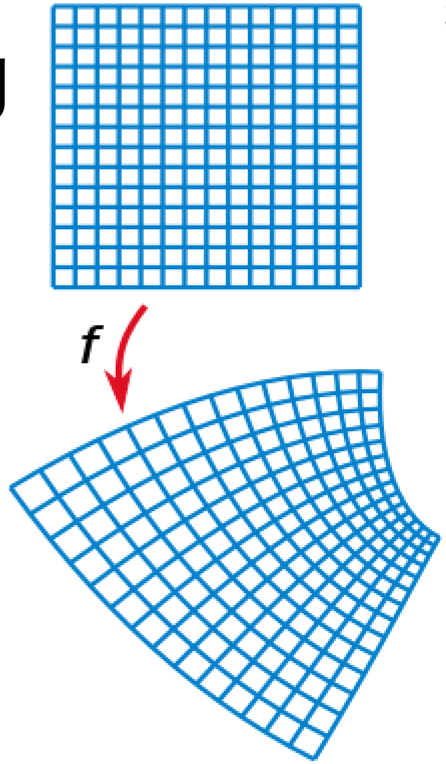
Chapter 13 – Guided Waves

Conformal Mapping

- In mathematics, a conformal map is a function that preserves angles locally.
- In the most common case, the function has a domain and an image in the complex plane.
- A complex function is called conformal (or angle-preserving) at a point if it preserves angles between directed curves as well as preserving orientation.
- Conformal maps preserve both angles and the shapes of infinitesimally small figures, but not necessarily their size or curvature.
- The bilinear transformation is a conformal mapping!

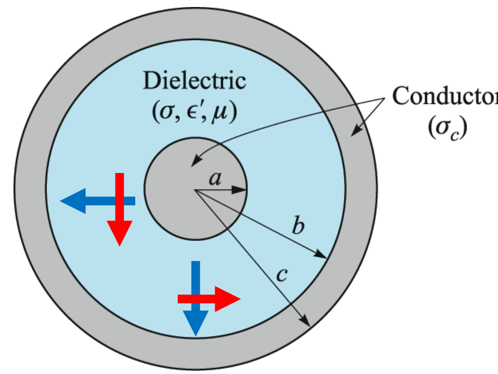


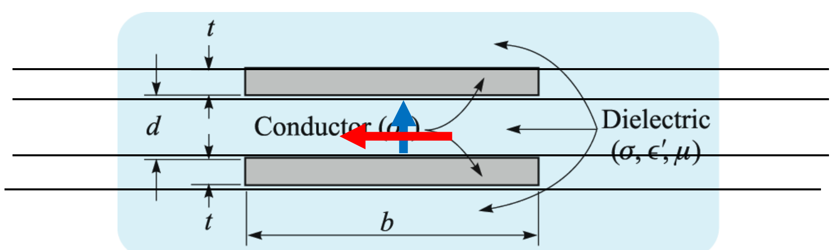
A representative conformal mapping. The grid lines are still orthogonal to each other after transformation.

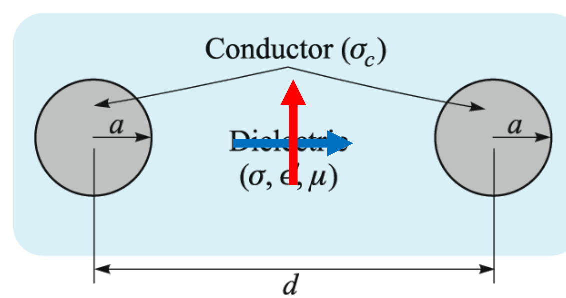
$$f(z) = \frac{1-z}{1+z}$$ where z is an arbitrary complex number

Conformal Mapping and Transmission Lines

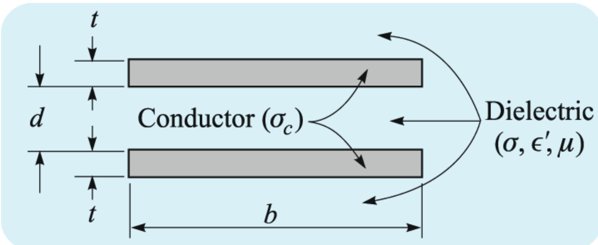
- Conformal mapping allows us to transform simple and well understood structures like the parallel-plate wave guide – and the structure of their TEM fields - into more complex but useful structures like coaxial lines and ladder line.
- This allows us to easily derive closed-form expressions for the ratio of their electric and magnetic fields, i.e., their *characteristic impedance.*
- *We will look at the details later!*

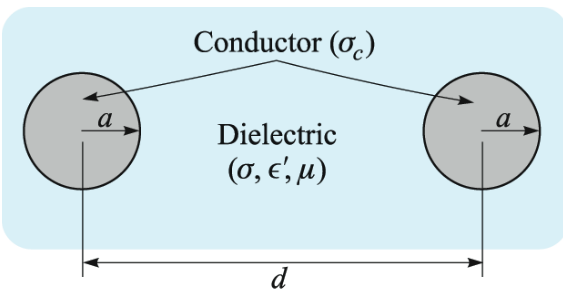


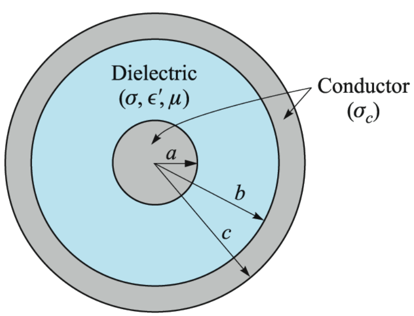




Properties of TEM Transmission Lines







The parallel-plate waveguide, ladder line, and coaxial line, seen here in cross section as viewed from along along the z-axis.

They

- consist of two conductors that are characterized by a surface current I flowing along them and a voltage V between them.
- support plane-like TEM or transverse electromagnetic waves that propagate in either the $+z$ or $-z$ directions where E and H are everywhere perpendicular to each other and the direction of propagation.

