a place of mind

ELEC 341: Systems and Control

Lecture 3

ODE solution via Laplace transform

© Siamak Najarian 2025

Course roadmap

Modeling	*Analysis*	*Design*
Laplace transform Transfer function Models for systems • Electrical • Electromechanical • Mechanical Linearization, delay	Stability • Routh-Hurwitz • Nyquist Time response • Transient • Steady state Frequency response • Bode plot	Design specs Root locus Frequency domain PID & Lead-lag Design examples

Matlab simulations

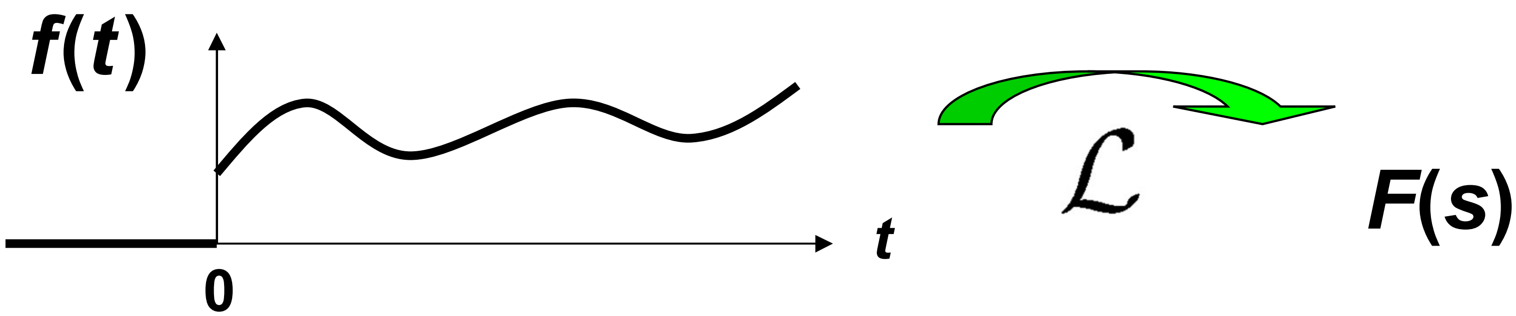
© Siamak Najarian 2025

Laplace transform (review)

- One of most important math tools in the course!
- Definition: For a function $f(t)$ ($f(t) = 0$ for $t < 0$),

$$\boxed{F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt}$$

(s: complex variable)



- We denote Laplace transform of $f(t)$ by $F(s)$.

Laplace transform table (review)

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin \omega t \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t \cdot u(t)$	$\frac{s}{s^2+\omega^2}$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$
$t^k f(t)$	$(-1)^k \frac{d^k F(s)}{ds^k}$

$\mathcal{L}$

$\mathcal{L}^{-1}$

Inverse Laplace Transform

(*u(t)* is often omitted.)

© Siamak Najarian 2025

Advantages of *s*-domain (review)

- We can transform an ordinary differential equation (ODE) into an algebraic equation which becomes easier to solve.

  (This lecture)

- It is easier to analyze and design interconnected (series, feedback etc.) systems.

  (Throughout the course)

