THE UNIVERSITY OF BRITISH COLUMBIA
Department of Electrical & Computer Engineering

ELEC 311 – Electromagnetic Fields and Waves

2025 W1

A Supplement to

Chapter 10 – Transmission Lines

in W. H. Hayt, Jr. and J. A. Buck, *Engineering Electromagnetics*, McGraw-Hill, 2019, pp. 303-368.

The purposes of this supplement are:

· to assist the reader in identifying key points to be recognized as they apply the SQ3R (Survey, Question, Read, Recite, Review) process to reading and reviewing the chapter and

· to provide comments and supplemental information that fill in apparent gaps in the textbook.

Introduction

While Chapters 11 and 12 will focus on waves propagating in space, this chapter focuses on waves that are guided from source to load by transmission lines. Transmission lines play an important role in all practical applications of electromagnetic waves from electrical power systems to digital systems to radio frequency and microwave systems to optical systems.

For the purposes of this chapter, transmission lines are two-wire structures that are uniform in cross-section along their length and support propagation of transverse electromagnetic (TEM) waves along their length. They may take the form of parallel wires or *ladder lines*, parallel plates, or coaxial cables. However, we won't consider the latter structures in detail until Chapter 13.

One important characteristic of TEM waves is that their group and phase velocities are identical at all frequencies. A consequence of this is that a uniform transmission line capable of supporting TEM modes can support them at any frequency "from DC to daylight."

In Chapter 13, structures that support multiple non-TEM modes along their length are also considered. One important characteristic of such modes is that their group and phase velocities are different. A consequence of this is that a uniform transmission line capable of supporting non-TEM modes can only support them above a certain cut-off frequency.

Antennas are devices that provide the transition between guided and propagating waves, and vice versa, and are the subject of the next course in this series, *ELEC 411 – Antennas and Propagation*. They are briefly discussed in Chapter 14 of this book.

10.1 Physical Description of Transmission Line Propagation

The section:

- considers the implications of finite velocity of propagation on the manner in which voltage and current advance along an initially uncharged two-wire transmission line
- notes that the voltage between the lines leads to energy stored in an electric field while the current traveling along the lines leads to energy stored in a magnetic field

- introduces a lumped-element model of a two-wire transmission line (with series inductances and shunt capacitances)
- notes that the manner in which the circuit elements charge and discharge defines the manner in which voltage and current propagate along the line

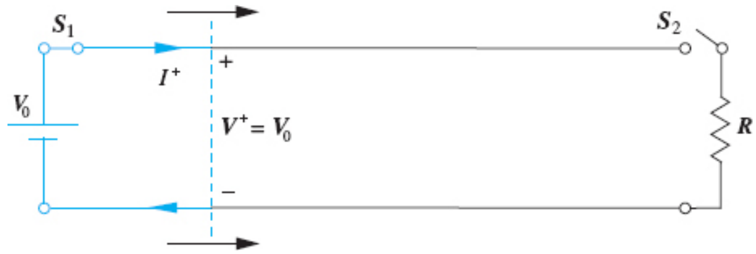


Figure 10.1 Basic transmission line circuit, showing voltage and current waves initiated by closing switch S_1.

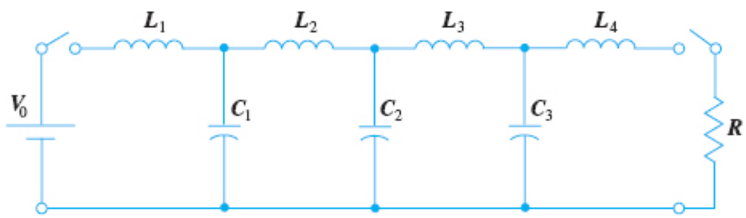


Figure 10.2 Lumped-element model of a transmission line. All inductance values are equal, as are all capacitance values.

Comments

The intent of this section is mostly to help the reader develop an intuitive understanding of the manner in which signals propagate along transmission lines before delving into the detailed mathematics of subsequent sections.