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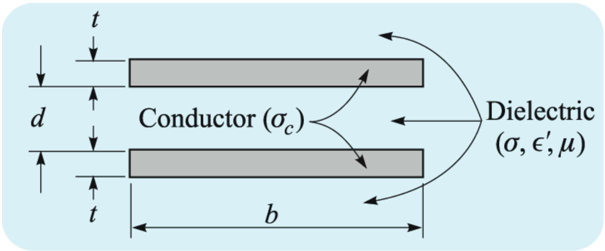
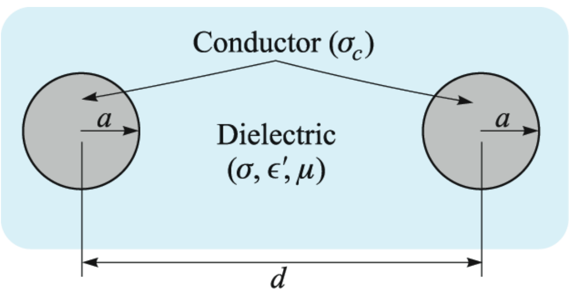
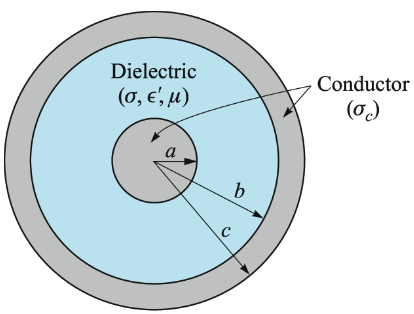
Properties of TEM Transmission Lines



- The characteristic impedance $Z_0 = E/H$ is determined by both the cross-sectional geometry of the transmission line and the properties of the dielectric.

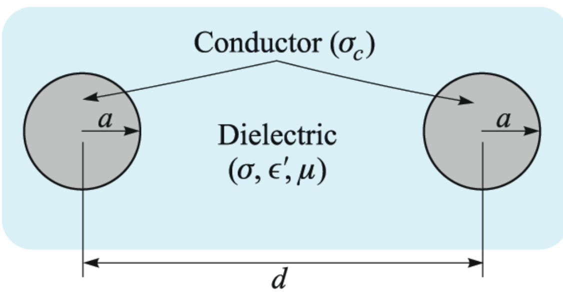


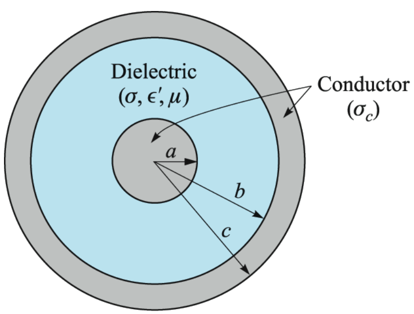


- We previously derived techniques for predicting the behaviour of plane waves at normal incidence to dielectric boundaries.
- These techniques also apply to TEM waves in transmission lines at discontinuities between:
  - characteristic impedances of two lines, or
  - a characteristic impedance and a load impedance (either a discrete load or a wave impedance).

Properties of TEM Transmission Lines

- We can show that
$$Z_0 = \sqrt{\frac{L}{C}}$$
where L and C are the inductance and capacitance per unit length of the structure.
- Because the field strength vectors lie exclusively in the x-y plane, this is a two-dimensional problem and can be solved using the following conformal mappings:





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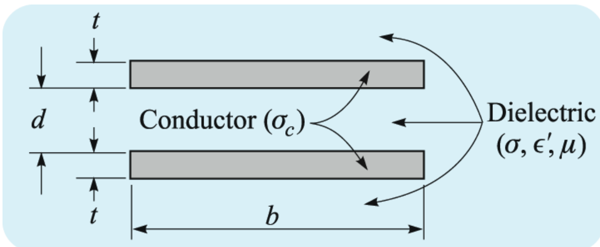
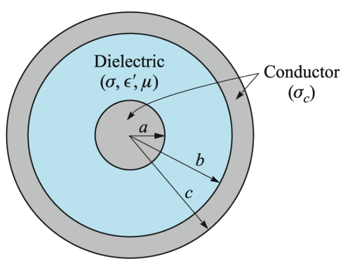
Coaxial line: $f(z) = A \ln z + B$

Ladder line: $f(z) = A \ln \frac{z+B}{z-B}$

Properties of TEM Transmission Lines



$$Z_0 = \sqrt{\frac{\mu}{\epsilon}}\frac{d}{b} = \sqrt{\frac{L}{C}}$$
$$Z_0 = \frac{1}{\pi}\sqrt{\frac{\mu}{\epsilon}}\ \cosh^{-1}\frac{d}{2a} = \sqrt{\frac{L}{C}}$$
If $d >> a$, $\cosh^{-1}\frac{d}{2a} \approx \ln\frac{d}{a}$
$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ \ln\frac{b}{a} = \sqrt{\frac{L}{C}}$$



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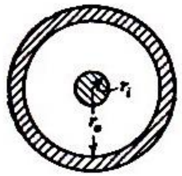
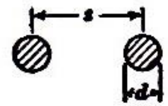
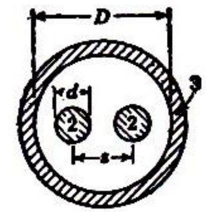
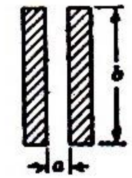
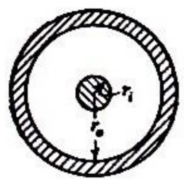
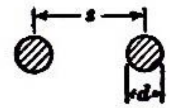
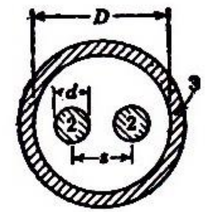
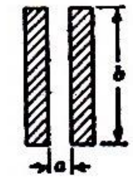
Transmission Line Parameters

