THE UNIVERSITY OF BRITISH COLUMBIA
Department of Electrical and Computer Engineering

ELEC 311 – Electromagnetic Fields & Waves
2025 W1

Drill Problems
Chapter 9 – Time-Varying Fields and Maxwell's Equations

The purpose of the seven drill problems in Chapter 9 is to help you master fundamental techniques used to analyze time-varying fields and apply Maxwell's equations. Many of the concepts will be critical to our study of plane waves, transmission lines and guided waves.

Answers should be short and to the point. Use sketches to explain your solution as required. Clarity, conciseness, and presentation all count. Solution = Intuition (strategy) + Execution (calculation). Make both explicit.

D9.1 Within a certain region, $\epsilon = 10^{-11}$ F/m and $\mu = 10^{-5}$ H/m. If $B_x = 2\times10^{-4}\cos(10^5 t)\sin(10^{-3}y)$ T: (*a*) use $\nabla\times\mathbf{H} = \epsilon\,\partial\mathbf{E}/\partial t$ to find **E**; (*b*) find the total magnetic flux passing through the surface $x = 0$, $0 < y < 40$ m, $0 < z < 2$ m, at $t = 1\ \mu$s; (*c*) find the value of the closed line integral of **E** around the perimeter of the given surface.

Strategy:

Given: In a region, $\epsilon = 10^{-11}$ F/m, $\mu = 10^{-5}$ H/m, and $B_x = 2\times10^{-4}\cos(10^5 t)\sin(10^{-3}y)$ T

Sought: (*a*) **E**; (*b*) Φ passing through $x = 0$, $0 < y < 40$ m, $0 < z < 2$ m, at $t = 1\ \mu$s; (*c*) the value of the closed line integral of **E** around the perimeter of the given surface

Steps:

1. Sketch and label the problem geometry as an aid to understanding.
2. To find **E,** we need to:
   a. convert **B** to **H**, and evaluate $\nabla\times\mathbf{H} = \partial\mathbf{D}/\partial t = \epsilon\,\partial\mathbf{E}/\partial t$,
   b. integrate $\partial\mathbf{E}/\partial t$ with respect to time and set the constant of integration to 0.
3. To find Φ, we need to integrate $\mathbf{B}(t)\cdot d\mathbf{S}$ across the surface S and evaluate the result at $t = 1\ \mu$s.
4. To find the value of the closed line integral of **E** around the perimeter of the given surface at $t = 1\ \mu$s, we can either evaluate the integral directly or recall the integral form of Faraday's law, $emf = \oint \mathbf{E}\cdot d\mathbf{L} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{S}$, and evaluate the right-hand side expression instead. Both will yield the same result.
5. We can quickly check the sign of the emf by visualizing the time dependence of B (and Φ). Where are they with respect to their peak values when $t = 1\ \mu$s given the value of the period T? Are they rising or falling? What is the implication for emf = $-d\Phi/dt$?

Consilium est demonstratum.

Answers: (*a*) $-20{,}000\sin(10^5 t)\cos(10^{-3}y)\mathbf{a}_z$ V/m; (*b*) 0.318 mWb; (*c*) −3.19 V

D9.2 With reference to the sliding bar shown below, let $d = 7$ cm, $\mathbf{B} = 0.3\mathbf{a}_z$ T, and $\mathbf{v} = 0.1\mathbf{a}_y e^{20y}$ m/s. Let $y = 0$ at $t = 0$. Find: (*a*) $v(t = 0)$; (*b*) $y(t = 0.1)$; (*c*) $v(t = 0.1)$; (*d*) V_{12} at $t = 0.1$.

