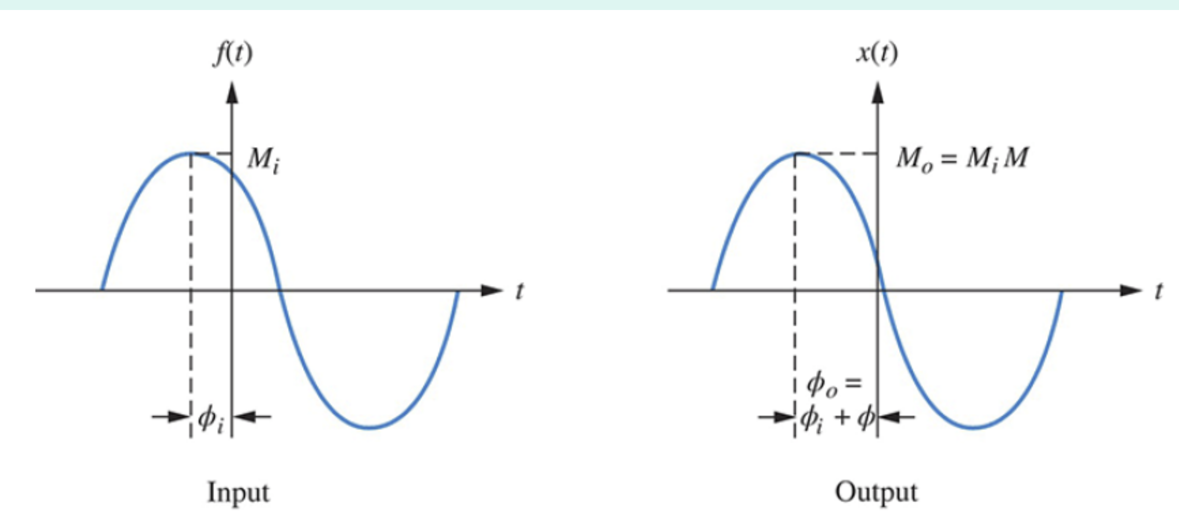
- Developed by Nyquist and Bode in the 1930s
- Advantages in system design and analysis:
  - modeling transfer functions from physical data
  - finding stability conditions and stability margins (gain margin, phase margin)
  - designing compensator networks to shape the desired response (steady-state error and transient response requirements)

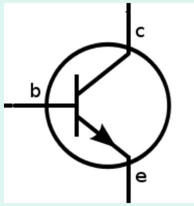
Frequency response

- Frequency response = steady-state response to a sinusoidal input signal
- Harmonic inputs to an LTI system generate harmonic response at the same frequency, but with differences in amplitude and phase angle from the input (these differences are functions of frequency)



Frequency response: $T(s)\big|_{s=j\omega} = T(j\omega) = \left|T(j\omega)\right|\angle\Phi(\omega) = M(\omega)\angle\Phi(\omega)$

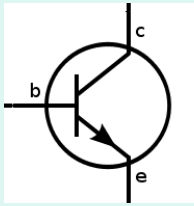




Simple Bode plot sketch example

- Draw the Bode plots for the following transfer function
- zero: z1=-10rad/s
- poles: p1=-1rad/s, p2=-100rad/s

$$H(s) = \frac{100(s+10)}{(s+1)(s+100)}$$



Transfer function - example

- Amplifier with all poles and zeros in the negative half-plane, real and distinct

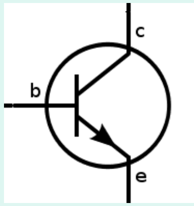
$$T(s) = K\frac{(s+\omega_{z1})(s+\omega_{z2})\ \dots\ (s+\omega_{zn})}{(s+\omega_{p1})(s+\omega_{p2})\ \dots\ (s+\omega_{pN})}$$

$$n < N$$

Frequency response:

$$T(j\omega) = K\frac{(j\omega+\omega_{z1})(j\omega+\omega_{z2})\ \dots\ (j\omega+\omega_{zn})}{(j\,\omega+\omega_{p1})(j\,\omega+\omega_{p2})\ \dots\ (j\,\omega+\omega_{pN})}$$

$$T(j\omega) = K\frac{M_{z1}(\omega)e^{j\tan^{-1}\frac{\omega}{\omega_{z1}}}\ M_{z2}(\omega)e^{j\tan^{-1}\frac{\omega}{\omega_{z2}}}\ \dots\ M_{zn}(\omega)e^{j\tan^{-1}\frac{\omega}{\omega_{zn}}}}{M_{p1}(\omega)e^{j\tan^{-1}\frac{\omega}{\omega_{p1}}}\ M_{p2}(\omega)e^{j\tan^{-1}\frac{\omega}{\omega_{p2}}}\ \dots\ M_{pN}(\omega)e^{j\tan^{-1}\frac{\omega}{\omega_{pN}}}}$$



