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UBC

ELEC 341: Systems and Control

Lecture 2

Laplace transform

Course roadmap

Modeling	*Analysis*	*Design*
Laplace transform Transfer function Models for systems • Electrical • Electromechanical • Mechanical Linearization, delay	Stability • Routh-Hurwitz • Nyquist Time response • Transient • Steady state Frequency response • Bode plot	Design specs Root locus Frequency domain PID & Lead-lag Design examples

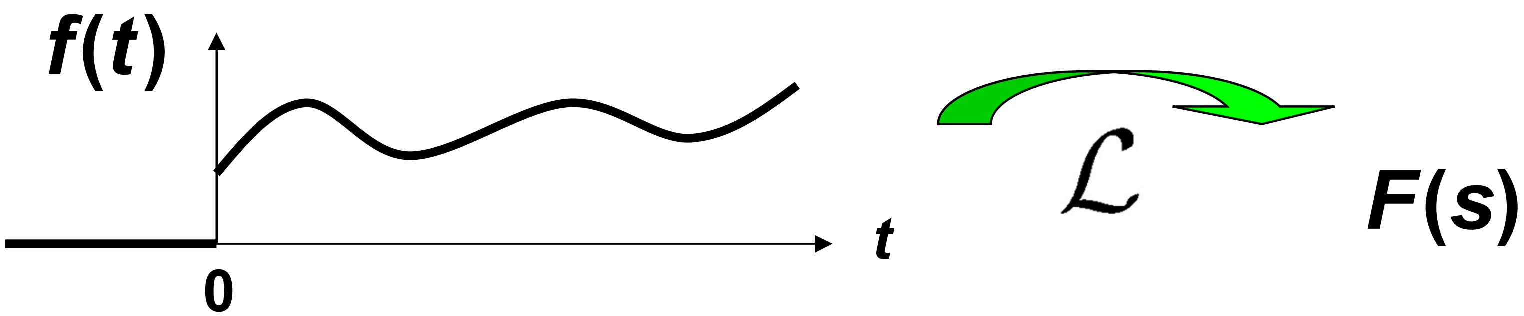
Matlab simulations

Laplace transform

- One of the most important math tools in the course!
- Definition: For a function $f(t)$ ($f(t) = 0$ for $t < 0$),

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

(*s*: complex variable)



- We denote Laplace transform of $f(t)$ by $F(s)$.

Advantages of *s*-domain

- We can transform an ordinary differential equation into an algebraic equation which is easy to solve.

  (Next lecture)

- It makes it easier to analyze and design interconnected (series, feedback, etc.) systems.

  (Throughout the course)

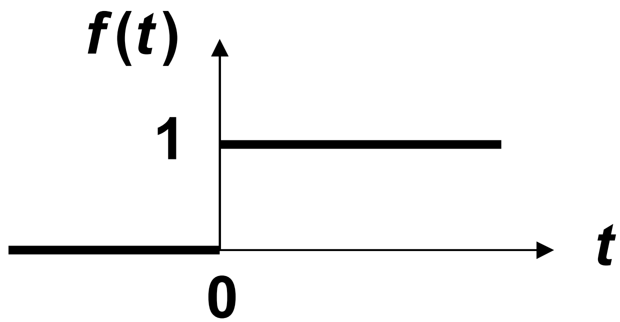
- Frequency domain information of signals can be dealt with.

  (Lectures for frequency responses)

Examples of Laplace transform

- Unit step function

$$f(t) = u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

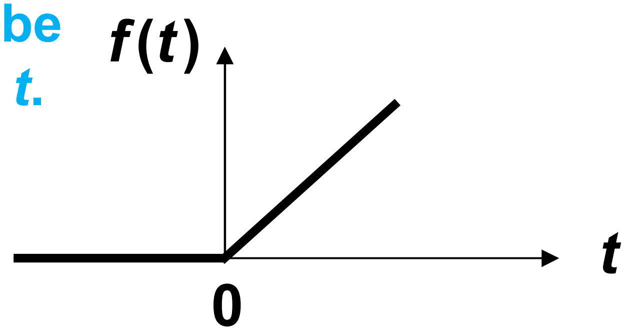


$$\Rightarrow \quad F(s) = \int_0^\infty 1 \cdot e^{-st} dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty = \boxed{\frac{1}{s}}$$

