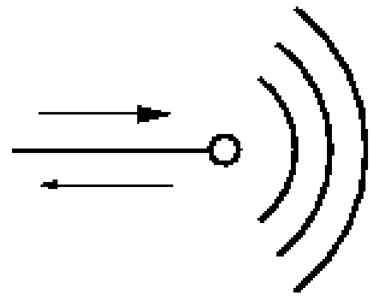


ELEC 311 - Electromagnetic Fields and Waves

Chapter 11
Uniform Plane Waves

Waves and propagation; Maxwell's equations; applications including transmission lines; impedance matching and Smith charts; reflection and refraction; waveguides and antennas. [4-0-0]

Outline

1. Time-Harmonic Electromagnetic Fields
2. The Wave Equation
3. The Plane Wave Solution
4. Perfect Dielectrics
5. Partially Conducting Media
6. Good Conductors and the Skin Effect
7. Spherical Waves
8. Power and the Poynting Vector

1 Time-Harmonic Electromagnetic Fields

- If one assumes a time-factor $e^{j\omega t}$ and a source-free region, Maxwell's equations reduce to:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{H} &= 0 \\ \nabla \times \mathbf{E} &= -j\omega \mathbf{B} = -j\omega\mu\mathbf{H} \\ \nabla \times \mathbf{H} &= \mathbf{J}_c + j\omega\mathbf{D} = (\sigma + j\omega\epsilon)\mathbf{E} \end{aligned}$$

- The coupling between the electric and magnetic fields is obvious!
- Look for a plane wave solution in which the wave propagates in the $+z$ direction, and for which $\mathbf{E}$ and $\mathbf{H}$ do not vary in x or y.

2 The Wave Equation

- Take the curl of the first two Maxwell's equations, yielding

$$\nabla \times (\nabla \times \mathbf{H}) = (\sigma + j\omega\epsilon)(\nabla \times \mathbf{E})$$

and

$$\nabla \times (\nabla \times \mathbf{E}) = -j\omega\mu(\nabla \times \mathbf{H})$$

- In Cartesian coordinates only,

$$\nabla^2 \mathbf{A} = (\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}.$$

This satisfies the identity

$$\nabla \times \nabla \times \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

- If $\nabla \cdot \mathbf{A} = 0$, then

$$\nabla \times \nabla \times \mathbf{A} \equiv -\nabla^2 \mathbf{A}$$

- Making the appropriate substitutions yields the vector wave equations

$$\nabla^2 \mathbf{H} = j\omega\mu(\sigma + j\omega\epsilon)\mathbf{H} \equiv \gamma^2 \mathbf{H}$$

and

$$\nabla^2 \mathbf{E} = j\omega\mu(\sigma + j\omega\epsilon)\mathbf{E} \equiv \gamma^2 \mathbf{E}\,.$$

- The propagation constant $\gamma = \alpha + j\beta$ in a given medium is the square root of γ^2 whose real and imaginary parts are positive:

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right)}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right)}$$

- The velocity of propagation is given by $v = \omega/\beta = f\lambda$.
- These quantities take on distinctive forms for media that are perfect dielectrics, partially conducting media, and good conductors.

3 The Plane Wave Solution

- The Helmholtz equations associated with a plane wave propagating in the $\pm z$ direction in a lossless medium are

