THE UNIVERSITY OF BRITISH COLUMBIA
Department of Electrical and Computer Engineering

ELEC 311 – Electromagnetic Fields & Waves
2025 W1

Strategies for Solving the Example Problems for
Chapter 9 – Time-Varying Fields and Maxwell's Equations

The purpose of these nine example problems is to help you master some of fundamental techniques used to analyze time-varying fields and apply Maxwell's equations. Many of the concepts will be critical to our study of plane waves, transmission lines and guided waves.

Try these problems before we review the solutions in class. Answers should be short and to the point. Use sketches to explain your solution as required. Clarity, conciseness, and presentation all count. Solution = Intuition (strategy) + Execution (calculation). Make both explicit.

9.X Justify or refute the claim in the caption of Fig. 9.2 that "An apparent increase in flux linkages does not lead to an induced voltage when one part of a circuit is simply substituted for another by opening the switch. No indication will be observed on the voltmeter."

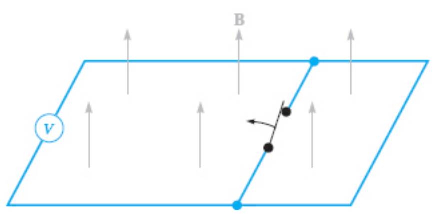

Fig. 9.2

Strategy:

We need to think "outside the box" for this problem.

The formulas that we were given in the textbook do not mention that we must overcome electromechanical force, i.e., supply energy, when changing the effective area of a loop, either by rotating it with respect to the magnetic field or by sliding a bar along parallel rails as in P9.5. It follows that we must supply energy to rotate the loop or slide the bar.

(Consider, for example, the considerable force that falling water must apply to the turbine blades in a hydroelectric generating station in order to cause the rotor to spin and electricity to be generated.)

Because simply opening or closing a switch does not require that we overcome such an electromechanical force, it follows that the action depicted in Fig. 9.2 cannot generate emf despite the fact that flux linkages apparently increase or decrease when the switch is opened or closed. If emf was in fact generated simply by opening or closing the switch, we would apparently be creating energy out of nothing, which is not possible.

For a more in-depth treatment, see the following, which is available on Canvas:

[1] L. V. Bewley, "Flux Linkages and Electromagnetic Induction in Closed Circuits," *Transactions of the American Institute of Electrical Engineers*, vol. 48, no. 2, pp. 327-337, April 1929, doi: 10.1109/T-AIEE.1929.5055221.

9.3 Given $\mathbf{H} = 300\mathbf{a}_z \cos(3\times10^8 t - y)$ A/m in free space, find the emf developed in the general $\mathbf{a}_\phi$ direction about the closed path having corners at (*a*) (0, 0, 0), (1, 0, 0), (1, 1, 0), and (0, 1, 0); and (*b*) (0, 0, 0) (2π, 0, 0), (2π, 2π, 0), and (0, 2π, 0).

Strategy:

Given: $\mathbf{H}(t, y)$, the loop geometry (which we note is confined to the $z = 0$ plane), $N = 1$ turn

Sought: emf generated as a result of the time-varying field for two different loop geometries.

Steps:

1. Sketch and label the problem geometry as an aid to understanding.
2. To find *emf*, we need to find $-N\, d\Phi/dt$; we know that $N = 1$ turn.
3. To find $d\Phi/dt$, we need to find $\Phi(t)$.
4. To find $\Phi(t)$, we need to find $\int_S \mathbf{B}(t) \cdot d\mathbf{S}$; the loop geometry which defines S is given for the two cases. They differ only by a scale factor. Note that the direction of current flow around the boundary of S defines both the polarity of the emf and the direction of **S** according to the right-hand rule.
5. To find $\mathbf{B}(t)$, we need to find $\mathbf{B} = \mu\mathbf{H}$; $\mathbf{H}(t, y)$ is given.

   Consilium est demonstratum.

9.5 The location of the sliding bar in Figure 9.5 is given by $x = 5t + 2t^3$, and the separation of the two rails is 20 cm. Let $\mathbf{B} = 0.8x^2\mathbf{a}_z$ T. Find the voltmeter reading at (*a*) $t = 0.4$ s; (*b*) $x = 0.6$ m.

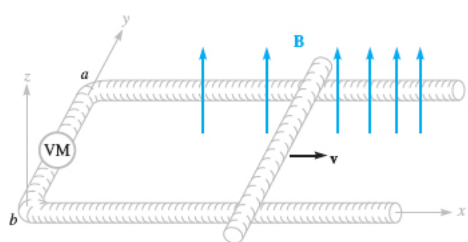

Fig. 9.5

Strategy:

Given: $\mathbf{B}(x)$, the loop geometry in the $z = 0$ plane, $N = 1$ loop, location of the sliding bar $x(t)$

Sought: emf at (*a*) $t = 0.4$ s; (*b*) $x = 0.6$ m.