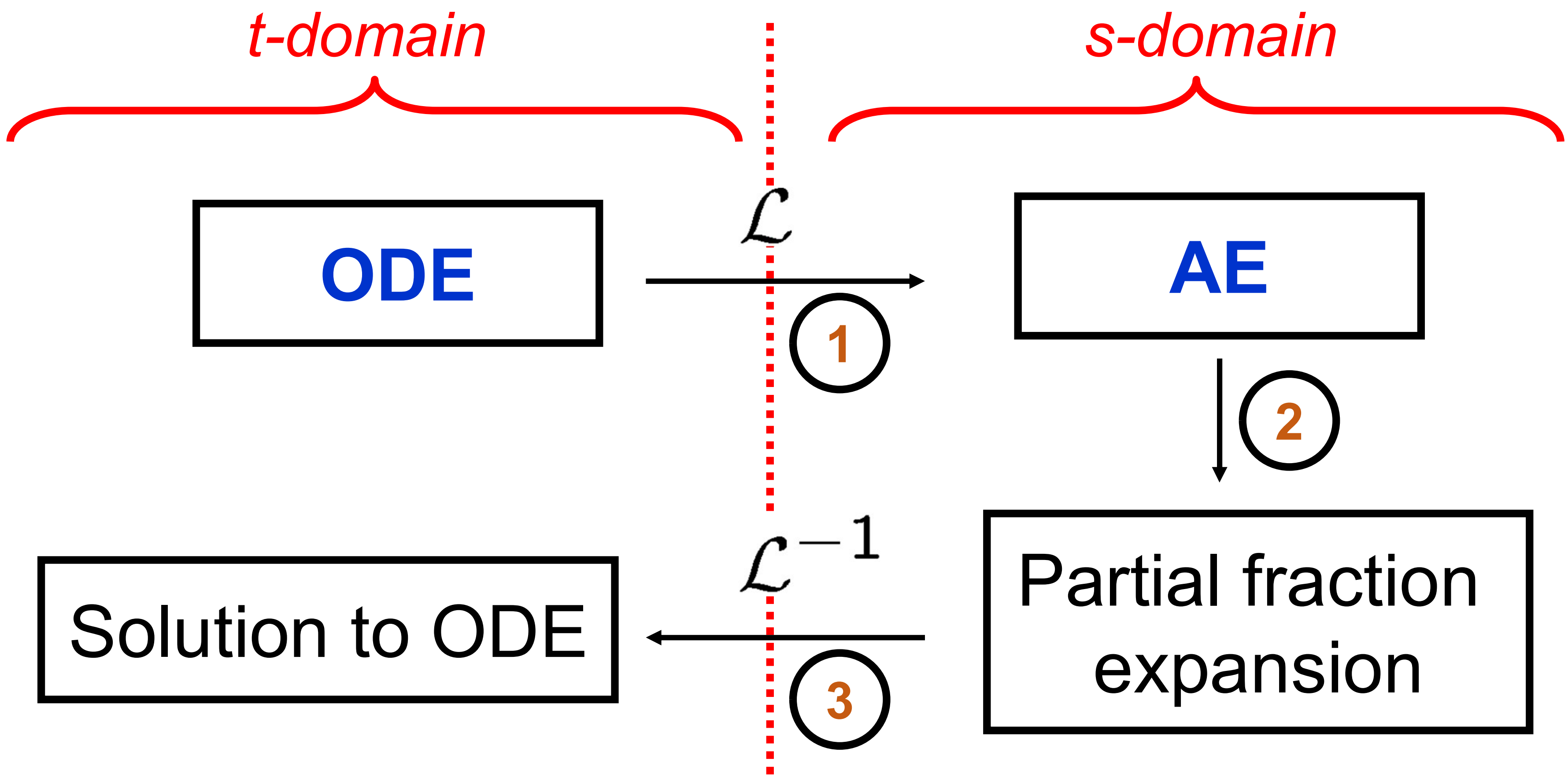
- Frequency domain information of signals can be dealt with.

  (Lectures for frequency responses)

© Siamak Najarian 2025

An advantage of Laplace transform

We can transform an ordinary differential equation (**ODE**) into an algebraic equation (**AE**).