10.2 *The Transmission Line Equations*

This section:

- introduces a more complete lumped element model of a two-wire transmission line that accounts for losses in the series inductances and shunt capacitances

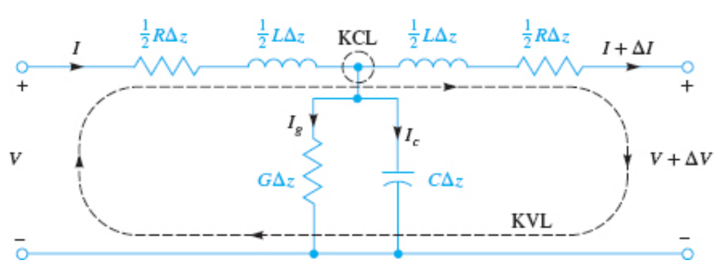


Figure 10.3 Lumped-element model of a short transmission line section with losses. The length of the section is Δz. Analysis involves applying Kirchhoff's voltage and current laws (KVL and KCL) to the indicated loop and node, respectively.

- applies Kirchhoff's voltage and current laws, as suggested by Figure 10.3, to derive a pair of differential equations, called the *telegraphist's equations,* that relate the change in voltage with distance to the change in current with time, and vice versa, *i.e.,*

$$\frac{\partial V}{\partial z} = -(RI + L\frac{\partial I}{\partial t})$$

$$\frac{\partial I}{\partial z} = -(GV + C\frac{\partial V}{\partial t})$$

- applies $\frac{\partial}{\partial z}$ to the first telegraphist's equation and $\frac{\partial}{\partial t}$ to the second telegraphist's equation and equates them via the common term that contains $\frac{\partial^2 I}{\partial t \partial z}$.

- uses this result to obtain the *general wave equations* for the voltage and current carried by transmission lines, *i.e.,*

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2} + (LG + RC)\frac{\partial V}{\partial t} + RGV$$

$$\frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2} + (LG + RC)\frac{\partial I}{\partial t} + RGI$$

- for the lossless case,

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2}$$

  where the velocity of propagation is given by $1/\sqrt{LC}$ and the ratio between V and I at any z and t is given by $Z_0 = \sqrt{L/C}$.

Comments

The general solution to the wave equation is given by $V(z,t) = (V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z})e^{j\omega t}$.

This section bridges the gap between the lumped- and distributed element models of transmission lines. The remainder of the chapter assumes a distributed element model where inductance, capacitance, resistance and conductance are measured per unit length.

This is one of many examples of a circuit model being used to solve what is ultimately an electromagnetic fields problem.

We saw another example of a pair of equations that related the space derivatives of **E** (or **H)** to the time derivatives of **H** (or **E)** were transformed into second-order differential equations in **E** (or **H)** in Problem 9.22. It was solved in exactly the same way.

10.3 Lossless Propagation

This section:

- considers propagation along lossless transmission lines, *i.e.,* $R = 0$ and $G = 0$
- introduces the wave equation for voltage,

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}$$

  and its forward and backward travelling solutions, *i.e.*,

$$V(z,t) = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right) = V^+ + V^-$$

- introduces the corresponding wave equation for current and recalls that voltage and current are linked through the telegraphist's equations
- notes key implications of the wave and telegraphist's equations:
  - the velocity of propagation of the wave is given by $v = \frac{1}{\sqrt{LC}}$
  - the ratio of voltage to current in a given wave is given by the characteristic impedance of the transmission line, $Z_0 = \sqrt{\frac{L}{C}}$

Comments

This may be regarded as the transmission line equivalent of wave propagation through a lossless dielectric, which we will see in Chapter 11.

10.4 Lossless Propagation of Sinusoidal Voltages

This section:

- introduces *real instantaneous* solutions to the wave equation that take the form of sinusoidal or time-harmonic functions of the form $V_0 \cos(\omega t \pm \beta z)$
- notes that:
  - ω is the angular frequency
  - β is the spatial frequency (but is more commonly referred to as the phase constant)
  - $\lambda = 2\pi/\beta$ is the wavelength or spatial interval over which the function is periodic
  - the argument for a forward traveling or incident wave takes the form $\omega t - \beta z$
  - the argument for a backward traveling or reflected wave takes the form $\omega t + \beta z$

Comments

Sinusoidal solutions are useful in Fourier analysis of more complex waveforms, including modulated signals used in radar or communications.