Steps:

1. Add labels to the given problem geometry as an aid to understanding.
2. To find *emf*, we need to find $-N\, d\Phi/dt$; we know that $N = 1$.
3. To find $d\Phi/dt$, we need to find $\Phi(t)$.
4. To find $\Phi(t)$, we need to find $\int_S \mathbf{B}(t) \cdot d\mathbf{S}$; the loop geometry and $\mathbf{B}(t)$ are given. Note that the direction of current flow around the boundary of S defines both the polarity of the emf and the direction of **S** according to the right-hand rule.
5. In (a), t is given so solving for x and dx/dt is straightforward. In (b), x is given but we are also need dx/dt. We need to find the t that corresponds to x.

   Consilium est demonstratum.

9.11 Let the internal dimensions of a coaxial capacitor be $a = 1.2$ cm, $b = 4$ cm, and $l = 40$ cm. The homogeneous material inside the capacitor has the parameters $\epsilon = 10^{-11}$ F/m, $\mu = 10^{-5}$ H/m, and $\sigma = 10^{-5}$ S/m. If the electric field intensity is $\mathbf{E} = (10^6/\rho) \cos 10^5 t \mathbf{a}_\rho$ V/m, find (*a*) **J**; (*b*) the total conduction current I_c through the capacitor; (*c*) the total displacement current I_d through the capacitor; (*d*) the ratio of the amplitude of I_d to that of I_c, the quality factor of the capacitor.

Given: $\mathbf{E}(\rho)$, the cross-sectional geometry and length of the coaxial capacitor, and the constitutive parameters of the material that fills the capacitor

Sought: a) **J**, b) the total conduction current I_c; (*c*) the total displacement current I_d; (*d*) the ratio of the amplitude of I_d to that of I_c, *i.e.,* the quality factor of the capacitor.

Steps:

1. Sketch and label the problem geometry as an aid to problem understanding.
2. To find **J**, we need to find $\sigma\mathbf{E}$. (The constitutive relations relate current (flux) density to electric or magnetic field strength.) Sketch **J** on the problem geometry.
3. To find I_c we need to find $\int_S \mathbf{J} \cdot d\mathbf{S}$ across any cylindrical shell that lies between the inner and outer conductors. Sketch the shell on the problem geometry.
4. To find I_d we need to find $\int_S \mathbf{J}_d \cdot d\mathbf{S}$ across the same cylindrical shell.
5. To find $\mathbf{J}_d$, we need to find $\partial\mathbf{D}/\partial t$.
6. To find **D**, we need to find $\epsilon_r\epsilon_0\mathbf{E}$
7. To find the quality factor Q of the capacitor, we need to find I_d/I_c .

   Consilium est demonstratum.

9.12 The magnetic flux density $\mathbf{B} = B_0 \cos(\omega t) \cos(k_0 z)\, \mathbf{a}_y$ Wb/m^2 exists in free space. B_0 and k_0 are constants. Find (*a*) the displacement current density; (*b*) the electric field intensity; (*c*) k_0.

Given: An expression for $\mathbf{B}(t,z)$ and knowledge that the fields exist in free space.

Sought: (*a*) the displacement current density; (*b*) the electric field intensity; (*c*) k_0.

Steps:

1. Sketch and label the problem geometry as an aid to problem understanding.
2. Recognize that the magnetic flux density is harmonic in both time *and* space.
3. First goal: The displacement current density
   a. To find $\mathbf{J}_d$, we need to find $\nabla \times \mathbf{H}$.
   b. To find **H**, we need to find $\mathbf{B}/\mu_0$.
4. Second goal: The electric field intensity
   a. To find **E**, we need to find $\mathbf{D}/\epsilon_r\epsilon_0$.
   b. To find **D**, we need to integrate $\partial\mathbf{D}/\partial t$ (which equals $\mathbf{J}_d$ from Step 3a) over time where the constant of integration can be assumed to be equal to zero because there is no DC component to the field.
5. To find k_0, compare: 1) the expression for **B** given in the problem to 2) the expression obtained by taking the curl of **E** from step 4. After obtaining $-\partial\mathbf{B}/\partial t$, integrate to obtain **B** where the constant of integration can be assumed to be equal to zero (why?). Find the value of k_0 that ensures that the magnitude of **B** is the same in both cases.

Consilium est demonstratum.

9.15 Use each of Maxwell's equations in point form to obtain as much information as possible about (*a*) **H**, if **E** = 0; (*b*) **E**, if **H** = 0.

Given: Maxwell's equations in point form subject to the condition that either **E** or **H** is zero.
Sought: As much information about **H** or **E** as possible.
Steps:

1. For each case, write the set of four Maxwell's equation in point form *with appropriate substitutions* that reflects the observation that either a) **E** = 0 or b) **H** = 0.
2. Among other things, recognize that cases a) and b) correspond to magnetostatic and electrostatic cases, respectively. Why? Magnetic and electric fields can only exist in isolation if they are static.