- Unit ramp function

Enforcing $f(t)$ to be zero for negative t.

$$f(t) = tu(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\Rightarrow \quad F(s) = \int_0^\infty t e^{-st} dt = -\frac{1}{s}\left[te^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt = \boxed{\frac{1}{s^2}}$$

(Integration by parts: see formula below)

$$\int u\, dv = uv - \int v\, du$$

Integration by parts

- Formula

$$\int f'(t)g(t)dt = f(t)g(t) - \int f(t)g'(t)dt$$

Why?

$$[f(t)g(t)]' = f'(t)g(t) + f(t)g'(t)$$

$$\Rightarrow \int [f(t)g(t)]'\,dt = \int \left[f'(t)g(t) + f(t)g'(t)\right] dt$$

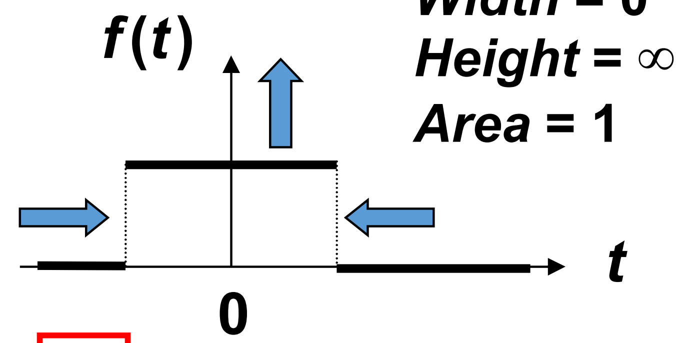
$$\Rightarrow f(t)g(t) = \int f'(t)g(t)dt + \int f(t)g'(t)dt \Rightarrow$$

$$\int f'(t)g(t)dt = f(t)g(t) - \int f(t)g'(t)dt$$

Ex. of Laplace transform (cont'd)

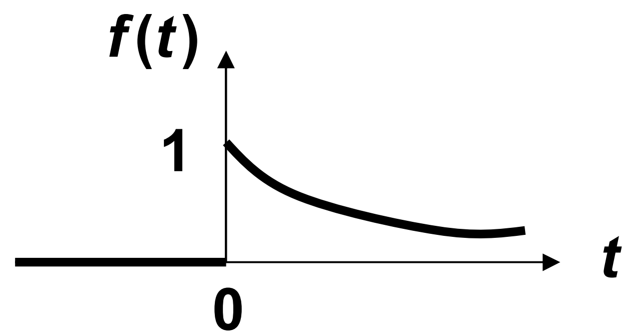
- Unit impulse function $f(t) = \delta(t)$

$$\int_{-\infty}^{\infty} \delta(t) g(t) dt = g(0)$$



$$\Rightarrow \quad F(s) = \int_0^{\infty} \delta(t) e^{-st} dt = e^{-s \cdot 0} = \boxed{1}$$

- Exponential function



$$f(t) = e^{-\alpha t} u(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\Rightarrow \quad F(s) = \int_0^{\infty} e^{-\alpha t} . e^{-st} dt = -\frac{1}{s+\alpha} \left[e^{-(s+\alpha)t} \right]_0^{\infty} = \boxed{\frac{1}{s+\alpha}}$$

Ex. of Laplace transform (cont'd)

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- Sine function

$$\mathcal{L}\{\sin \omega t \cdot u(t)\} = \frac{\omega}{s^2 + \omega^2}$$

- Cosine function

$$\mathcal{L}\{\cos \omega t \cdot u(t)\} = \frac{s}{s^2 + \omega^2}$$

Remark: Instead of computing Laplace transform for each function, you can use the **Laplace transform table**.

Laplace transform table

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin \omega t \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t \cdot u(t)$	$\frac{s}{s^2+\omega^2}$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$
$t^k f(t)$	$(-1)^k \frac{d^k F(s)}{ds^k}$

$\mathcal{L}$ →

← $\mathcal{L}^{-1}$

Inverse Laplace Transform

(u(t) is often omitted.)

