THE UNIVERSITY OF BRITISH COLUMBIA
Department of Electrical and Computer Engineering

ELEC 311 – Electromagnetic Fields and Waves
2025 W1

Practice Midterm 3 - In-Class – Strategies + Solutions

Chapter 11 – Uniform Plane Waves
Chapter 12 – Plane Wave Reflection and Dispersion

The purpose of this midterm exam is to assess your mastery of the fundamental techniques used to analyze uniform plane waves and plane wave reflection and dispersion.

Answers should be short and to the point. Use sketches to explain your solution as required. Clarity, conciseness, and presentation all count. Solution = Intuition (strategy) + Execution (calculation). Make both explicit.

1. Uniform Plane Waves

A 1 MHz plane wave travels in the *z*-direction in a non-magnetic conducting medium in which the conductivity is 58 MS/m.

a. Calculate the intrinsic impedance, propagation constant, wave velocity, skin depth and penetration depth. Briefly explain your strategy and reasoning for each calculation. Use sketches where required. Briefly explain the physical significance of each quantity. [10]

Strategy + *Solution*:

1. Recall that the propagation constant γ is a parameter of the vector Helmholtz equation, $\nabla^2\mathbf{E} = \gamma^2\mathbf{E}$, where $\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$. This means that $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$. Of course, $\gamma = \alpha + j\beta$ = 15,124 Np/m + j 15,124 rad/m is the complex propagation constant. (Rather than use the general formula, we used the simplified expression derived in the second part to this problem.)

2. Recall that $\mathbf{H} = \frac{j}{\omega\mu}\nabla\times\mathbf{E}$ so the intrinsic impedance $\eta = E/H = \sqrt{j\omega\mu/(\sigma + j\omega\epsilon)}$. Thus, $\eta =$ (Rather than use the general formula, we use the simplified expression derived in the second part to this problem.)

3. Recall that $\sigma \gg \omega\epsilon$ for a good conductor, i.e., the loss tangent $\sigma/\omega\epsilon$ is very high, which allows the expressions for γ and η to be greatly simplified. This is explored in more detail in the second part to this problem.

4. Recall that the velocity of propagation is given by ω/β = 415 m/s and the wavelength is given by $2\pi/\beta$ = 415 microns.

5. Recall that when the attenuation constant α is expressed in Np/m, the skin depth $\delta = 1/\alpha = 66.1$ microns and the penetration depth = $5\delta\ = \ 330.6$ microns.

b. Explain how the general expressions for the intrinsic impedance and propagation constant can be greatly simplified in this case. [5]

Strategy + *Solution*:

1. Recall that $\sigma \gg \omega\epsilon$, the $j\omega\epsilon$ term in the general expression for intrinsic impedance

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

can be neglected so the intrinsic impedance becomes complex and is given by

$$\eta = \sqrt{j\omega\mu/\sigma},$$

and the maximum lag of $\theta = 45$ degrees is observed.

2. Recall that in the general case,

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right)}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right)}$$

but if $\sigma \gg \omega\epsilon$, the '1' terms in the general expressions for α and β can be neglected and α and β both simplify to the same expression albeit with different units (Np/m and rad/m, respectively).

$$\alpha = \sqrt{\pi f\mu\sigma} = \ 15124 \text{ Np/m.}$$

c. A plane wave with a power density of 100 W/m^2 is observed at $z = 0$. What power density will be observed at $z = 10$ microns = 10^{-5} m? [5]

Strategy + *Solution*: No time-variation is mentioned so we're being asked for the time-averaged power density. Because power density is determined from the product of the electric and magnetic field strengths, it decays with distance twice as quickly as electric and magnetic field strength. Thus, if S_0 = 100 W/m^2 is the power density at $z = 0$, and we take great care to express everything in consistent units, the power density at z is given by

$$S(z) = \ S_0 e^{-2\alpha z} = 73.9 \text{ W/m}^2$$

2. Uniform Plane Waves

a. A plane wave with electric field strength of 10 V/m and frequency of 5 GHz is travelling in the positive z direction through a perfect dielectric with relative permittivity = 2.5 and relative permeability = 1. Give the corresponding Helmholtz equations and find expressions for the field components of the wave and intrinsic impedance of the medium. [5]

Strategy:

1. Recall that the propagation constant γ is a parameter of the vector Helmholtz equations, $\nabla^2\mathbf{E} = \gamma^2\mathbf{E}$ and $\nabla^2\mathbf{H} = \gamma^2\mathbf{H}$, where $\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$ and $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \alpha + j\beta$ is the complex propagation constant. For a perfect dielectric, $\alpha = 0$ Np/m and $\beta = \omega\sqrt{\mu\epsilon}$ rad/m. Here, $E_0 = 10$ V/m, $\omega = 3.14 \times 10^{10}$ rad/s and $\beta = 165.6$ rad/m.