			$p = \frac{s}{d}$ $q = \frac{s}{D}$ 	 Formulas for $a \ll b$
Capacitance C, farads/meter	$\frac{2\pi\epsilon}{\ln\left(\frac{r_0}{r_i}\right)}$	$\frac{\pi\epsilon}{\cosh^{-1}\left(\frac{s}{d}\right)}$	----	$\frac{\epsilon b}{a}$
External inductance L, henrys/meter	$\frac{\mu}{2\pi}\ln\left(\frac{r_0}{r_i}\right)$	$\frac{\mu}{\pi}\cosh^{-1}\left(\frac{s}{d}\right)$	----	$\mu\frac{a}{b}$
Conductance G, siemens/meter	$\frac{2\pi\sigma}{\ln\left(\frac{r_0}{r_i}\right)} = \frac{2\pi\omega\epsilon''}{\ln\left(\frac{r_0}{r_i}\right)}$	$\frac{\pi\sigma}{\cosh^{-1}\left(\frac{s}{d}\right)} = \frac{\pi\omega\epsilon''}{\cosh^{-1}\left(\frac{s}{d}\right)}$	----	$\frac{\sigma b}{a} = \frac{\omega\epsilon'' b}{a}$
Resistance R, ohms/meter	$\frac{R_s}{2\pi}\left(\frac{1}{r_0}+\frac{1}{r_i}\right)$	$\frac{2R_s}{\pi d}\left[\frac{s/d}{\sqrt{(s/d)^2-1}}\right]$	$\frac{2R_{s2}}{\pi d}\left[1+\frac{1+2p^2}{4p^4}(1-4q^2)\right] + \frac{8R_{s3}}{\pi D}q^2\left[1+q^2-\frac{1+4p^2}{8p^4}\right]$	$\frac{2R_s}{b}$
Internal inductance L_i, henrys/meter (for high frequency)	← $\frac{R}{\omega}$ →			

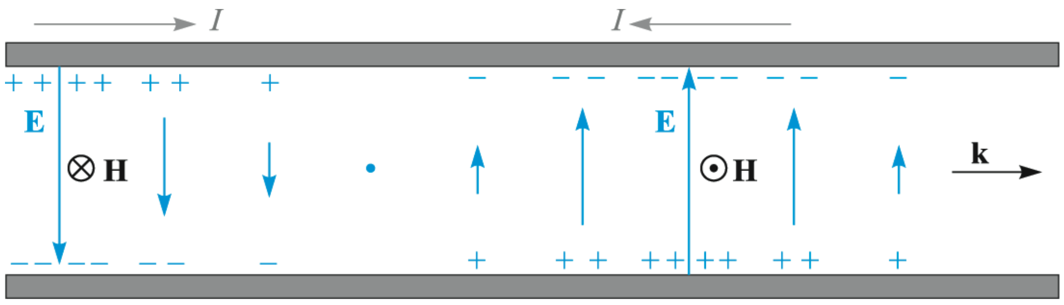
S. Ramo et al., *Fields and Waves in Communication Electronics*, Wiley, 1965, 1984.

Transmission Line Parameters

			$p = \frac{s}{d}$, $q = \frac{s}{D}$ 	 Formulas for $a \ll b$
Characteristic impedance at high frequency Z_0, ohms	$\frac{\eta}{2\pi}\ln\left(\frac{r_0}{r_i}\right)$	$\frac{\eta}{\pi}\cosh^{-1}\left(\frac{s}{d}\right)$	$\frac{\eta}{\pi}\left\{\ln\left[2p\left(\frac{1-q^2}{1+q^2}\right)\right] - \frac{1+4p^2}{16p^4}(1-4q^2)\right\}$	$\eta\frac{a}{b}$
Z_0 for air dielectric	$60\ln\left(\frac{r_0}{r_i}\right)$	$120\cosh^{-1}\left(\frac{s}{d}\right) \cong 120\ln\left(\frac{2s}{d}\right)$ if $s/d \gg 1$	$120\left\{\ln\left[2p\frac{(1-q^2)}{(1+q^2)}\right] - \frac{1+4p^2}{16p^4}(1-4q^2)\right\}$	$120\pi\frac{a}{b}$
Attenuation due to conductor α_c	← $\frac{R}{2Z_0}$ →			
Attenuation due to dielectric α_d	← $\frac{GZ_0}{2} = \frac{\sigma\,\eta}{2}$ →			
Total attenuation dB/meter	← $8.686(\alpha_c + \alpha_d)$ →			
Phase constant for low-loss lines β	← $\omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}$ →			

A Plane Wave Constrained by Parallel Plates

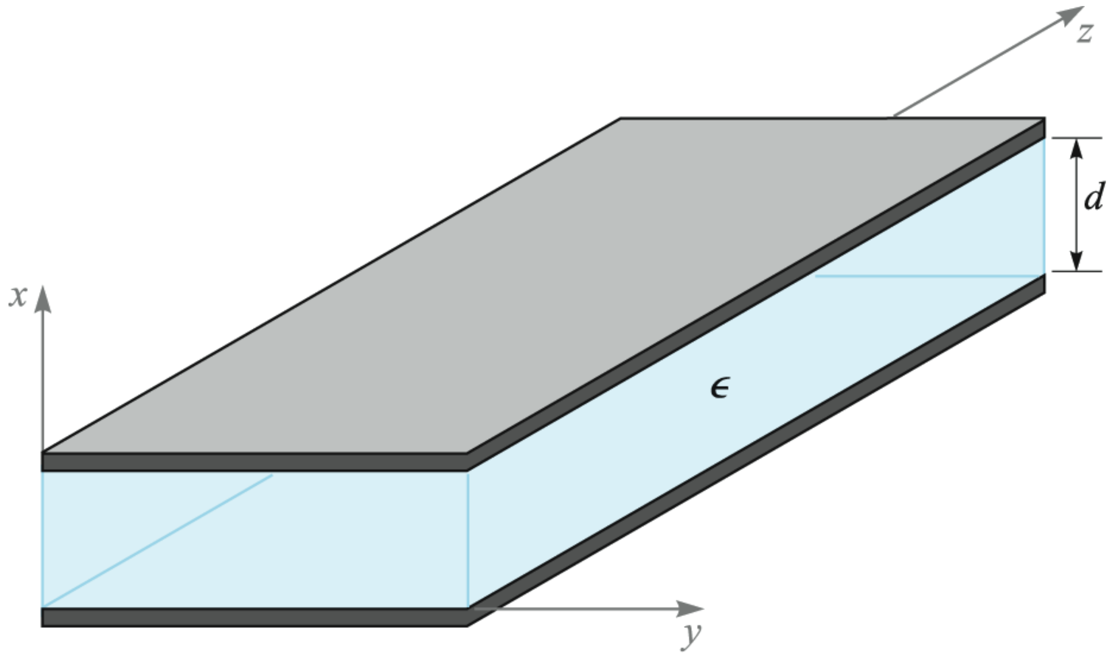
- A uniform plane wave, aka TEM wave, is infinite in extent and, therefore, unrealizable in practice.
- Consider a vertically polarized plane wave propagating in the z-direction.
- If PEC plates are placed at $x = a$ and $x = b$, a TEM wave will propagate between the plates but not extend either above or below.
- This is referred to as a *parallel-plate waveguide*.
- This structure will NOT support a horizontally polarized plane wave. Why not?