Properties of Laplace transform
1. Linearity

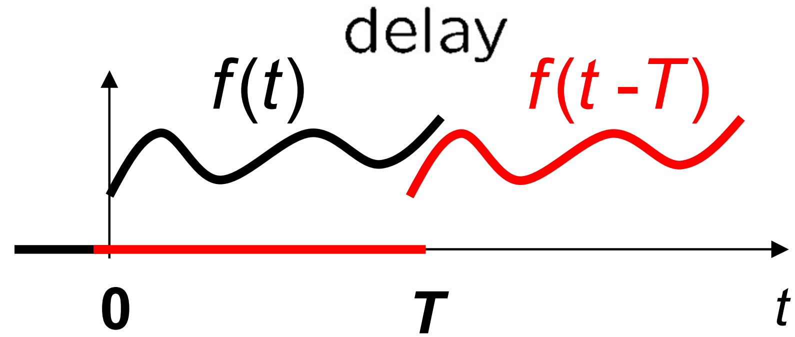
$$\mathcal{L}\left\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\right\} = \alpha_1 F_1(s) + \alpha_2 F_2(s)$$

Proof.
$$\begin{aligned}\mathcal{L}\left\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\right\} &= \int_0^\infty (\alpha_1 f_1(t) + \alpha_2 f_2(t)) e^{-st} dt \\ &= \alpha_1 \underbrace{\int_0^\infty f_1(t) e^{-st} dt}_{F_1(s)} + \alpha_2 \underbrace{\int_0^\infty f_2(t) e^{-st} dt}_{F_2(s)}\end{aligned}$$

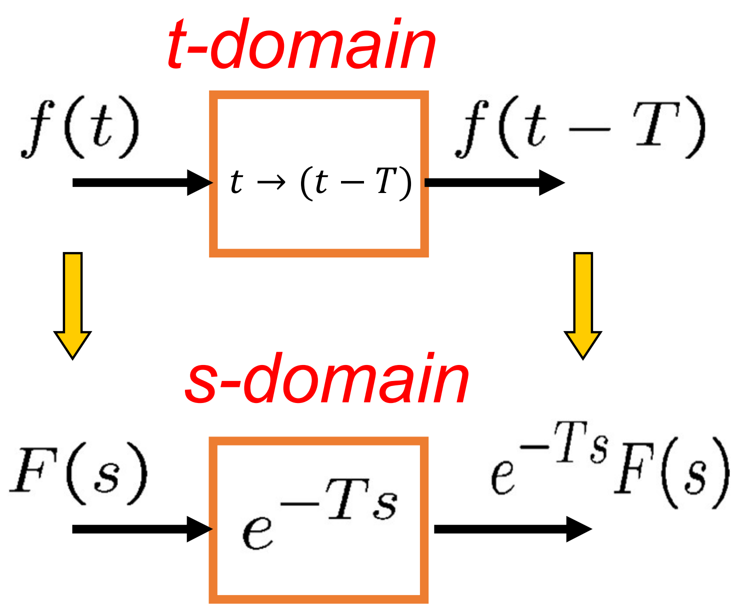
Ex.
$$\mathcal{L}\left\{5u(t) + 3e^{-2t}\right\} = 5\mathcal{L}\left\{u(t)\right\} + 3\mathcal{L}\left\{e^{-2t}\right\} = \frac{5}{s} + \frac{3}{s+2}$$