2. Recall that for a uniform plane wave travelling in the z direction, the wave is uniform in x and y and varies only with z. Without loss of generality, assume that the electric field vector is pointed in the x direction. Then, $\mathbf{E} = E_0 e^{j(\omega t - \beta z)}\hat{\mathbf{x}}$.

3. Recall that $\mathbf{H} = \frac{j}{\omega\mu}\nabla \times \mathbf{E}$ so the intrinsic impedance $\eta = E/H = \sqrt{j\omega\mu/(\sigma + j\omega\epsilon)}$. That is, $H = E/\eta$ and $\mathbf{H} = H_0 e^{j(\omega t - \beta z)}\hat{\mathbf{y}}$. Here, $\eta = E/H = \sqrt{\mu/\epsilon} = 238\ \Omega$

b. A medium has relative permittivity = 2.5, relative permeability = 1 and conductivity = 50 S/m. Find the intrinsic impedance of the medium and the velocity of propagation, wavelength, loss tangent, and complex propagation constant of a 50 MHz plane wave that is travelling through it. [5]

Strategy:

1. Recall that $\mathbf{H} = \frac{j}{\omega\mu}\nabla \times \mathbf{E}$ so
   the intrinsic impedance $\eta = E/H = \sqrt{j\omega\mu/(\sigma + j\omega\epsilon)} = 2.81 \measuredangle 45°\Omega$.

2. Recall that the propagation constant γ is a parameter of the vector Helmholtz equation, $\nabla^2\mathbf{E} = \gamma^2\mathbf{E}$, where $\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$ and $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \alpha + j\beta = 99.3\,\frac{\text{Np}}{\text{m}} + j\,99.3\,\frac{\text{rad}}{\text{m}}$ is the complex propagation constant.

3. Recall that the loss tangent $\tan\delta = \sigma/\omega\epsilon = 7193$. It is zero for a perfect dielectric and infinite for a perfect conductor.

4. Recall that the velocity of propagation is given by $\frac{\omega}{\beta} = 3.164 \times 10^6$ m/s and the wavelength is given by $2\pi/\beta = 0.0633$ m.

c. Consider an AWG 30 copper wire of length 15 cm. What is the skin depth and resistance at 2 GHz? How deeply does the current penetrate? What are the attenuation and phase constants? [5]

Strategy:

1. Recall that AWG 30 wire has a diameter of 0.2546 mm. (We would provide this number on an exam.)

2. Recall that annealed copper has a conductivity of 5.8 MS/m.

3. Recall that for a good conductor, $\sigma \gg \omega\epsilon$, $\alpha = \sqrt{\pi f \mu \sigma} = 6.77 \times 10^5$ Np/m.

4. Recall that when the attenuation constant α is expressed in Np/m, the skin depth $\delta = \frac{1}{\alpha} = 1.478\ \mu$m and the penetration depth = $5\delta = 7.390\ \mu$m.

5. Recall that the effective cross-sectional area of a conductor at high frequencies is well approximated by the area of an annulus with outer circumference πd and thickness δ.

6. Recall that $R = L/\sigma A = 2.18\ \Omega$.

d. A plane wave with electric field strength of 10 V/m and frequency of 5 GHz is travelling in free space in the positive z direction. Calculate the peak and time averaged power density that passes through $z = 0$ and the total power that passes through an aperture of dimensions 50 cm x 50 cm. [5]

Strategy:

1. Recognize that the instantaneous power density is given by the Poynting vector,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}\ .$$

2. Recognize that the time-averaged power density in a perfect dielectric is given by

$$\langle S \rangle = \frac{1}{2}\mathrm{Re}(\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2\eta} E_0^2\ .$$

3. Recognize that the total power (in W) that passes through a given area is given by the product of the power density (in W/m^2) and the area (in m^2) where it is assumed that the power density is uniform, as it always is for a uniform plane wave.

Accordingly, S = 0.265 W/m2, $\langle S \rangle$ = 0.1325 W/m2, and P = 0.331 W.

3. Plane Wave Reflection and Dispersion

a. What boundary conditions must be satisfied when a plane wave is incident upon the interface between two different material media? [5]

Strategy:

1. Recall that at the interface between two different material media, the following fields are present: 1) the sum of the incident and reflected electric field and 2) the sum of the incident and reflected magnetic field on one side of the boundary, and 3) the transmitted electric field and 4) the transmitted magnetic field on the other side.

2. Recall that at the boundary, we can use Stokes' theorem to prove that *the tangential components of the electric and magnetic field strength must each be continuous across the boundary*.