Consilium est demonstratum.

9.18 The parallel-plate transmission line shown in Figure 9.7 has dimensions b = 4 cm and d = 8 mm, while the medium between the plates is characterized by $\mu_r = 1$, $\epsilon_r = 20$, and $\sigma = 0$. Neglect fields outside the dielectric. Given the field $\mathbf{H} = 5\cos(10^9 t - \beta z)\mathbf{a}_y$ A/m, use Maxwell's equations to help find (*a*) β, if $\beta > 0$; (*b*) the displacement current density at $z = 0$; (*c*) the total displacement current crossing the surface $x = 0.5d$, $0 < y < b$, $0 < z < 0.1$ m in the $\mathbf{a}_x$ direction.

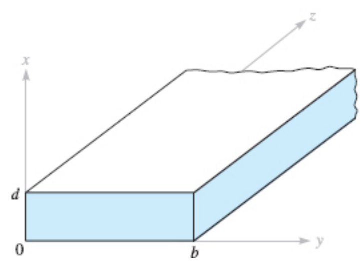

Fig. 9.7

Given: $\mathbf{H}(t,z)$, the parallel-plate geometry, and the nature of the dielectric material.
Sought: (*a*) β, if $\beta > 0$; (*b*) the displacement current density at $z = 0$; (*c*) the total displacement current crossing the surface $x = 0.5d$, $0 < y < b$, $0 < z < 0.1$ m in the $\mathbf{a}_x$ direction.
Steps:

1. Add labels to the given problem geometry as an aid to understanding.
2. To find β, compare: 1) the expression for **H** given in the problem to 2) the expression obtained by:
   a. taking the curl of **H** to yield an expression for $\partial\mathbf{E}/\partial t$,
   b. integrating this to yield E,
   c. taking the curl of **E** to yield an expression for $\partial\mathbf{H}/\partial t$, and then
   d. integrating this to yield an expression for H.

   Find the value of β that ensures that the magnitude of **H** is the same in both cases.
3. Step 2a yields an expression for displacement current density.
4. Integrating $\mathbf{J}_d$ across the surface defined by $x = 0.5d$, $0 < y < b$, $0 < z < 0.1$ m in the $\mathbf{a}_x$ direction yields the total displacement current requested in (c).

   Consilium est demonstratum.

9.22 In a sourceless medium in which **J** = 0 and $\rho_v = 0$, assume a rectangular coordinate system in which **E** and **H** are functions only of z and t. The medium has permittivity ϵ and permeability μ. (a) If $\mathbf{E} = E_x\mathbf{a}_x$ and $\mathbf{H} = H_y\mathbf{a}_y$, begin with Maxwell's equations and determine the second-order partial differential equation that E_x must satisfy. (b) Show that $E_x = E_0 \cos(\omega t - \beta z)$ is a solution of that equation for a particular value of β. (c) Find β as a function of given parameters.

Strategy:

Given: **E** and **H** are functions only of z and t, $\mathbf{E} = E_x\mathbf{a}_x$ and $\mathbf{H} = H_y\mathbf{a}_y$, the medium is sourceless, and the medium has permittivity ϵ and permeability μ.

Sought: a) the second-order partial differential equation that *Ex* must satisfy. (b) that $E_x = E_0 \cos(\omega t - \beta z)$ is a solution of that equation for a particular value of β. (c) β as a function of given parameters.

Steps:

1. Sketch and label the problem geometry as an aid to understanding.
2. Find $\nabla \times \mathbf{E}$ (a spatial derivative) to yield an expression for the rate of change of **H** with time.
3. Find $\nabla \times \mathbf{H}$ (a spatial derivative) to yield an expression for the rate of change of **E** with time.
4. Take the $\partial/\partial z$ of the first expression and the $\partial/\partial t$ of the second expression so that both resulting expressions are equal to the $\partial^2 H_y/\partial t \partial z$.
5. Equate the second-order spatial derivative of **E** to a constant ($\mu\epsilon$) multiplied by the second-order temporal derivative of **E** and thereby satisfy (a).
6. Substitute the expression in (b) into this second-order PDE in order to demonstrate that it is indeed a solution provided that $\beta^2 = \omega^2\mu\epsilon$ and thereby satisfy (b) and (c).

   Consilium est demonstratum.

9.26 Write Maxwell's equations in point form in terms of **E** and **H** as they apply to a sourceless medium, where **J** and ρ_v are both zero. Replace ϵ with μ, μ with ϵ, **E** with **H**, and **H** with −**E**, and show that the equations are unchanged. This is a more general expression of the *duality principle* in circuit theory.

Strategy:

This problem presents explicit directions for the derivation. All we need do is follow them.

Consilium est demonstratum.