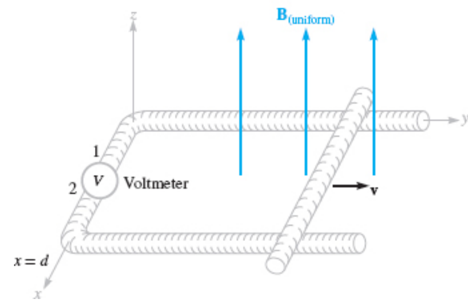

Figure 9.1 An example illustrating the application of Faraday's law to the case of a constant magnetic flux density **B** and a moving path. The shorting bar moves to the right with a velocity **v**, and the circuit is completed through the two rails and an extremely small high-resistance voltmeter. The voltmeter reading is $V_{12} = -Bvd$.

Strategy:

Given: The geometry of the fixed and moving wires, an expression for the velocity v of the moving wire, the position of the moving wire at $t = 0$. Assume MKS units.

Sought: (*a*) $v(t = 0)$; (*b*) $y(t = 0.1)$; (*c*) $v(t = 0.1)$; (*d*) V_{12} at $t = 0.1$.

Steps:

1. Sketch and label the problem geometry as an aid to understanding.
2. To find $v(t = 0)$, we need to evaluate $v(y)$ using the given value of $y(t = 0)$.
3. To find $y(t = 0.1)$, we need to express $v = 0.1e^{20y}$ as $dy/dt = 0.1e^{20y}$ and transform this to $dt = \int 10\, e^{-20y} dy$. Solving this and adding a constant of integration that sets $y = 0$ at $t = 0$ allows us to find y at $t = 0.1$ or $y(t = 0.1)$.
4. To find $v(t = 0.1)$, we need to use the value of $y(t = 0.1)$ obtained in step 3 and substitute it into the given expression for $v(y)$.
5. To find V_{12} at $t = 0.1$, we can apply equation (9) as given in the figure caption above.
   Rationale: $\Phi = \mathbf{B}\cdot\mathbf{A} = Byd$ so $-d\Phi/dt = -Bd\, dy/dt = -Bvd$

   Consilium est demonstratum.

Answers: (*a*) 0.1 m/s; (*b*) 1.12 cm; (*c*) 0.125 m/s; (*d*) −2.63 mV

D9.3 Find the amplitude of the displacement current density: (*a*) adjacent to an automobile antenna where the magnetic field intensity of an FM signal is $H_x = 0.15\cos[3.12(3\times10^8 t - y)]$ A/m; (*b*) in the airspace at a point within a large power distribution transformer where $\mathbf{B} = 0.8\cos[1.257\times10^{-6}(3\times10^8 t - x)]\mathbf{a}_y$ T; (*c*) within a large, oil-filled power capacitor where $\epsilon_r = 5$ and $\mathbf{E} = 0.9\cos[1.257\times10^{-6}(3\times10^8 t - 5z)]\mathbf{a}_x$ MV/m; (*d*) in a metallic conductor at 60 Hz, if $\epsilon = \epsilon_0$, $\mu = \mu_0$, $\sigma = 5.8\times10^7$ S/m, and $\mathbf{J} = \sin(377t - 117.1z)\mathbf{a}_x$ MA/m^2.

Strategy:

Given: Expressions for time-varying fields and fluxes in a variety of situations.

Sought: The *amplitude* of the displacement current density in each case.

Steps:

1. Sketch and label the problem geometry in each case as an aid to understanding.
2. For case (a), the environment is free space and **H** is given. To find the displacement current density, find the curl of **H**.
3. For case (b), the environment is air-filled and **B** is given. To find the displacement current density, find the curl of **H**. To find **H**, take $\mathbf{B}/\mu_0$.
4. For case (c), the environment is oil-filled and **E** is given. To find the displacement current density, find $\epsilon\, \partial\mathbf{E}/\partial t$.
5. For case (d), the environment is a good conductor and **J** is given. The ratio of conduction current density to displacement current density is $\sigma/\omega\epsilon$. Thus, to find the displacement current density, find the conduction current density and divide by $\sigma/\omega\epsilon$.

   Consilium est demonstratum.