3. Recall that at the boundary, we can use the divergence theorem to prove that *the normal components of the electric and magnetic flux density must each be continuous across the boundary*.

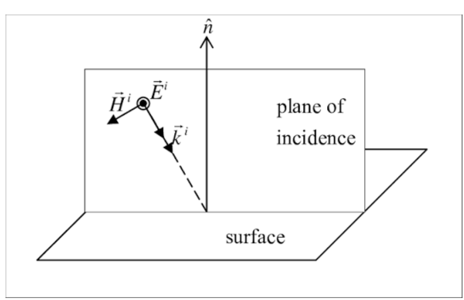
Bonus

4. Recall that for normal incidence, there are no components of the electric or magnetic field that are normal to the boundary so only the tangential components of the electric and magnetic field strength must each be continuous across the boundary.

5. Recall that for oblique incidence, there are components of the electric or magnetic field that are normal to the boundary so we must account for the fact the normal components of the electric and magnetic flux density must each be continuous across the boundary as well.

b. What is meant by TM polarization? How is it different from TE polarization? Use a sketch to explain your answer. [5]

Strategy: Recall that TM = Transverse Magnetic, TE = Transverse Electric, and Transverse implies transverse or perpendicular to the *plane of incidence*. In the sketch below, **E** is perpendicular to the plane of incidence so the wave is TE polarized. If **H** was perpendicular to the plane, the wave would be TM polarized.



c. Design a surface that will not reflect TM polarized waves when the angle of incidence is 81 degrees. Explain your strategy and reasoning for each calculation. Use sketches where required. What name do we give to this angle of incidence? [5]

Strategy + *Solution*:

1. Recall that

$$\Gamma_{TM} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

   and for TM-polarized wave and a given ratio $\sqrt{\eta_2/\eta_1} = \sqrt{\epsilon_1/\epsilon_2}$, there exists an angle $\theta_i = \theta_B$ (Brewster's angle) such that $\Gamma_{TM} = 0$. No such angle exists for Γ_{TE}.

2. Recall that Brewster's angle $\theta_B = \tan^{-1}\sqrt{\epsilon_{r2}/\epsilon_{r1}}$.

3. Solve for ϵ_{r2} given that region 1 is free space with permittivity $\epsilon_{r1} = 1$. (We're not explicitly told this but it's a reasonable assumption.) A flat surface with permittivity ϵ_{r2} will display the desired property.

Numerical Solutions:

$\epsilon_{r2} = \tan^2 81 = 39.86.$

This will be a very dense material, likely a ceramic.

d. What will the reflection coefficient be for a TE polarized wave incident at the same angle? [5]

Strategy + *Solution*:

1. Recall that

$$\Gamma_{TE} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}.$$

b. Recall that Snell's law for transmission (or refraction) is given by

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{v_1}{v_2}$$

   where region 1 is the region of incidence and reflection and region 2 is the region of transmission and v is the velocity of the wave in each region.

c. We are given η_1, θ_i. Calculate η_2, θ_t, and finally, Γ_{TE} = -0.9511 .

4. Plane Wave Reflection and Dispersion

a. Derive the boundary conditions for the normal and tangential components of static electric and magnetic fields at the boundaries between two media. [5]

Strategy:

1. Recall from Chapter 9 that at the boundary, we can use Stokes' theorem to prove that the tangential components of the electric and magnetic field strength must each be continuous across the boundary.

2. Recall from Chapter 9 that at the boundary, we can use the divergence theorem to prove that the normal components of the electric and magnetic flux density must each be continuous across the boundary.

b. A plane wave with electric field strength of 10 V/m and frequency of 5 GHz is travelling in the positive z direction. At z = 0, it is incident upon a perfect dielectric with relative permittivity = 10 and relative permeability = 1. Derive expressions for and determine the strength of the transmitted and reflected waves. [5]

Strategy + Solution:

1. Recall that the intrinsic impedance of a perfect dielectric, including free space, is given by

$$\eta = \sqrt{\mu/\epsilon} = 377/\sqrt{\epsilon_r} \ .$$

Here, the intrinsic impedance of free space is 377 Ω while the intrinsic impedance of the perfect dielectric is 119.2 Ω.