$$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0$$

$$\frac{\partial^2 E_y}{\partial z^2} + k^2 E_y = 0$$

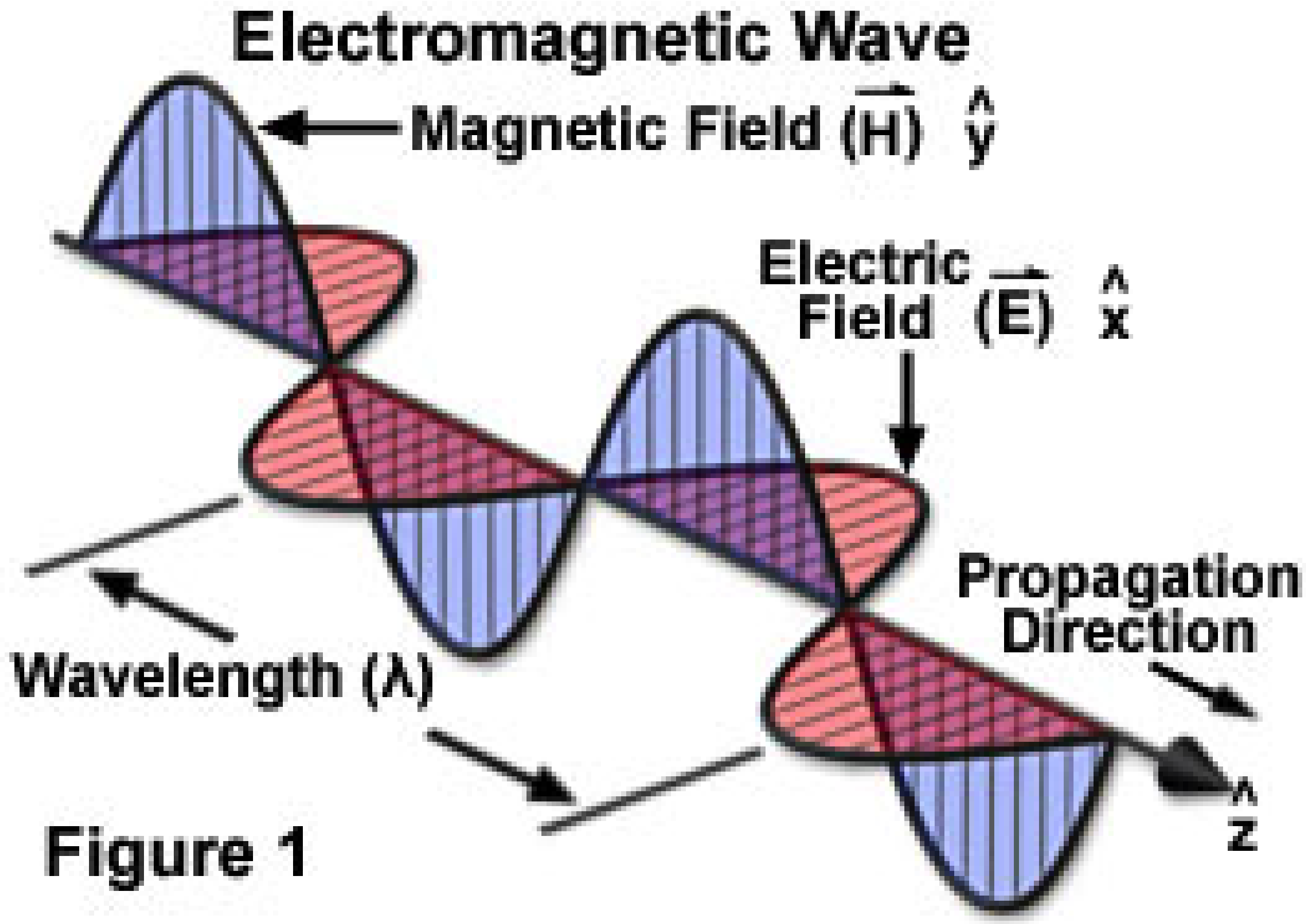
  where $k = \omega\sqrt{\mu_0 \epsilon_0} = \omega/c = 2\pi/\lambda = \beta$.

- The two cases refer to horizontally and vertically polarized waves, respectively. Use superposition of the results to obtain the general case - an arbitrarily polarized wave.

- The time-harmonic solution to the first equation is

$$\begin{aligned} E_x &= A\, e^{-jkz} + B\, e^{+jkz} \ \text{V/m} \\ H_y &= \frac{1}{\eta_0}\left(A\, e^{-jkz} + B\, e^{+jkz}\right) \ \text{A/m.} \end{aligned}$$

- The e^{-jkz} solution is a plane wave travelling in the $+z$ direction.
- The e^{+jkz} solution is a plane wave travelling in the $-z$ direction.
- For each wave separately, $|E_x/H_y| = \eta_0$.