Answers: (*a*) 0.468 A/m^2; (*b*) 0.800 A/m^2; (*c*) 0.0150 A/m^2; (*d*) 57.6 pA/m^2

D9.4 Let $\mu = 10^{-5}$ H/m, $\epsilon = 4 \times 10^{-9}$ F/m, $\sigma = 0$, and $\rho_v = 0$. Find k (including units) so that each of the following pairs of fields satisfies Maxwell's equations:
(a) $\mathbf{D} = 6\mathbf{a}_x - 2y\mathbf{a}_y + 2z\mathbf{a}_z$ nC/m^2, $\mathbf{H} = kx\mathbf{a}_x + 10y\mathbf{a}_y - 25z\mathbf{a}_z$ A/m;
(b) $\mathbf{E} = (20y - kt)\mathbf{a}_x$ V/m, $\mathbf{H} = (y + 2 \times 10^6 t)\mathbf{a}_z$ A/m.

Strategy:

Given: Expressions for **D** and **H**, and for **E** and **H**, in cartesian coordinates.

Sought: k in each case.

Steps:

The first case represents *static* fields in a source-free region, so recall Maxwell's equations in point form for this special case.

1. To find k, we need to find the expansion of the divergence of **D** and **B** (= μ **H)** in rectangular coordinates, and set each to 0.

   Observations:

   Here, the divergence of **D** = 0 regardless of the value of k. Accounting for the divergence of **B** is sufficient to define k.

   The curl of **D** and **B** must also be zero, but a quick check shows that they satisfy that condition regardless of the value of k.

   The second case represents *time-varying* fields in a source-free region, so recall Maxwell's equations in point form for this special case.

2. To find k, we need to find the expansion of the curl of **E** and **H** (in rectangular coordinates, and set each to $-\partial\mathbf{B}/\partial t$ and $\partial\mathbf{D}/\partial t$, respectively. This will yield two equations in k but they will be consistent, *i.e.,* have the same solution.

   Observation:

   The divergence of **D** and **B** must also be zero, but a quick check shows that they satisfy that condition regardless of the value of k.

Consilium est demonstratum.

Answers: (a) $k = 15$ A/m^2; (b) $k = -2.5\times10^8$ V/(m · s)

D9.5 The unit vector $\mathbf{n} = 0.64\mathbf{a}_x + 0.6\mathbf{a}_y - 0.48\mathbf{a}_z$ is directed from region 2 ($\epsilon_r = 2$, $\mu_r = 3$, $\sigma_2 = 0$) toward region 1 ($\epsilon_{r1} = 4$, $\mu_{r1} = 2$, $\sigma_1 = 0$). If $\mathbf{B}_1 = (\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z) \sin 300t$ T at point P in region 1 adjacent to the boundary, find the amplitude at P of: (*a*) $\mathbf{B}_{N1}$; (*b*) $\mathbf{B}_{t1}$; (*c*) $\mathbf{B}_{N2}$; (*d*) $\mathbf{B}_2$.

Strategy:

Given: The constitutive parameters for regions 1 and 2, expressions for $\mathbf{B}_1$ and the point P, and a unit vector that defines the boundary between regions 1 and 2.

Sought: The amplitude at P of: (*a*) $\mathbf{B}_{N1}$; (*b*) $\mathbf{B}_{t1}$; (*c*) $\mathbf{B}_{N2}$; (*d*) $\mathbf{B}_2$.