Side-view

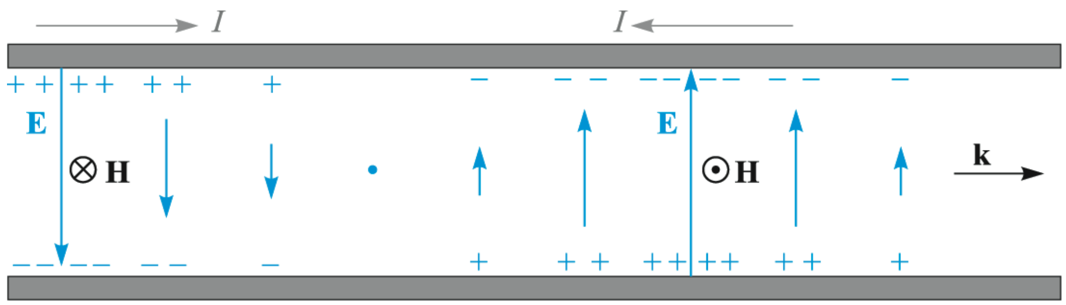
Parallel-Plate Waveguides

- In general, TM-polarized (Transverse Magnetic) waves can propagate between parallel-plates.



Parallel-Plate Waveguides

- TM-polarized (Transverse Magnetic) waves can propagate between parallel-plates.

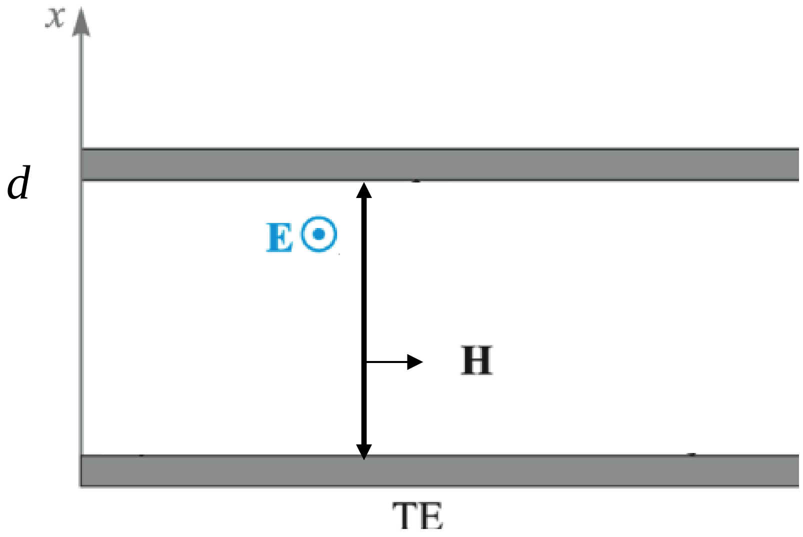


Side-view of a TM-polarized wave

- How can we support TE-polarized (Transverse Electric) waves between parallel-plates with spacing d?
- If we exchange E and H, we have $E_{tan} \neq 0$ at both the upper and lower plates, a violation of the boundary conditions at the surface of a perfect electrical conductor (PEC).
- Is there a solution?

Parallel-Plate Waveguides

- How can we support TE-polarized (Transverse Electric) waves between parallel-plates with spacing d?
- What if the wave was propagating in the x direction?
- Under what conditions would the boundary conditions be met?

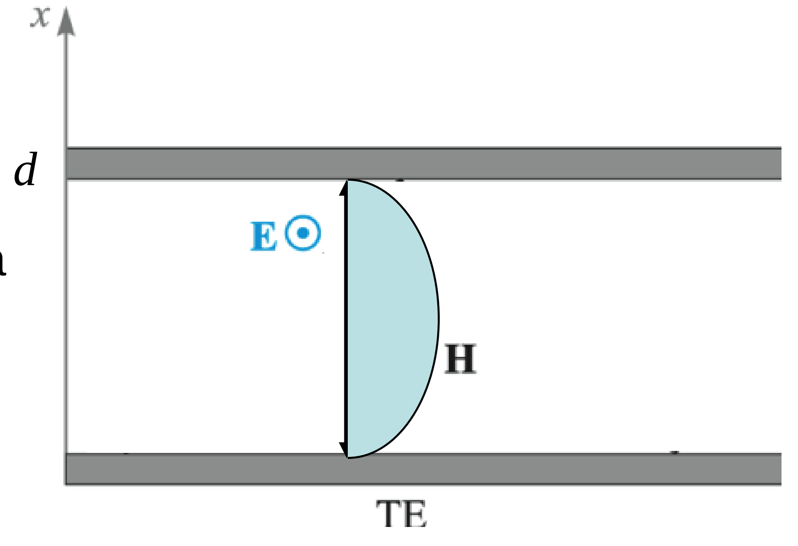


Side-view of a TE-polarized wave

Parallel-Plate Waveguides

- What if the wave was propagating in the x direction?
- Under what conditions would the boundary conditions be met?
- Ans. If the spacing between the plates is a multiple of λ/2, a *standing wave* will form such that $E_{\tan} = 0$ at both the upper and lower plates.