Electromagnetic Wave
Magnetic Field (H) ŷ
Electric Field (E) x̂
Propagation Direction
Wavelength (λ)
ẑ
Figure 1

- The power flux density is given by the time-averaged Poynting vector,

$$\mathbf{S} = \frac{1}{2}\,\mathrm{Re}\{\mathbf{E} \times \mathbf{H}^*\}\ (\mathrm{W/m^2}).$$

- For the e^{-jkz} solution,

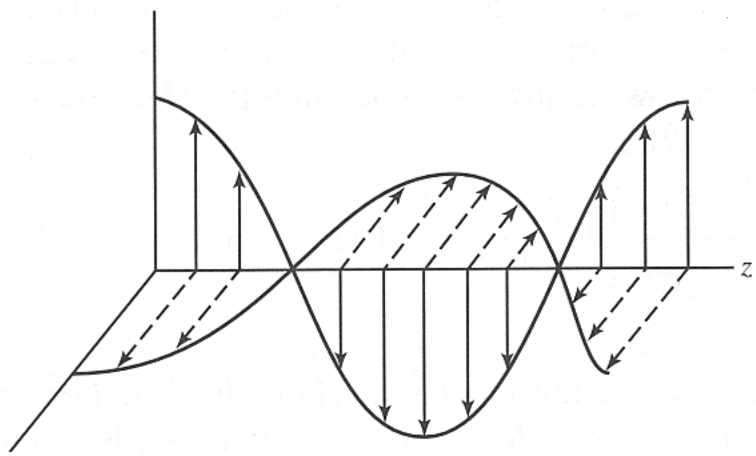
$$\mathbf{S} = \frac{A^2}{2\eta_0}\ \hat{z}\ .$$

- For the e^{+jkz} solution,

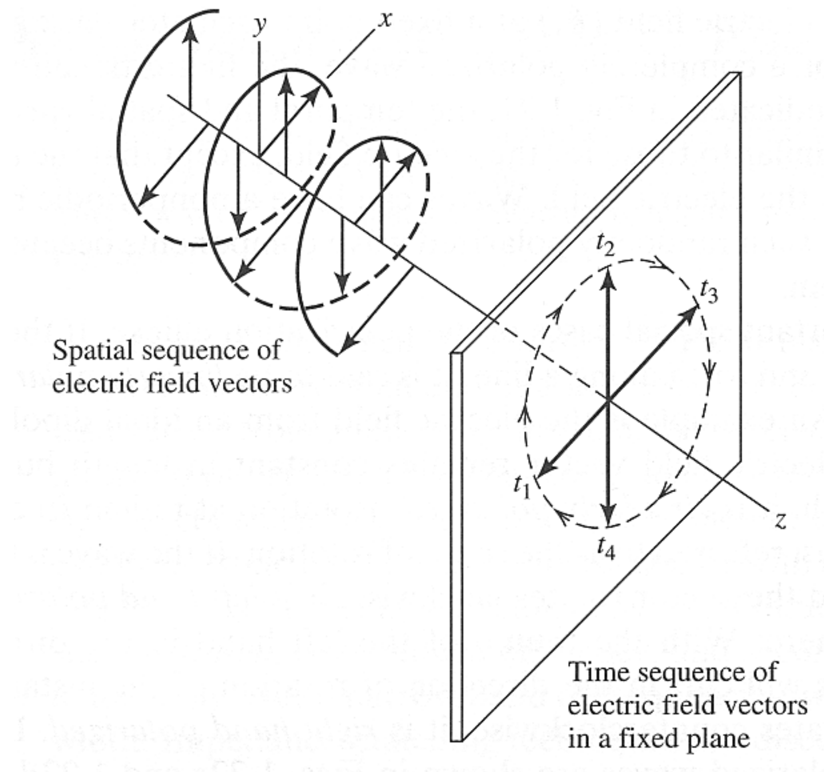
$$\mathbf{S} = -\frac{B^2}{2\eta_0}\ \hat{z}\ .$$

Polarization of a Plane Wave

- In general, an arbitrarily polarized plane wave can be decomposed into components with electric field vectors that point in the vertical and horizontal directions.