10.5 Complex Analysis of Sinusoidal Waves

This section:

- introduces solutions to the wave equation that take the form of complex exponentials

- distinguishes between:
  - the complex instantaneous voltage which accounts for dependence on both *z* and *t*, and,
  - the phasor voltage which suppresses the *t* dependence
- notes that the phasor form applies only under sinusoidal steady-state conditions
- shows how to transform between complex instantaneous voltage form and phasor form
- notes that one obtains the real sinusoidal voltage wave by multiplying the phasor voltage by $e^{j\omega t}$ (reincorporating the time dependence) and taking the real part of the resulting expression

Comments

It is essential that one masters these relations and their meaning before proceeding further.

10.6 *Transmission Line Equations and Their Solutions in Phasor Form*

This section:

- expresses wave equation in terms of the phasor voltage, *i.e.,* suppresses the time variation factor $e^{j\omega t}$
- defines the propagation constant $\gamma = \alpha + j\beta$ in terms of *R, L, G* and *C, i.e.,*

  $$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{ZY}$$

- proposes a trial solution for the voltage wave in terms of an incident and reflected wave, *i.e.,*

  $$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

- uses the telegraphist's equations to establish the relationship between voltage and current
- defines the characteristic impedance Z_0 in terms of *R, L, G* and *C, i.e.,*

  $$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{Z}{Y}}$$

- uses knowledge of β to determine wavelength and phase velocity, *i.e.,* $v = \frac{\omega}{\beta}$

Comments

These results are pivotal to the remainder of this chapter.

10.7 *Low-Loss Propagation*

This section:

- considers how the expressions of the previous section simplify under low-loss conditions, *i.e.,* $R << \omega L$, $G << \omega C$
- uses the first three terms in a binomial series to simplify the general expression for γ for the low-loss case
- obtains expressions for α, β and Z_0 in terms of *R, L, G* and *C* for the low-loss case
- shows that *Heaviside's condition* ($R/L = G/C$) yields phase and group velocities that are constant with frequency and eliminates distortion of wideband signals
- considers the impact of skin effect on *R* with frequency

Comments

The same general approach can also be applied to uniform plane waves travelling under low-loss conditions.

10.8 Power Transmission and the Use of Decibels in Loss Characterization

This section uses the expressions for voltage and current to derive expressions for:

- instantaneous power
- time-averaged power
- time-averaged power as a function of distance, $P(z) = P_0 e^{-2\alpha z}$

and gives expressions for

- power loss in dB, $P(\text{dB}) = 10 \log P_o/P_i$
- the conversion factor between attenuation (or gain) in decibels and Nepers,

  1 Np = 8.686 dB or 1 dB = 0.1151 Np

Comments

The same general approach can also be applied to uniform plane waves.

It is important to use the correct units for attenuation or gain. If the base is 10, use dB. If the base is *e*, use Np.

10.9 Wave Reflection at Discontinuities

The concept of wave reflection was introduced in §10.1 and is similar to that considered in Chapter 12 for plane waves incident on material boundaries.

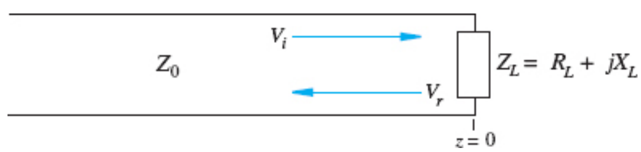


Figure 10.5 Voltage wave reflection from a complex load impedance.