Properties of Laplace transform
2. Time delay

$$\mathcal{L}\{f(t-T)u(t-T)\} = e^{-Ts}F(s)$$



delay means: $t \rightarrow (t-T)$



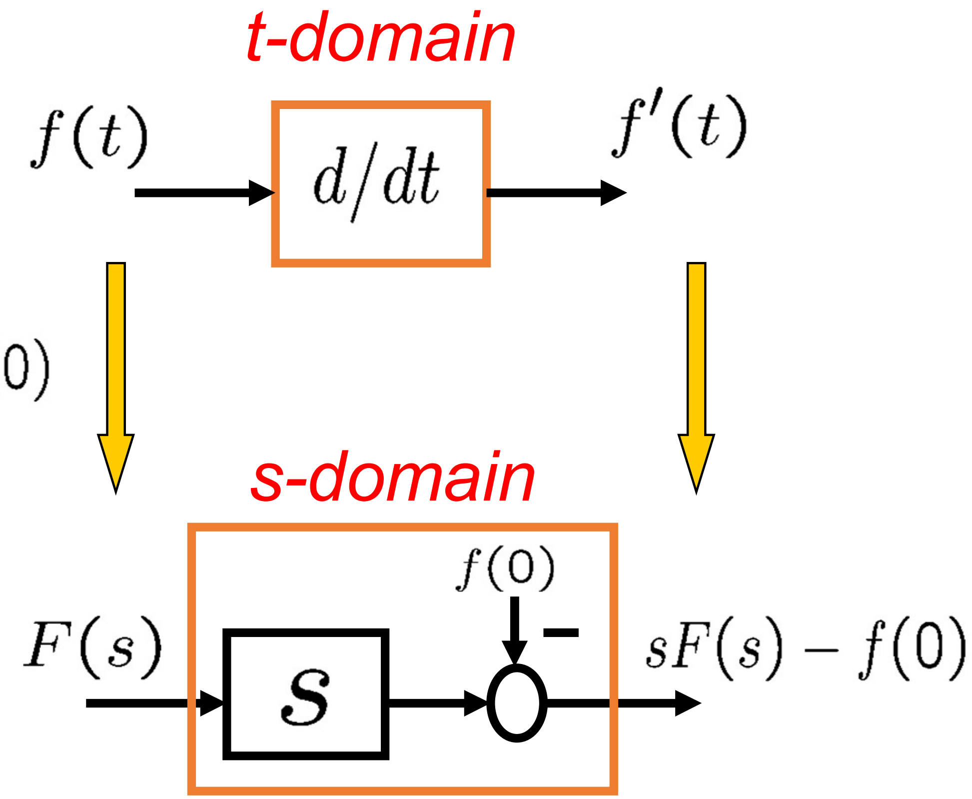
Proof.

$$\mathcal{L}\{f(t-T)u(t-T)\}$$
$$= \int_T^{\infty} f(t-T)e^{-st}dt$$
$$= \int_0^{\infty} f(\tau)e^{-s(T+\tau)}d\tau = e^{-Ts}F(s)$$

Ex. $\mathcal{L}\left\{e^{-0.5(t-4)}u(t-4)\right\} = \dfrac{e^{-4s}}{s+0.5}$

Properties of Laplace transform
3. Differentiation

$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$$



Proof.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt$$
$$= \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}dt = sF(s) - f(0)$$

Ex.

$$\mathcal{L}\{(\cos 2t)'\} = s\mathcal{L}\{\cos 2t\} - 1$$
$$= \frac{s^2}{s^2+4} - 1 = \frac{-4}{s^2+4}$$
$$(= \mathcal{L}\{-2\sin 2t\})$$

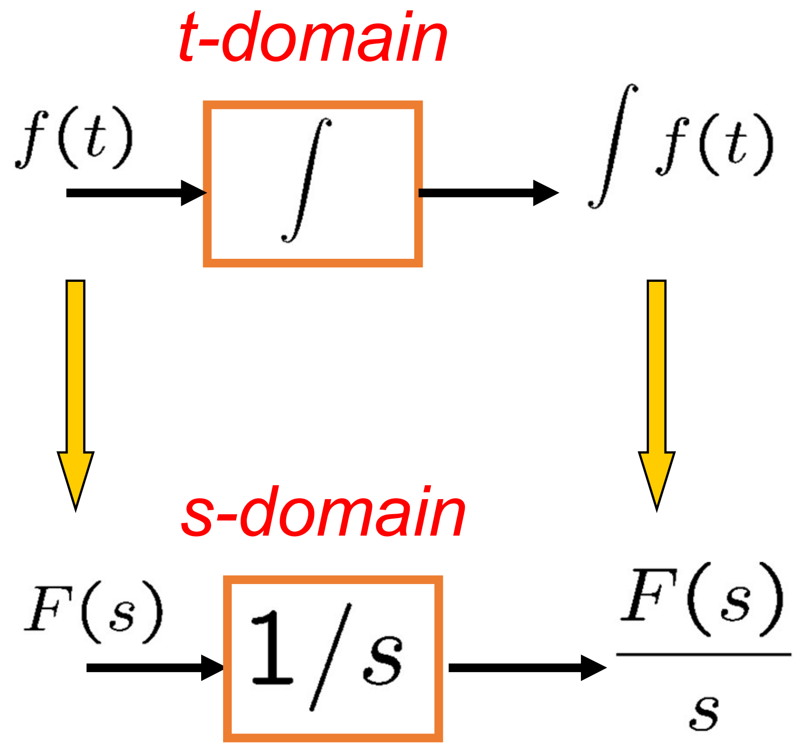
Properties of Laplace transform
4. Integration

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

Proof.