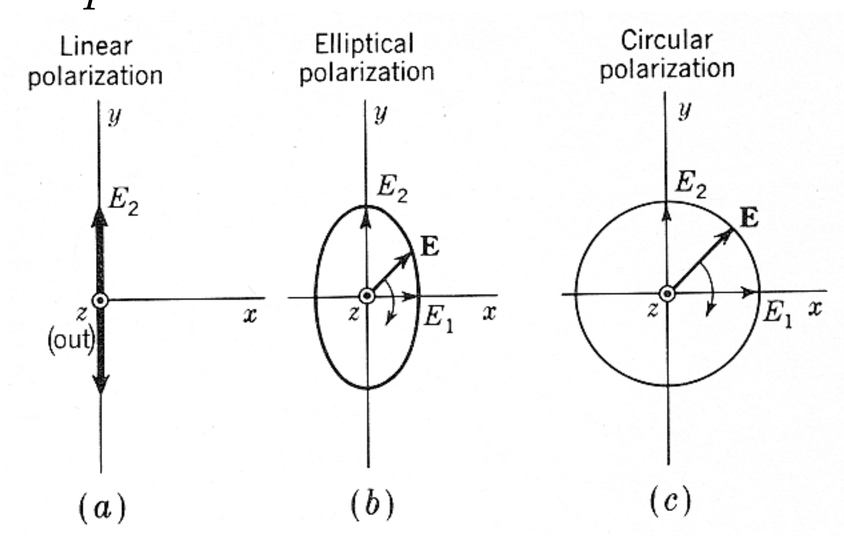


- Depending upon the relative amplitude and phase of the two components, the amplitude and orientation of the electric vector will change over time.



Definition of Polarization

- The polarization of an electromagnetic wave refers to the behaviour of the electric field over time as observed at a fixed point in space.
- In general, the electric field vector will trace an elliptical locus; special cases include linear and circular. The shape of the locus, and the direction in which $\mathbf{E}$ is rotating (clockwise or left-handed in the figure below), specify the *polarization state* of the wave.



Generalization to Arbitrary Waves

- Using Fourier synthesis, we can generalize the time-harmonic solution to waves of arbitrary shape.

- This is consistent with the notion that the scalar wave equation in one dimension,

$$\frac{\partial^2 F}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2}$$

has solutions of the form

$$F = f_1(z - vt) \quad \text{and} \quad f_2(z + vt)$$

where f_1 and f_2 are arbitrary functions.

- These waves travel with speed v in the $+z$ and $-z$ directions, respectively.

4 Perfect Dielectrics

- A material is classified as a perfect dielectric if $\sigma = 0$.
- In this case,

$$\alpha = 0 \qquad \beta = \omega\sqrt{\mu\epsilon} \qquad \eta = \sqrt{\frac{\mu}{\epsilon}}\ .$$

- Implications:
  - waves do not suffer attenuation.
  - $\mathbf{E}$ and $\mathbf{H}$ are in phase and everywhere perpendicular to each other..
  - $v = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{\mu\epsilon}}$.
- A material is considered a good dielectric if $\sigma/\omega\epsilon < 0.1$.

5 Partially Conducting Media

- For a region which is slightly conductive, one can assume an electric field of the form:

$$\mathbf{E} = E_0 \, e^{-\gamma z} \, \hat{x} \; .$$

- Because

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

  it can be shown that

$$\mathbf{H} = \sqrt{\frac{\sigma + j\omega\epsilon}{j\omega\mu}} \, E_0 \, e^{-\gamma z} \, \hat{y} \; .$$

- Here, the ratio E_x/H_y gives the intrinsic impedance of the medium.

- In this case, the intrinsic impedance is complex and can be expressed in the form $|\eta|\angle\theta$.

- We can express the electric and magnetic fields in the form

$$\begin{aligned}\mathbf{E}(z,t) &= E_0\, e^{-\alpha z} e^{j(\omega t - \beta z)}\hat{x} \\ \mathbf{H}(z,t) &= \frac{E_0}{|\eta|}\, e^{-\alpha z} e^{j(\omega t - \beta z - \theta)}\hat{y}\end{aligned}$$

- Exercise: What is the physical significance of the parameters α and θ?

- Exercise: What are the velocity of propagation and wavelength in terms of μ, ω, ϵ, and σ?