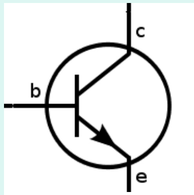
Transfer function exm (2)

- rewrite the transfer function, separating phasor magnitudes from phasor phases

$$T(j\omega) = K \frac{M_{z1}(\omega) e^{j\tan^{-1}\frac{\omega}{\omega_{z1}}} \; M_{z2}(\omega) e^{j\tan^{-1}\frac{\omega}{\omega_{z2}}} \; \ldots \; M_{zn}(\omega) e^{j\tan^{-1}\frac{\omega}{\omega_{zn}}}}{M_{p1}(\omega) e^{j\tan^{-1}\frac{\omega}{\omega_{p1}}} \; M_{p2}(\omega) e^{j\tan^{-1}\frac{\omega}{\omega_{p2}}} \; \ldots \; M_{pN}(\omega) e^{j\tan^{-1}\frac{\omega}{\omega_{pN}}}}$$

$$T(j\omega) = K \frac{M_{z1}(\omega) M_{z2}(\omega) \ldots M_{zn}(\omega) e^{j\left(\tan^{-1}\frac{\omega}{\omega_{z1}} + \tan^{-1}\frac{\omega}{\omega_{z2}} + \ldots + \tan^{-1}\frac{\omega}{\omega_{zn}}\right)}}{M_{p1}(\omega) M_{p2}(\omega) \ldots M_{pN}(\omega) e^{j\left(\tan^{-1}\frac{\omega}{\omega_{p1}} + \tan^{-1}\frac{\omega}{\omega_{p2}} + \ldots + \tan^{-1}\frac{\omega}{\omega_{pN}}\right)}}$$

$$T(j\omega) = K \frac{M_{z1}(\omega) M_{z2}(\omega) \ldots M_{zn}(\omega)}{M_{p1}(\omega) M_{p2}(\omega) \ldots M_{pN}(\omega)} e^{j\left(\tan^{-1}\frac{\omega}{\omega_{z1}} + \ldots + \tan^{-1}\frac{\omega}{\omega_{zn}} - \tan^{-1}\frac{\omega}{\omega_{p1}} - \ldots - \tan^{-1}\frac{\omega}{\omega_{pN}}\right)}$$



Transfer function (3)

$$T(j\omega) = K\,\frac{M_{z1}(\omega)M_{z2}(\omega)\ldots M_{zn}(\omega)}{M_{p1}(\omega)M_{p2}(\omega)\ldots M_{pN}(\omega)}\,e^{j\left(\tan^{-1}\frac{\omega}{\omega_{z1}} + \ldots + \tan^{-1}\frac{\omega}{\omega_{zn}} - \tan^{-1}\frac{\omega}{\omega_{p1}} - \ldots - \tan^{-1}\frac{\omega}{\omega_{pN}}\right)}$$

$$M_{z1} = \sqrt{\omega^2 + \omega_{z1}^2}\,,\ M_{z2} = \sqrt{\omega^2 + \omega_{z2}^2}\,,\ \ldots\,,\ M_{zn} = \sqrt{\omega^2 + \omega_{zn}^2}$$

and

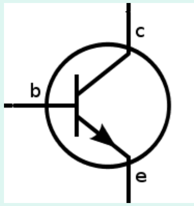
$$M_{p1} = \sqrt{\omega^2 + \omega_{p1}^2}\,,\ M_{p2} = \sqrt{\omega^2 + \omega_{p2}^2}\,,\ \ldots\,,\ M_{pN} = \sqrt{\omega^2 + \omega_{pN}^2}$$

Separate components: magnitude and phase of T(jω)

$$20\log|T(j\omega)| = 20\log|K|$$

$$+\ 20\log\sqrt{\omega^2 + \omega_{z1}^2} + 20\log\sqrt{\omega^2 + \omega_{z2}^2} + \ldots + 20\log\sqrt{\omega^2 + \omega_{zn}^2}$$

$$-\ 20\log\sqrt{\omega^2 + \omega_{p1}^2} - 20\log\sqrt{\omega^2 + \omega_{p2}^2} - \ldots - 20\log\sqrt{\omega^2 + \omega_{pN}^2}$$



Magnitude and phase separation

Magnitude:

$$20\log|T(j\omega)| = 20\log|K|$$

$$+\ 20\log\sqrt{\omega^2+\omega_{z1}^2} + 20\log\sqrt{\omega^2+\omega_{z2}^2} + \ldots + 20\log\sqrt{\omega^2+\omega_{zn}^2}$$

$$-\ 20\log\sqrt{\omega^2+\omega_{p1}^2} - 20\log\sqrt{\omega^2+\omega_{p2}^2} - \ldots - 20\log\sqrt{\omega^2+\omega_{pN}^2}$$