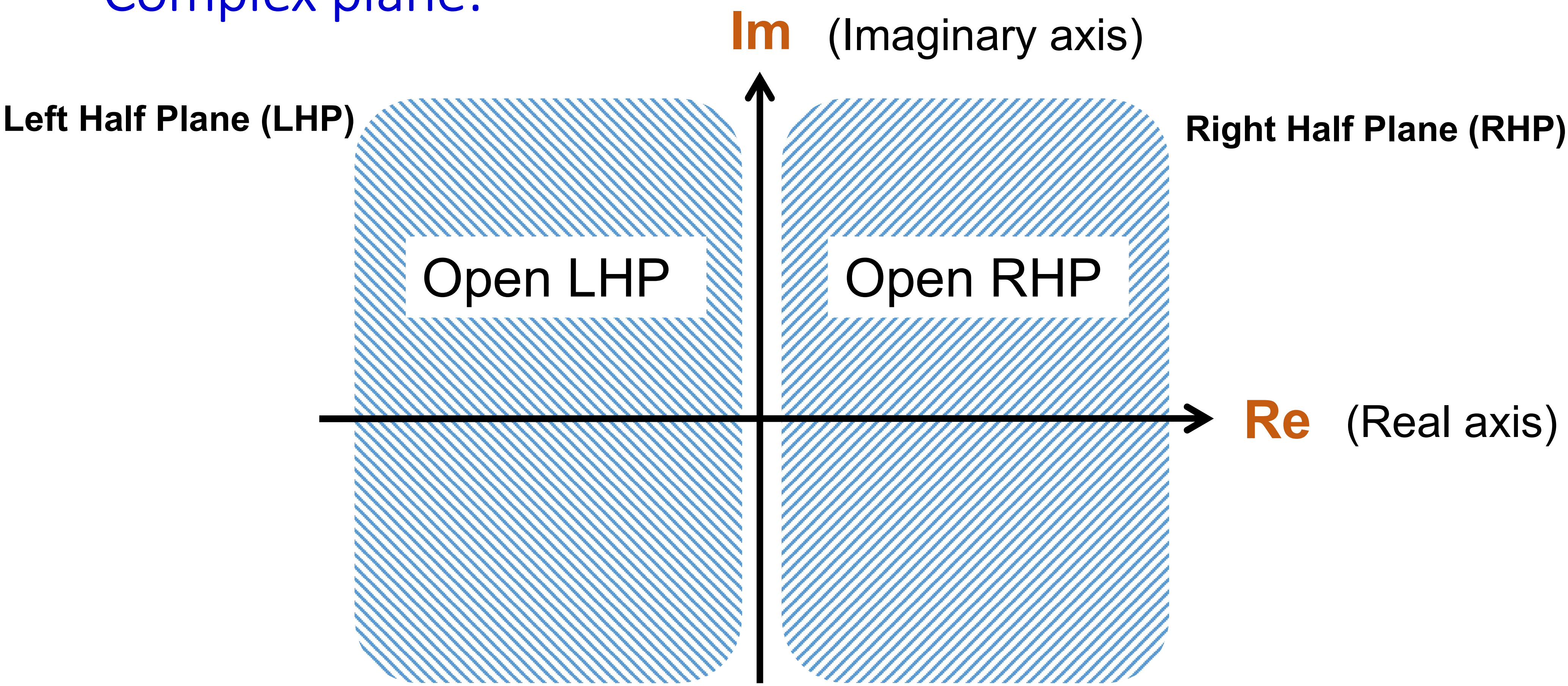
$$\begin{aligned}\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] &= \int_0^\infty \left(\int_0^t f(\tau)d\tau\right) e^{-st}dt \\ &= -\frac{1}{s}\left[\left(\int_0^t f(\tau)d\tau\right) e^{-st}\right]_0^\infty \\ &\quad +\frac{1}{s}\int_0^\infty f(t)e^{-st}dt \\ &= \frac{F(s)}{s}\end{aligned}$$