This section:

- demonstrates that the relationship between V_i and V_r is determined by the need for $V_i/I_i = V_r/I_r = Z_0$ but $V_L/I_L = Z_L$
- for the general case of a lossy line, derives expressions for
  - the voltage reflection and transmission coefficients Γ and τ and
  - the power reflection and transmission coefficients $|\Gamma|^2$ and $1 - |\Gamma|^2$
- notes that the power transmission coefficient is ***not*** given by $|\tau|^2$ (why not?)
- generalizes the result to the case of two semi-infinite lines with different characteristic impedances that are connected in series

Comments

The power reflection coefficient $|\Gamma|^2 = P_r/P_i$ can be expressed in dB as 20 log $|\Gamma|$. This is often expressed in terms of its reciprocal, P_i/P_r. Commonly referred to as the *return loss*, it is expressed in dB as -20 log $|\Gamma|$.

10.10 Voltage Standing Wave Ratio

This section:

- introduces the *standing wave* that forms from the superposition of an incident wave and wave reflected from an interface (see Figure 10.6 below)
- defines the *voltage standing wave ratio* (VSWR) associated with such a standing wave as

$$s = \frac{V_{\max}}{V_{\min}}$$

- derives expressions for the locations of minima and maxima of the voltage standing wave with respect to the discontinuity
- derives an expression for the VSWR in terms of the voltage reflection coefficient associated with the discontinuity that gave rise to the reflected wave,

$$s = \frac{V_{\max}}{V_{\min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

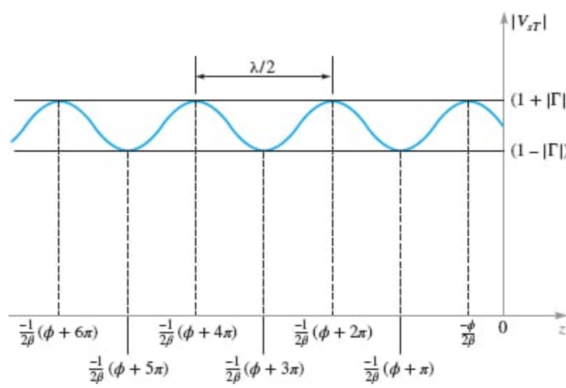


Figure 10.6 Plot of the magnitude of V_{sT} as found from Eq. (85) as a function of position, z, in front of the load (at $z = 0$). The reflection coefficient phase is ϕ, which leads to the indicated locations of maximum and minimum voltage amplitude, as found from Eqs. (86) and (89).

Comments

Standing waves are a result of interference between a wave that is incident upon a boundary or discontinuity and the wave reflected from that boundary or discontinuity. They are associated with waves travelling along transmission lines and propagating through space.

10.11 Transmission Lines of Finite Length

This section:

- considers the standing wave that forms within a transmission line of finite length that is not impedance matched

- defines the *wave impedance* as the ratio of the total phasor voltage $V_{sT}(z)$ (sum of the voltages associated with the forward and backward travelling waves) to the total phasor current $I_{sT}(z)$ (sum of the currents associated with the forward and backward travelling waves)
- derives an expression for the wave impedance Z_w

$$Z_w(z) \equiv \frac{V_{sT}(z)}{I_{sT}(z)} = \frac{V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}}{I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z}}$$

leading to

$$Z_w(z) \equiv \frac{e^{-j\beta z} + \Gamma e^{j\beta z}}{e^{-j\beta z} - \Gamma e^{j\beta z}}$$

and

$$Z_w(z) = Z_0 \frac{Z_L \cos(\beta z) - jZ_0 \sin(\beta z)}{Z_0 \cos(\beta z) - jZ_L \sin(\beta z)}$$

or, alternatively,

$$Z_w(\ell) = Z_0 \frac{Z_L \cos(\beta \ell) + jZ_0 \sin(\beta \ell)}{Z_0 \cos(\beta \ell) + jZ_L \sin(\beta \ell)}$$

where z is a given position along the transmission line
$\ell = -z$ is a distance from $z = 0$ back towards the source
Z_0 is the characteristic impedance of the line
β is the phase constant of the line and
Z_L is the load impedance

- notes that wave impedance is, in general complex, even if Z_0 and Z_L are both real
- demonstrates that a transmission line of length λ/2 will transform a real load impedance into an identical wave impedance at the input, *i.e.,* an identical input impedance, regardless of the characteristic impedance of the line
- shows that a transmission line of length λ/4 will transform a real load impedance into an input impedance given by $Z_{\text{in}} = Z_0^2 / Z_L$

Comments

Half-wave and quarter-wave transformers or matching sections play important roles in both optical and radio frequency systems.