Side-view of a TE-polarized wave

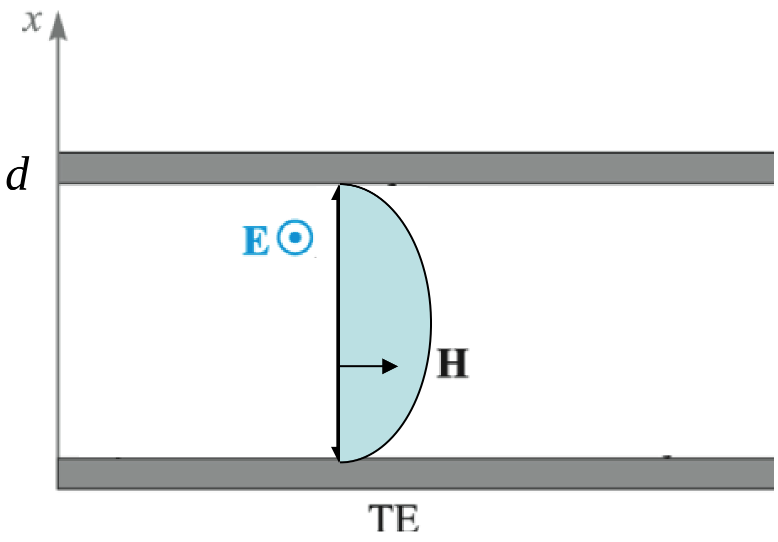


- Reflection is perfect so the upward and downward travelling waves are of equal strength.

Parallel-Plate Waveguides

- This doesn't seem too useful.
- Neither the upward or downward travelling wave is propagating in the z-direction and this only works at discrete frequencies such that $d = n\lambda/2$.
- But it suggests that reflecting waves might yield a solution.

Side-view of a TE-polarized wave

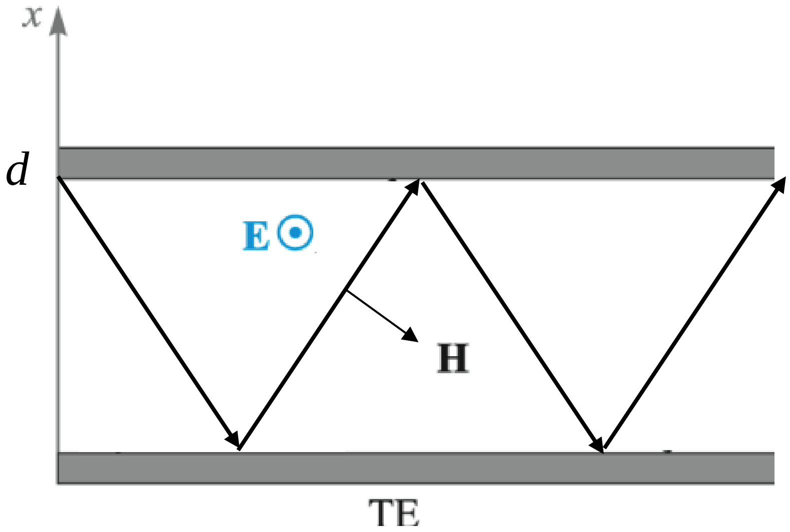


- This is still a TEM wave!

Parallel-Plate Waveguides

- Suppose we launch the plane wave at an angle such that the wave bounces obliquely off the upper and lower plates.
- Assuming that we only want nulls at the upper and lower plates, for each frequency there is only one angle of reflection that will yield a standing wave.

Side-view of a TE-polarized wave

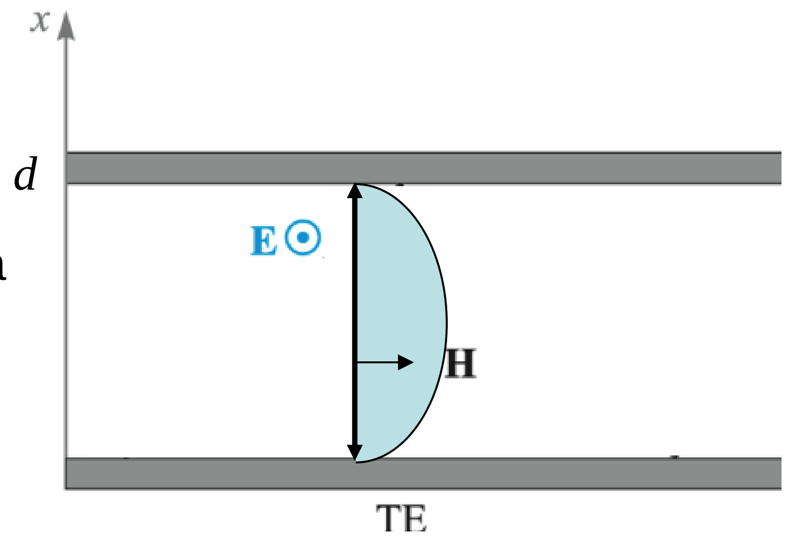


- This is no longer a TEM wave because we now have an H_z component that is parallel to the direction of propagation!

Parallel-Plate Waveguides

- When θ = 0, the wave only propagates up and down (transverse to the direction of propagation) and the waveguide is said to be *cut-off*.
- The frequency f_c that corresponds to $d = \lambda_c/2$ is the *lowest* frequency that can be supported by the structure and is referred to as the *cut-off* frequency of the wave guide.