Ex. $\mathcal{L}\left\{\int_0^t u(\tau)d\tau\right\} = \frac{\mathcal{L}\{u(t)\}}{s} = \frac{1}{s^2}$



Properties of Laplace transform
5. Final value theorem (FVT)

Complex plane:



- ***"Open"*** *means that it* ***does not*** *include imaginary axis.*
- ***"Closed"*** *means that it* ***does*** *include imaginary axis.*

Properties of Laplace transform
5. Final value theorem (FVT)

If **all** the poles of ***s.F*(*s*)** are in open left half plane (LHP), with possibly one simple pole at the origin, then we have:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

The term “with possibly one simple pole at the origin” means that even if ***s.F*(*s*)** has an ***s*** in its denominator, you can still use the FVT.

Ex. $F(s) = \dfrac{5}{s(s^2+s+2)}$ ⇒ $\displaystyle\lim_{t\to\infty} f(t) = \lim_{s\to 0} \frac{5}{s^2+s+2} = \frac{5}{2}$

Poles of *sF*(*s*) are in open LHP, so final value theorem applies.

(*poles = roots of the denominator*)

Ex. $F(s) = \dfrac{4}{s^2+4}$ ⇒ $\displaystyle\lim_{t\to\infty} f(t) \neq \lim_{s\to 0} \frac{4s}{s^2+4} = 0$

Since the poles of *sF*(*s*) are not in open LHP (i.e., they are on imaginary axis), final value theorem **does NOT** apply.

Properties of Laplace transform
6. Initial value theorem (IVT)

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) \quad \text{if the limits exist.}$$

Remark: In this theorem, it does not matter if pole location of *sF*(*s*) is in LHP or not.

Ex. $F(s) = \dfrac{5}{s(s^2+s+2)}$ ⇒ $\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = 0$

Ex. $F(s) = \dfrac{4}{s^2+4}$ ⇒ $\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = 0$

Properties of Laplace transform
7. Convolution

$F_1(s) = \mathcal{L}\{f_1(t)\}$
$F_2(s) = \mathcal{L}\{f_2(t)\}$

$\mathcal{L}\left\{\int_0^t f_1(\tau) f_2(t-\tau) d\tau\right\}$ or $\mathcal{L}\left\{\int_0^t f_1(t-\tau) f_2(\tau) d\tau\right\}$ = ?

$$\mathcal{L}\left\{\int_0^t f_1(\tau) f_2(t-\tau) d\tau\right\} = \mathcal{L}\left\{\int_0^t f_1(t-\tau) f_2(\tau) d\tau\right\} = F_1(s)F_2(s)$$

The above is called **convolution theorem**.