Steps:

1. Sketch and label the problem geometry as an aid to understanding.
2. To find $\mathbf{B}_{N1}$, we need to take the dot product of $\mathbf{B}$ and the unit vector $\mathbf{n}$ and multiply the result by $\mathbf{n}$.
3. To find $\mathbf{B}_{t1}$, we need to evaluate $\mathbf{B} - \mathbf{B}_{N1}$.
4. To find $\mathbf{B}_{N2}$, we need to apply the boundary condition which states that $\mathbf{B}_{N1} = \mathbf{B}_{N2}$.
5. To find $\mathbf{B}_2$, we need to find $\mathbf{B}_{N2}$ and $\mathbf{B}_{t2}$.
6. To find $\mathbf{B}_{t2}$, we need to apply the relevant boundary condition: $\mathbf{H}_{t2} = \mathbf{H}_{t1}$.
7. To find $\mathbf{H}_{t2} = \mathbf{H}_{t1}$, take $\mathbf{B}_{t1}/\mu_0\mu_{r1}$.
8. To find $\mathbf{B}_{t2}$, take $\mu_0\mu_{r2}\,\mathbf{H}_{t2}$.
9. To find $\mathbf{B}_2$, take $\mathbf{B}_{t2} + \mathbf{B}_{n2}$.

   Consilium est demonstratum.

Answers: (*a*) 2.00 T; (*b*) 3.16 T; (*c*) 2.00 T; (*d*) 5.15 T

D9.6 The surface $y = 0$ is a perfectly conducting plane, whereas the region $y > 0$ has $\epsilon_r = 5$, $\mu_r = 3$, and $\sigma = 0$. Let $\mathbf{E} = 20\cos(2\times10^8 t - 2.58z)\mathbf{a}_y$ V/m for $y > 0$, and find at $t = 6$ ns; (*a*) ρ_S at $P(2, 0, 0.3)$; (*b*) **H** at *P*; (*c*) **K** at *P*.

Strategy:

Given: A description of the two regions, **E** in y > 0

Sought: At $t = 6$ ns; (*a*) ρ_S at $P(2, 0, 0.3)$; (*b*) **H** at *P*; (*c*) **K** at *P*.

Steps:

1. Sketch and label the problem geometry as an aid to understanding. Note that the region for $y > 0$ is a perfect dielectric while $P(2, 0, 0.3)$ lies on the perfectly conducting plane.
2. To find ρ_S, the surface charge density in C/m^2, we need to apply equation (39) which is a consequence of applying the integral form of Gauss's law (eqn (35)) at the boundary.
3. To find **H**, we need to find **B.**
4. To find **B**, we need to expand the curl of **E** in rectangular coordinates and integrate $\partial\mathbf{B}/\partial t$ with respect to time while setting the constant of integration to 0.
5. To find **K**, the surface current density in A/m, we need to apply equation (42) which is a consequence of applying the integral form of Ampère's law (34) at the boundary.

Consilium est demonstratum.

Answers: (*a*) $\rho_S = 0.81$ nC/m^2; (*b*) $\mathbf{H} = -62.3\mathbf{a}_x$ mA/m; (*c*) $\mathbf{K} = -62.3\mathbf{a}_z$ mA/m

D9.7 A point charge Q_1 of 4 cos($10^8\pi t$) μC is located at $P_1(0, 0, 1.5)$, whereas $Q_2 = -4\cos(10^8\pi t)$ μC is located at $P_2(0, 0, -1.5)$, both in free space. Find V at $P(r = 450, \theta, \phi = 0)$ at $t = 15$ ns for $\theta =$: (*a*) 0°; (*b*) 90°; (*c*) 45°.

Strategy:

Given: A pair of time-varying point charges at specified locations.

Sought: V at $P(r = 450, \theta, \phi = 0)$ at $t = 15$ ns for $\theta =$: (*a*) 0°; (*b*) 90°; (*c*) 45°.

Steps:

1. Sketch and label the problem geometry as an aid to understanding.
2. To find $V(x, y, z, t)$, apply equation (45). The integral simplifies to a summation,

$$V(x, y, z, t) = \frac{Q_1(t')}{4\pi\epsilon R_1} + \frac{Q_2(t')}{4\pi\epsilon R_2}$$

where, to account for the effect of distance to the observation point, the retarded time $t' = t - R/v$ replaces t in order to account for the finite length of time that it takes the effect to propagate.

Consilium est demonstratum.

Answers: (a) 159.8 V; (b) 0; (c) 143 V