Side-view of a cut-off TE-polarized wave



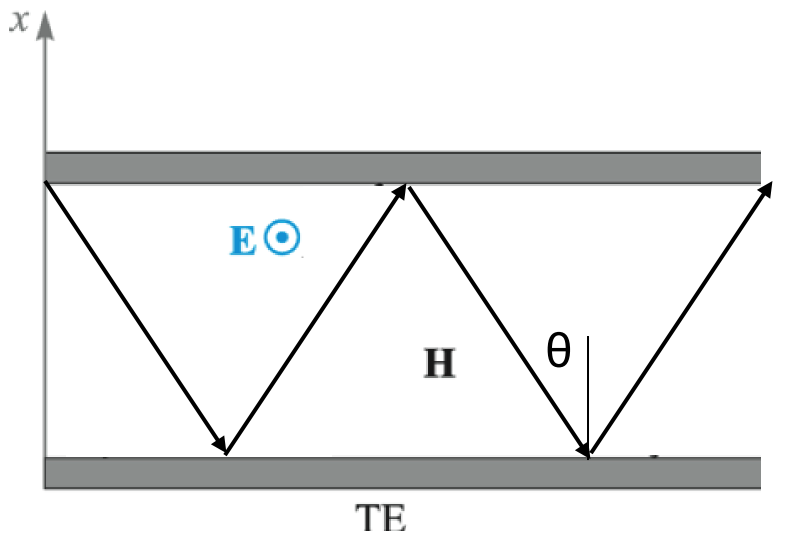
$$\lambda_c = 2d$$

Higher-order modes are possible*!*

Parallel-Plate Waveguides

- As the frequency gradually increases above f_c, the angle θ will gradually increase such that the transverse standing wave forms with nulls at $x = 0$ and $x = d$.
- How to calculate that angle?

Side-view of a TE-polarized wave

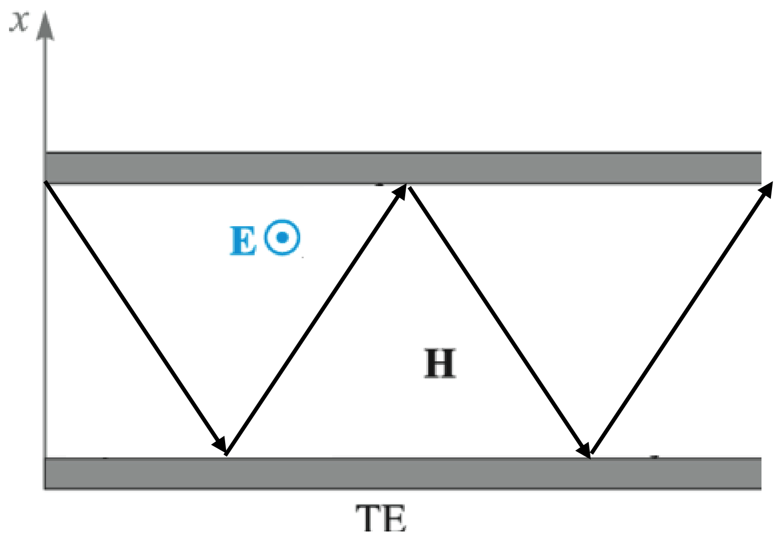


- As usual, the angle θ is measured from the normal to the surface!

Parallel-Plate Waveguides

- An obvious consequence: the wave will seem to travel more slowly because it is taking a longer path.
- The extra length is directly related to the angle θ.
- The resulting velocity is referred to as the *group velocity* and is always slower that the velocity of light in that medium.
- At cut off, the group velocity is 0.

Side-view of a TE-polarized wave

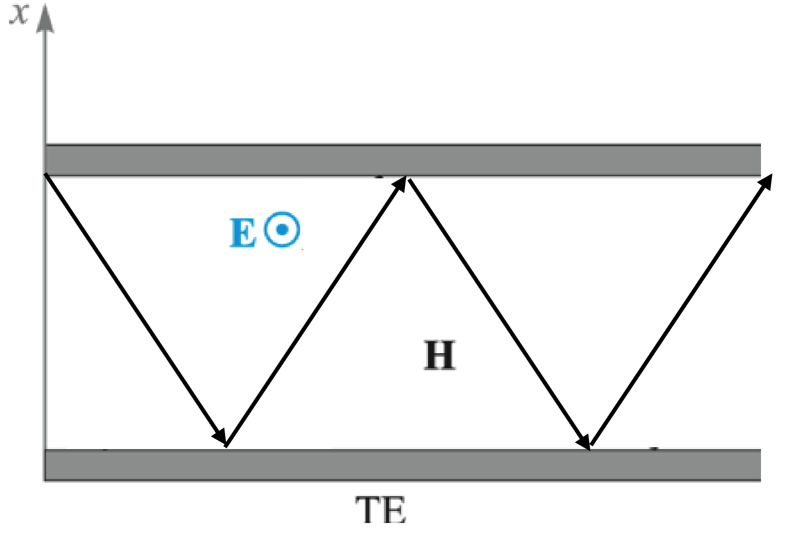


- Group velocity increases with frequency and reaches c/n in the limit..

Parallel-Plate Waveguides

- If a TE standing wave has one complete half-cycle, we refer to it as a TE_1 mode.
- If a TM standing wave has two complete half-cycles, we refer to it as a TM_2 mode.
- Higher-order modes have higher cut-off frequencies.
- At a given frequency, multiple modes may co-exist.