IMPORTANT REMARK

$$\mathcal{L}\{f_1(t) f_2(t)\} \neq F_1(s)F_2(s)$$

Properties of Laplace transform
8. Frequency shift theorem

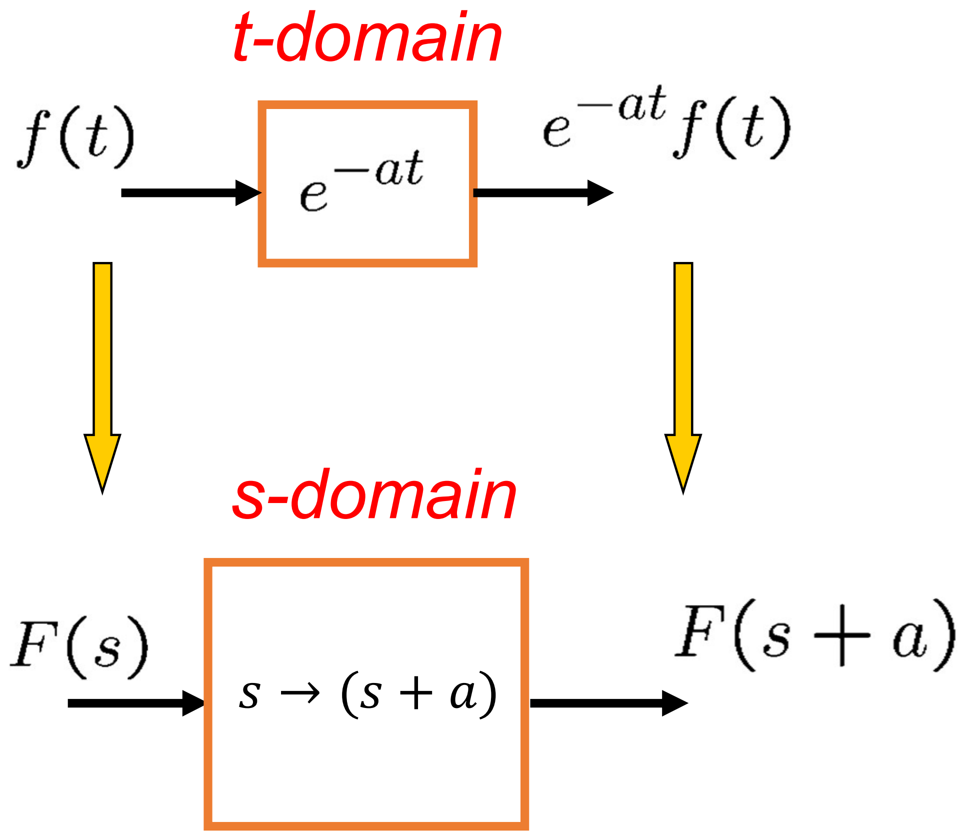
$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a)$$

Proof.

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = \int_0^\infty e^{-at}f(t)e^{-st}dt$$
$$= \int_0^\infty f(t)e^{-(s+a)t}dt = F(s+a)$$

Ex.

$$\mathcal{L}\left\{te^{-2t}\right\} = \frac{1}{(s+2)^2}$$



Example 1

$$\mathcal{L}\{\delta(t-2T)\} = ?$$

$$f(t) = \delta(t)$$

$$\begin{cases} \mathcal{L}\{\delta(t)\} = \mathcal{L}\{f(t)\} = F(s) = 1 \\ \mathcal{L}\{f(t-2T)\} = \mathcal{L}\{\delta(t-2T)\} = e^{-2Ts}F(s) = e^{-2Ts} \cdot 1 = e^{-2Ts} \end{cases}$$

$$\Rightarrow \quad \boxed{\mathcal{L}\{\delta(t-2T)\} = e^{-2Ts}}$$

Example 2

$$\mathcal{L}\left\{\sin 2t \cos 2t\right\} = ?$$

$$\begin{aligned}\mathcal{L}\left\{\sin 2t \cos 2t\right\} &= \mathcal{L}\left\{\frac{1}{2}\sin 4t\right\} \\ &= \frac{1}{2}\mathcal{L}\left\{\sin 4t\right\} \\ &= \boxed{\frac{1}{2}\cdot\frac{4}{s^2+4^2}}\end{aligned}$$

Euler's formula

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$\begin{cases} e^{j\theta} = \cos\theta + j\sin\theta \\ e^{-j\theta} = \cos\theta - j\sin\theta \end{cases}$$

$$\Rightarrow \begin{cases} \cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2} \\ \sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j} \end{cases}$$

Example 3

$$\mathcal{L}\{t \sin 2t\} = ?$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\begin{aligned}
\mathcal{L}\{t\sin 2t\} &= \mathcal{L}\left\{t \cdot \frac{e^{2jt} - e^{-2jt}}{2j}\right\} \\
&= \frac{1}{2j}\left\{\mathcal{L}\left\{te^{2jt}\right\} - \mathcal{L}\left\{te^{-2jt}\right\}\right\} \\
&= \frac{1}{2j}\left\{\frac{1}{(s-2j)^2} - \frac{1}{(s+2j)^2}\right\} \\
&= \frac{1}{2j} \cdot \frac{(s+2j)^2 - (s-2j)^2}{(s^2+4)^2} = \boxed{\frac{4s}{(s^2+4)^2}}
\end{aligned}$$

An alternative method is to use the following formula:

$$L\left\{t^k f(t)\right\} = (-1)^k \frac{d^k F(s)}{ds^k}$$

Summary

- Laplace transform (an important math tool!)
  - Definition
  - Laplace transform table
  - Properties of Laplace transform
- Next
  - Solution to ODEs via Laplace transform