6 Good Conductors and the Skin Effect

- A material is classified as a good conductor if $\sigma >> \omega\epsilon$.
- Exercise: What is the significance of the ratio $\sigma/\omega\epsilon$?
- In this case,

$$\gamma = \alpha + j\beta$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma}$$

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ$$

- We can express the fields in the form

$$\begin{aligned}\mathbf{E}(z,t) &= E_0\, e^{-\alpha z} e^{j(\omega t - \beta z)}\hat{x} \\ \mathbf{H}(z,t) &= \frac{E_0}{|\eta|}\, e^{-\alpha z} e^{j(\omega t - \beta z - \pi/4)}\hat{y}\end{aligned}$$

- Exercise: What are the velocity of propagation and wavelength in terms of μ, ω, ϵ, and σ?

- What is the significance of the skin depth, $\delta = \dfrac{1}{\sqrt{\pi f \mu \sigma}} = \dfrac{1}{\alpha}$?

- *Ans.* Skin depth is the the distance over which the fields decay by $1/e$. At 5δ, the magnitude is 0.67% of its initial value, or essentially nil.

- Exercise: How else is skin depth significant?

7 Spherical Waves

- Plane waves are idealizations that cannot exist in practice. Why?
- Practical antennas radiate spherical waves.
- In spherical coordinates, the Helmholtz equation is given by:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial^2\psi}{\partial\phi^2}\right)+k^2\psi=0.$$

- Solve using the method of separation of variables. We begin by assuming that the solutions takes the form

$$\psi = R(r)\Theta(\theta)\Phi(\phi)\ .$$

- We can show that:
  - The R equation gives rise to spherical Bessel functions.
  - The Θ equation gives rise to associated Legendre functions.
  - The Φ equation is the familiar harmonic equation.
- In the simplest case, we assume spherical symmetry and ignore spherical harmonics.
- The result is satisfied by solutions of the form

$$\Psi(r) = C\,\frac{e^{-j\beta r}}{r} \qquad \text{and} \qquad \Psi(r) = D\,\frac{e^{j\beta r}}{r}\,.$$

- The first solution corresponds to an outward travelling wave while the second corresponds to an inward travelling wave (of no practical interest).

8. Power and the Poynting Vector

- In §3, we referred to the Poynting vector. Here, we use vector calculus to prove the validity of the concept.
- For a region with conductivity σ,

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

- If $\mathbf{E}$ is dotted with each term,

$$\mathbf{E} \cdot \nabla \times \mathbf{H} = \sigma E^2 + \mathbf{E} \cdot \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

- Apply the vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

to yield

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \sigma E^2 + \mathbf{E} \cdot \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

- By Maxwell's second equation,

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) = \mathbf{H} \cdot \left(-\mu \frac{\partial \mathbf{H}}{\partial t} \right) = -\frac{\mu}{2} \frac{\partial H^2}{\partial t}$$

- Similarly,

$$\mathbf{E} \cdot \left(-\epsilon \frac{\partial \mathbf{E}}{\partial t} \right) = \frac{\epsilon}{2} \frac{\partial E^2}{\partial t}$$

$$\sigma E^2 = -\frac{\epsilon}{2} \frac{\partial E^2}{\partial t} - \frac{\mu}{2} \frac{\partial H^2}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H})$$

- Integration of this equation thoughout an arbitrary volume v and using the divergence theorem to convert the last term to a surface integral gives

$$\int_v \sigma E^2 dv = -\int_v \left(\frac{\epsilon}{2} \frac{\partial E^2}{\partial t} + \frac{\mu}{2} \frac{\partial H^2}{\partial t} \right) dv - \oint_s (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s}$$

- Finally, we obtain

$$P(t) = \oint_s (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = \oint_s \mathbf{S} \cdot d\mathbf{s}$$

where $\mathbf{S}$ is the Poynting vector.

- If $\mathbf{E}$ and $\mathbf{H}$ are expressed in complex form and have the common time-dependence $e^{j\omega t}$ then the time-average of $\mathbf{S}$ is given by

$$\mathbf{S}_{avg} = \frac{1}{2}\mathrm{Re}(\mathbf{E} \times \mathbf{H}^*)$$

- For plane waves, the Poynting vector gives both the direction of energy flow and the direction of propagation.

Summary

We have reviewed (or, in several cases, introduced):

- Plane wave propagation in free space, perfect dielectrics, partially conducting media, and good conductors.
- Spherical wave propagation in free space.
- The Poynting vector

References

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