The function and operation of half-wave and quarter-wave transformers or matching sections are very clear when illustrated on a Smith Chart (Section 10.13).

10.12 Some Transmission Line Examples

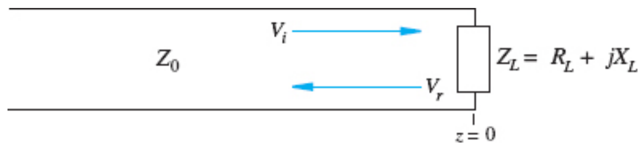
This section applies many of the results that were obtained in the previous sections to several example transmission line problems and simplifies the task by considering only lossless lines

Comments

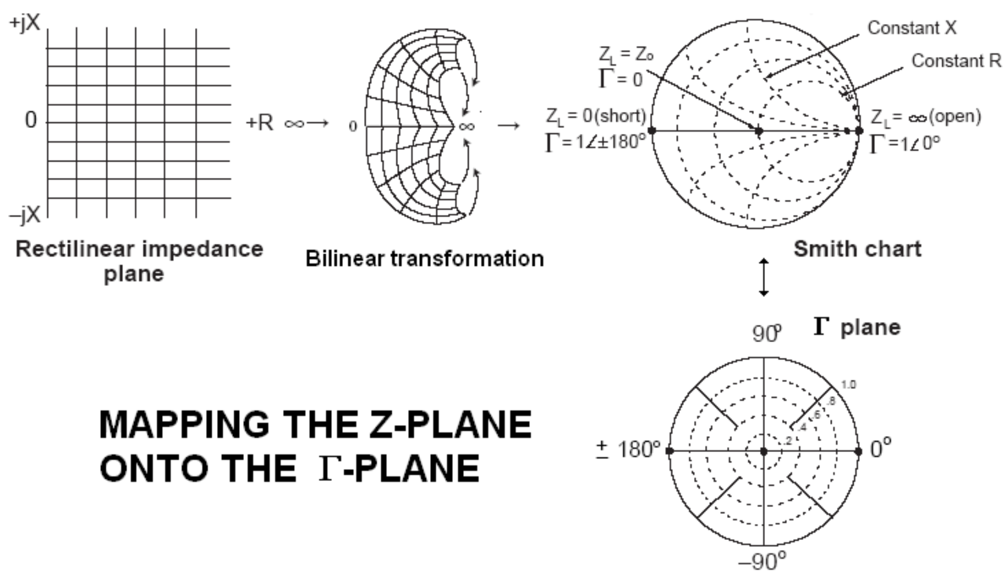
If we need to consider lossy lines, we can modify our expression for wave impedance by replacing the sine and cosine functions by their hyperbolic counterparts and the phase constant β by the full complex propagation constant γ.

10.13 Graphical Methods: The Smith Chart

Consider a transmission line with characteristic impedance Z_0 that has been terminated by a complex load Z_L:



This section introduces the Smith Chart, a mapping of the complex or rectilinear impedance plane onto the complex reflection coefficient or Γ plane that allows one to visualize the manner in which Γ and Z evolve as one moves forward and backward along the transmission line.



To simplify interpretation of the Smith Chart, the load and wave impedances are generally normalized to the characteristic impedance Z_0 of the line before it is plotted on the Smith chart. The original values $Z = R + jX$ are generally expressed in uppercase. The normalized values $z = r + jx$ are generally expressed in lowercase.