2. Since we aren't given the angle of incidence, it is reasonable to assume that the wave is normally incident. Recall that the voltage reflection and transmission coefficients are the result of applying the following conditions:

$$E_i + E_r = E_t$$

$$H_i + H_r = H_t$$

These equalities reflect the boundary conditions at the interface between regions 1 and 2.

$$\frac{E_i}{H_i} = \eta_1$$

$$\frac{E_r}{H_r} = \eta_1$$

$$\frac{E_t}{H_t} = \eta_2$$

These equalities reflect the material properties of the media in regions 1 and 2.

yielding the following results

$$\Gamma = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -0.520$$

These are the voltage reflection and transmission coefficients, respectively.

$$\tau = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_2 + \eta_1} = 0.481$$

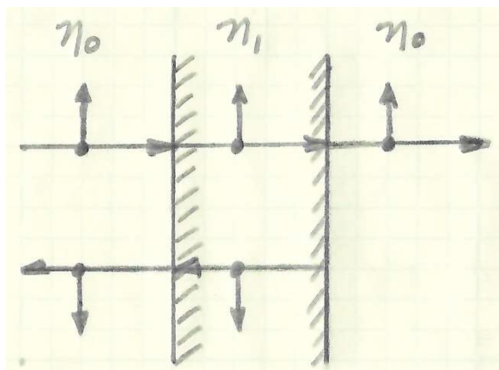
c. A plane wave with electric field strength of 30 V/m and frequency of 10 GHz is travelling in the positive x direction. At $x = 0$, it is incident upon a perfect dielectric slab, relative permittivity = 10 and relative permeability = 1. Derive expressions for and determine the strength and power density of the transmitted and reflected waves when:

1. the thickness of the slab is 0.75 cm. [2.5]

2. the thickness of the slab is 0.237 cm. [2.5]

Strategy + *Solution*:

1. Recognize that this is the problem geometry:



2. Recognize that at the second boundary,

$$\Gamma_2 = \frac{E_r}{E_i} = \frac{\eta_0 - \eta_1}{\eta_0 + \eta_1}$$

3. Recognize that at the first boundary,

$$\Gamma_1 = \frac{E_r}{E_i} = \frac{\eta_w - \eta_0}{\eta_w + \eta_0}$$

where η_w is the wave impedance in region 2,

$$\eta_w(z) = \frac{E_i(z) + E_r(z)}{H_i(z) + H_r(z)}$$

with appropriate care given to ensure that the correct sign is used for magnetic field strength. After some manipulation, this becomes

$$\eta_w(z) = \eta_1 \frac{\eta_0 - j\eta_1 \tan(\beta z)}{\eta_1 - j\eta_0 \tan(\beta z)}$$

4. Recognize that for lossless media, the power reflected back into the first region is given by $P_r = P_i\,|\Gamma_1|^2$ while the power transmitted into the third region is given by $P_t = P_i\,(1 - |\Gamma_1|^2)$. This is a consequence of conservation of energy and power.

   For the first slab, the incident, reflected, and transmitted powers are 1.194, 0.364, and 0.829 W/m^2, respectively.

   For the second slab, the incident, reflected, and transmitted powers are 1.194, 0.799, and 0.395 W/m^2, respectively.

d. A circularly polarized wave is obliquely incident upon a pure dielectric with relative permittivity of 4. If the angle of incidence is 63.43 degrees, what is the angle of refraction? What is the electric and magnetic strength and polarization of the reflected wave? [5]

   Strategy:

   1. Recall that a circularly polarized wave can be regarded as the sum of TM and TE polarized plane waves of equal amplitude but a 90-degree phase shift.

   2. Recall that the intrinsic impedance of a perfect dielectric with permeability μ and permittivity ϵ, including free space, is given by

   $$\eta = \sqrt{\mu/\epsilon} = 377/\sqrt{\epsilon_r}.$$

   Here, the intrinsic impedance of free space is 377 Ω while the intrinsic impedance of the pure (perfect) dielectric is 188.5 Ω.

   3. Recall that Snell's law for transmission (or refraction) is given by

   $$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{v_1}{v_2}$$

   where region 1 is the region of incidence and reflection and region 2 is the region of transmission and v is the velocity of the wave in each region. This allows us to calculate θ_t. Here, the angle of refraction $\theta_t = 26.56$ degrees.

   4. Recall that the voltage reflection coefficient Γ for both TE and TM polarization is a function of the intrinsic impedances of both the incident and transmission medium, and the angle of incidence, *e.g.*,

   $$\Gamma_{TM} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}.$$

5. Recognize that for TM-polarized waves only, there exists Brewster's angle, an angle $\theta_B = \tan^{-1}\sqrt{\epsilon_2/\epsilon_1}$, for which $\Gamma = 0$. Here, $63.43^o = \tan^{-1}\sqrt{4}$

6. Recognize that for this case, the TM-polarized wave is completely transmitted so the reflected component has only a TE-polarized component. To find the amplitude and phase of the reflected wave relative to the TE-polarized component of the incident wave, apply the expression

$$\Gamma_{TE} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

to yield Γ_{TE}= -0.60.