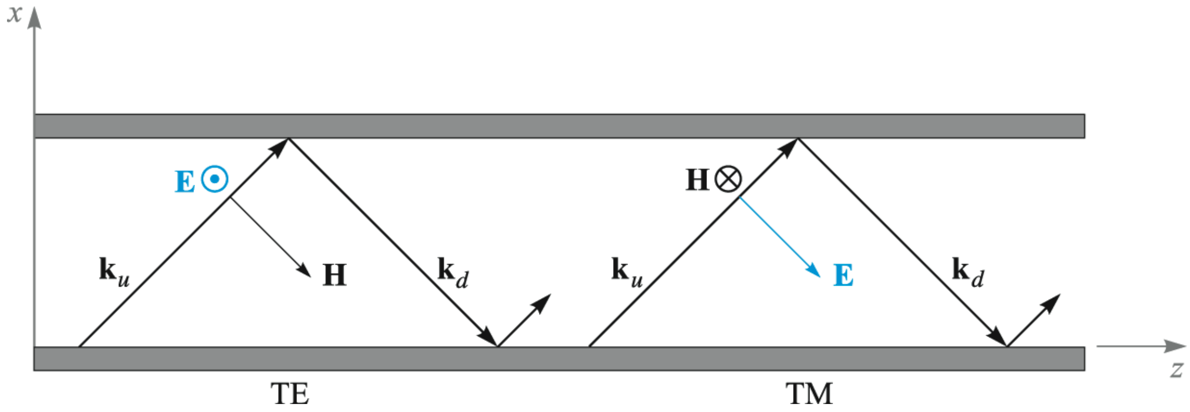
Side-view of a TE-polarized wave



- In general, we want to avoid multiple mode operation so we set the plate spacing d so that we pass just one mode over the band of interest.

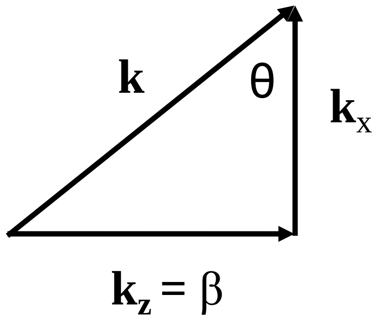
Parallel-Plate Waveguides

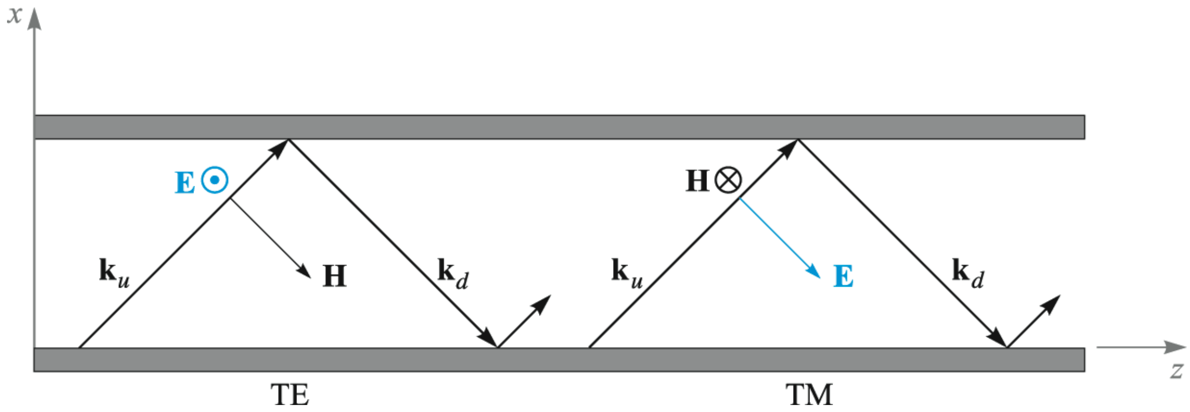
- The TM-polarized TEM-wave in a parallel-plate can be referred to as a TM_0 mode.
- There is no TE_0 mode.
- Higher order TM-modes (TM_1 , TM_2 , etc.) are quite possible.



Parallel-Plate Waveguides

- The wave vector $\mathbf{k}$ is the key to solving parallel-plate wave guide problems.
- Here $\mathbf{k} = \mathbf{k}_u = \mathbf{k}_d = 2\pi/\lambda$ and $\mathbf{k}_x = 2\pi/\lambda_c$
- This is sufficient to solve for $\mathbf{k}_z = \beta = 2\pi/\lambda_g$ and θ.
- Details are in the practice problems.



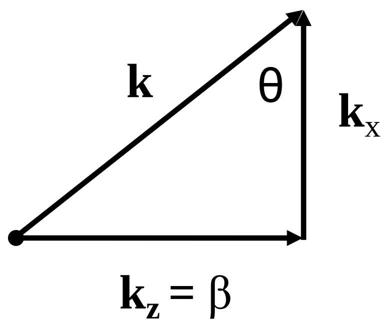


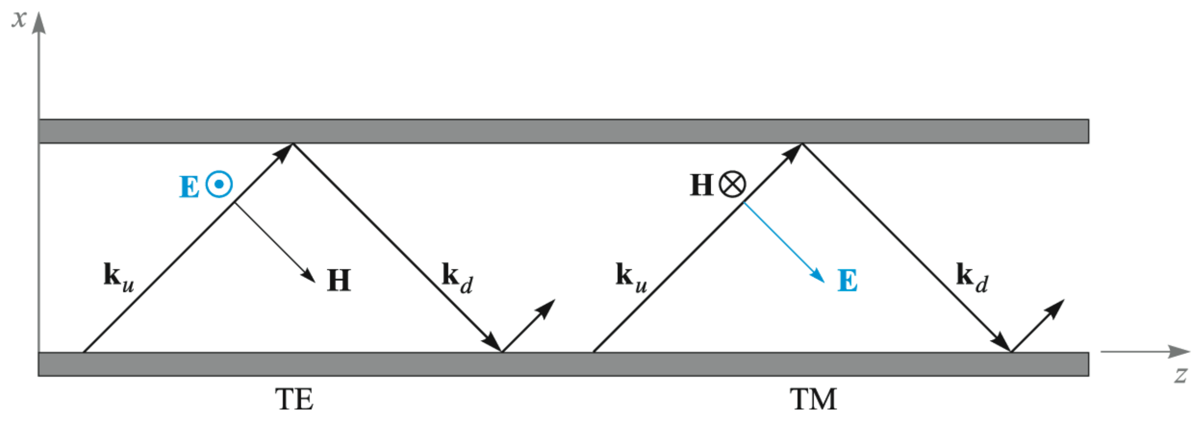
Parallel-Plate Waveguides

- The phase velocity is given by $\frac{\omega}{\beta} = \frac{c}{n \ \sin\theta_m}$

n is the refractive index of the dielectric between the plates

- The group velocity is given by $\frac{\partial\omega}{\partial\beta} = \frac{c\sin\theta_m}{n}$.
- It is the rate at which the envelope of the wave travels through space.