This section:

- gives the precise mapping of $z = r + jx$ where $z_0 = 1$ onto the complex reflection coefficient or Γ plane
- shows that the straight lines corresponding to constant r or x in the complex or rectilinear impedance plane transform into circles in the complex reflection coefficient or Γ plane
- shows that such a mapping is characteristic of the bilinear transformation that relates Γ to z_L
- recalls that Γ varies as $\Gamma_0 e^{j2\beta z}$ so that a complete rotation of the Smith Chart corresponds to one half rather than a full wavelength of travel along the transmission line

In addition, this section shows how:

- the variation in impedance with distance back from the load is plotted

- the impedance at a given distance can be read from the Smith chart
- impedance can be transformed to admittance
- among other things, the Smith chart can be used to
  - visualize the function and operation of quarter and half-wave sections
  - design a single-stub matching network
  - design a double-stub matching network

Comments

The Smith Chart often intimidates those who are encountering it for the first time. In the lecture slides, we reduce this section to a detailed set of performance objectives:

1. Given Γ_0, plot $\Gamma(\ell)$ on a Smith chart.
2. Given Z_0 and Z, plot Z/Z_0 (normalized Z) on a Smith chart.
3. Given Z_0 and Z, find Γ using a Smith chart.
4. Given Z_0, Z and d, find $\Gamma(d)$ and $\Gamma(-d)$ using a Smith chart.
5. Given Y_0, plot Y/Y_0 (normalized Y) on a Smith chart.
6. Given Z, plot Y on a Smith chart.
7. Analyze $\lambda/2$ and $\lambda/4$ transformers or sections using a Smith chart.
8. Given a short- or open-circuit stub of length ℓ, find Y_{in} & Z_{in}.
9. Given Z_L, find d and ℓ required to achieve a single-stub match using a shunt stub.
10. Given Z_L, find d, ℓ_1 and ℓ_2 required to achieve a double-stub match using shunt stubs.

Until the advent of pocket calculators in the early 1970's, engineers relied on a variety of graphical aids based upon mathematical transformations to reduce their calculation burden. The Smith Chart was only one of many such aids. It is, however, one of the best known.

With the rise of the vector network analyzer (VNA) during the 1970's and 1980's, the Smith Chart achieved new importance as an overlay that aids in the interpretation of complex reflection measurements returned by such instruments.

Here, we focused on the manner in which the complex reflection coefficient varies with distance from the load. The Smith Chart plays an equally important role in interpreting the manner in which the complex reflection coefficient varies with frequency.

10.14 Transient Analysis

While previous sections have focused on the operation of transmission lines under steady state conditions, this section focuses on transient behaviour, *i.e.,* how voltage and current evolve when a pulse is applied to transmission lines that is initially uncharged.

These concepts form the basis for time domain reflectometry (TDR), a very useful radar-like technique for determining the location of discontinuities in long transmission lines. In TDR, a voltage source is used to apply a step function to the input to a transmission line and an oscilloscope is used to observe the result over time.

This section:

- introduces the notion of applying a step function to an uncharged transmission line by applying a voltage source and then closing a switch
- notes that a step function contains a broad spectrum of frequency components and propagates along the transmission lone at the group velocity (rather than the phase velocity)
- presents the voltage reflection diagram as a method of tracking the voltage at any point in the line
- presents the current reflection diagram as a method of tracking the current at any point in the line
- considers the transient problem in which the line is initially charged

Comments

ELEC 311 students are expected to have a good basic awareness of the underlying principles but will *not* be required to solve detailed problems involving transient analysis.

References

Notably absent from the list of references are:

[1] P. H. Smith, *Electronic Applications of the Smith Chart*. Kay Electric, 1969, 222 pp.

Although no longer in print, this classic work by the inventor of the Smith Chart is available from many used book sellers. A second edition with a new Chapter 15 was published by SciTech Publishing in 1995. The Table of Contents is given below.

[2] R. A. Chipman, *Transmission Lines.* Schaum's Outline Series, McGraw-Hill, 1968, 248 pp.

Although no longer in print, this classic work is available from many used book sellers. While many books in the Schaum's Outline Series have been dismissed as superficial or lacking in detail, this particular volume presents a very detailed and thorough treatment of its subject. The Table of Contents is given below.