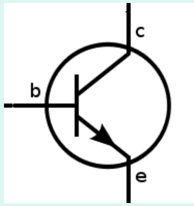
Phase: we must add 0 if K>0 and π if K is negative

$$\phi(\omega) = \tan^{-1}\frac{\omega}{\omega_{z1}} + \tan^{-1}\frac{\omega}{\omega_{z2}} + \ldots + \tan^{-1}\frac{\omega}{\omega_{zn}}$$

$$-\tan^{-1}\frac{\omega}{\omega_{p1}} - \tan^{-1}\frac{\omega}{\omega_{p2}} - \ldots - \tan^{-1}\frac{\omega}{\omega_{pN}}$$

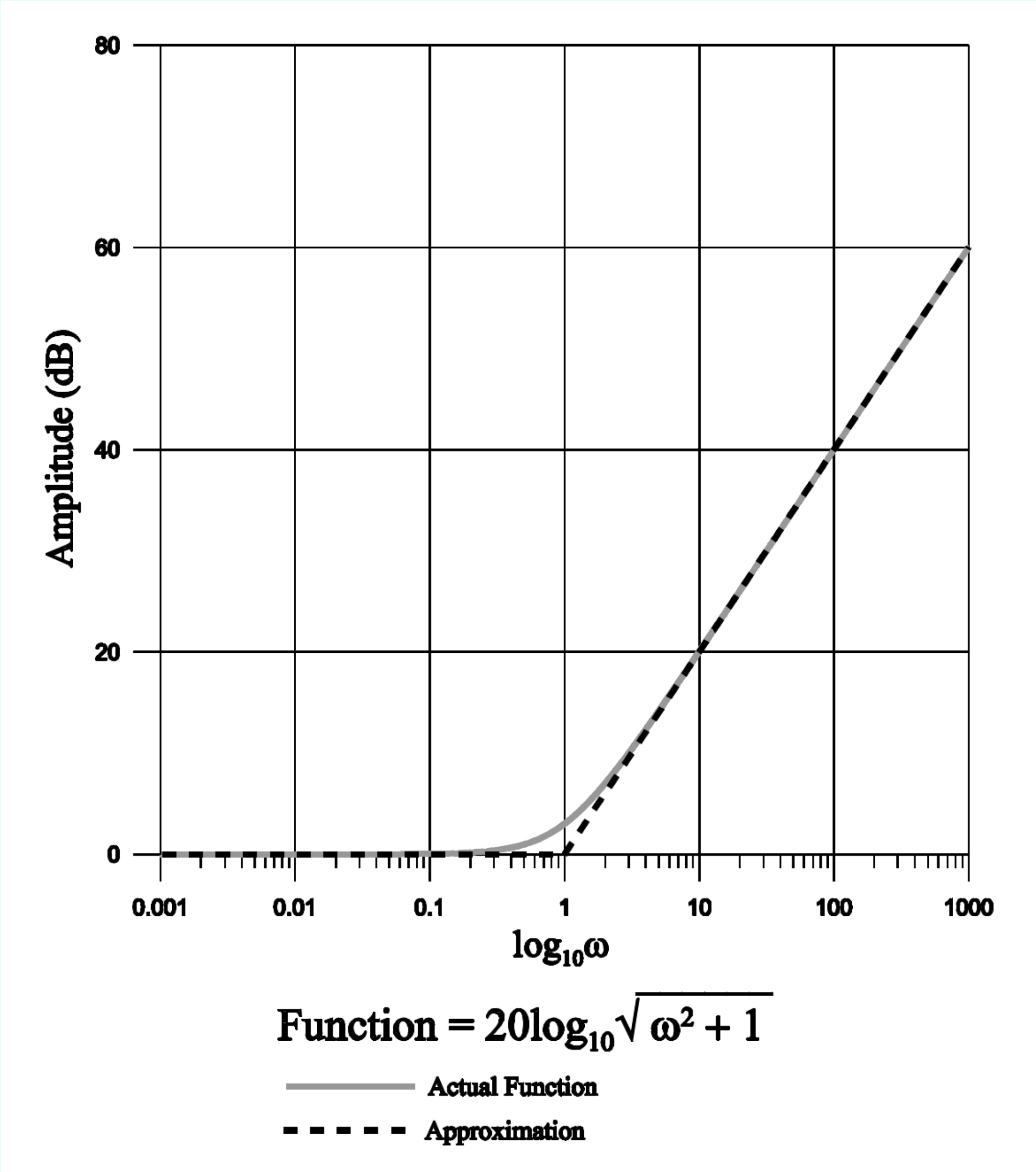
Intermediate remarks

- The log operation separates the frequency response into additive primitive components
- While related, we can separate the visual representations for magnitude and phase
- We only need to identify the patterns of variations in the magnitude - phase representation for the primitive components (poles/zeros)



Simple zero z1=-1

- Assume a simple zero in the transfer function



Effects of a single zero on magnitude

- Zero at ω_z
- Global effect felt for $\omega>\omega_z$
- Magnitude ($20\log|H(j\omega)|$ increase rate of +20dB/dec