Parallel-Plate Waveguides

- The phase velocity is given by $\frac{\omega}{\beta} = \frac{c}{n\ \sin\theta_m}$

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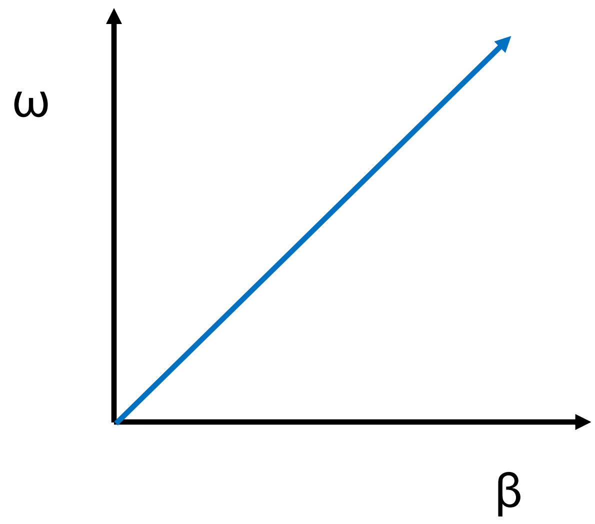
- The group velocity is given by $\frac{\partial\omega}{\partial\beta} = \frac{c\sin\theta_m}{n}$.



For TEM waves, plotting ω vs. β yields a straight line.

- This implies that the phase and group velocities are identical.

What does it look like for a wave guide?

- What are the implications?

Parallel-Plate Waveguides

- The phase velocity is given by $\frac{\omega}{\beta} = \frac{c}{n \ \sin\theta_m}$

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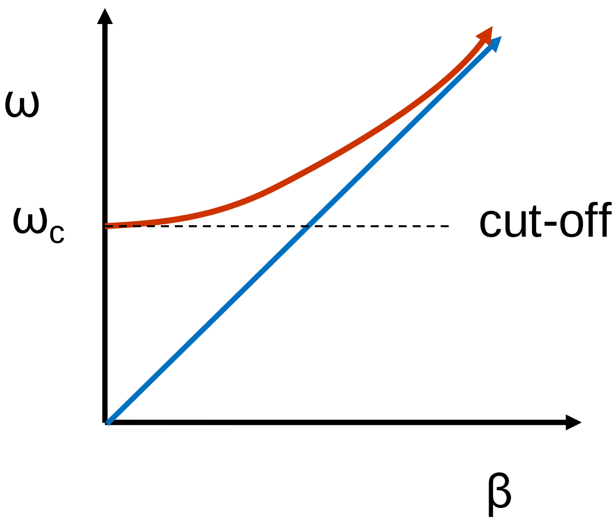
- The group velocity is given by $\frac{\partial\omega}{\partial\beta} = \frac{c \sin\theta_m}{n}$.



For TEM waves, plotting ω vs. β yields a straight line.

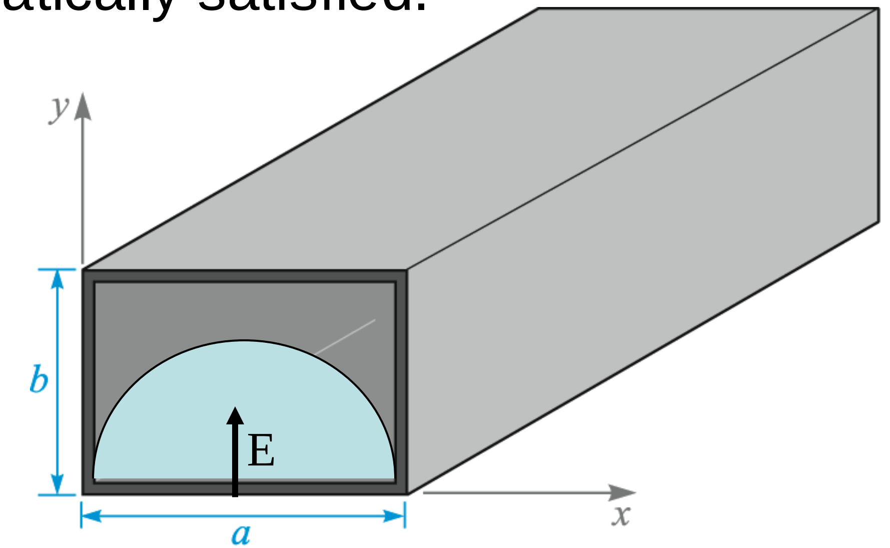
- This implies that the phase and group velocities are identical.

What does it look like for a wave guide?

- What are the implications?
- What is *group velocity* at cut-off, i.e., ω_c?

Rectangular Waveguides

- The same principles apply to rectangular waveguide with horizontal dimension a and vertical dimension $b < a$.
- This can treated as a parallel-plate waveguide turned on its side.
- For TE modes, the boundary conditions at $y = 0$ and $y = b$ are automatically satisfied.



For the dominant TE_{10} mode,

$\lambda_c/2 = a$

$\lambda_c = 2a$

a and b are the *interior* dimensions!

Performance Objectives

1. Given a parallel-plate waveguide with separation d, calculate the cut-off frequency of the nth mode.
2. Given a parallel-plate waveguide with separation d through which a wave with frequency f propagates, determine which modes are propagating.
3. For each propagating mode in such a waveguide, calculate:
   - the angle of reflection, θ,
   - the group velocity, v_g,
   - the phase velocity, v_p ,
   - the group delay, and
   - the guide wavelength, λ_g.
4. Apply knowledge of the parallel-plate waveguide to the TE or TM_{m0} modes of rectangular waveguide.