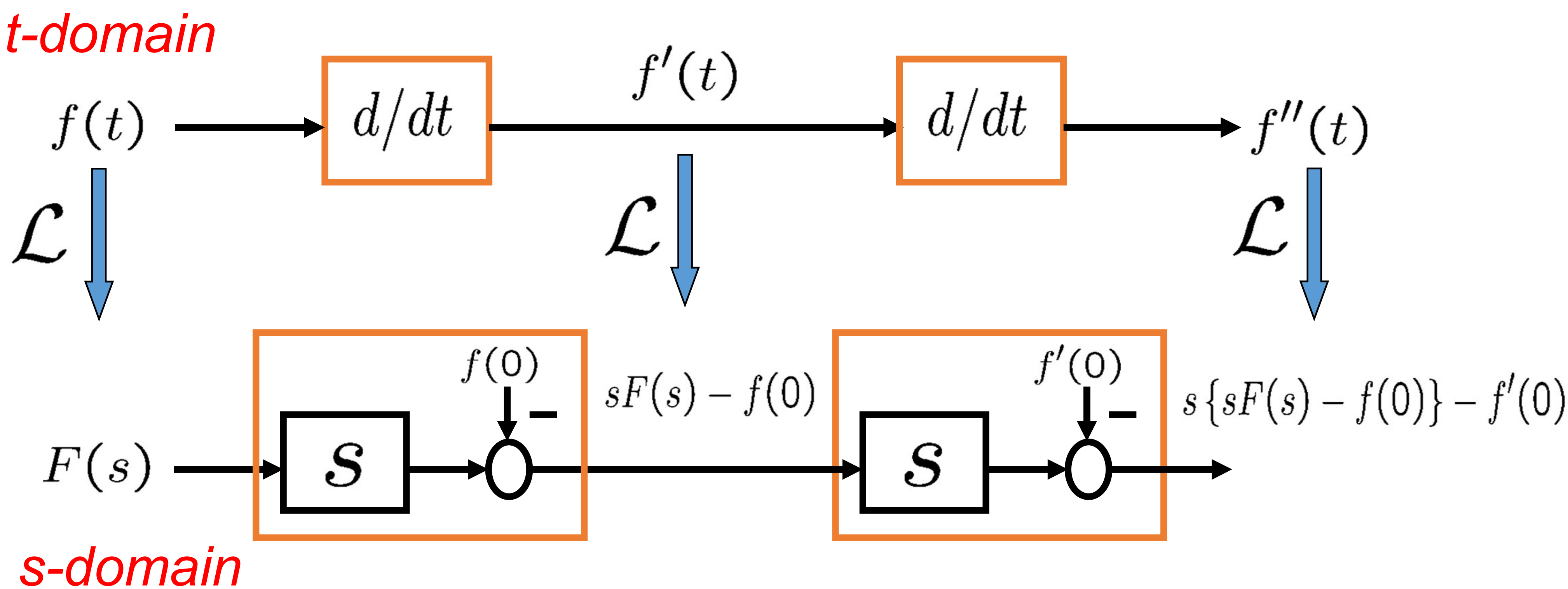


© Siamak Najarian 2025

Properties of Laplace transform
Differentiation (extended)

$$\boxed{\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)}$$

Let us extend this …



Or, in general: $\boxed{\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)}$

© Siamak Najarian 2025

Example 1 (distinct roots)

ODE with initial conditions (ICs):

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=5u(t),\ y(0)=-1,\ y'(0)=2$$

1. Laplace transform

$$\underbrace{s^2Y(s)-sy(0)-y'(0)}_{\mathcal{L}\left\{y''(t)\right\}}+3\underbrace{\left\{sY(s)-y(0)\right\}}_{\mathcal{L}\left\{y'(t)\right\}}+2Y(s)=\frac{5}{s}$$

$$\Rightarrow\quad Y(s)=\frac{-s^2-s+5}{s(s+1)(s+2)}$$

distinct roots

© Siamak Najarian 2025

Example 1 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Multiply both sides by $s(s+1)(s+2)$:

$$-s^2 - s + 5 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

Compare coefficients:

$$\begin{array}{lcl} s^2\text{-term} & : & -1 = A + B + C \\ s^1\text{-term} & : & -1 = 3A + 2B + C \\ s^0\text{-term} & : & 5 = 2A \end{array} \Rightarrow \begin{cases} A = \dfrac{5}{2} \\ B = -5 \\ C = \dfrac{3}{2} \end{cases}$$

© Siamak Najarian 2025

Example 1 (cont'd)

Note: We can also use a modified version of **Residue Method** for finding the coefficients.

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Find A:

Multiply both sides by the denominator of $\frac{A}{s}$, i.e., multiply by s:

$$\frac{-s^2 - s + 5}{(s+1)(s+2)} = A + \frac{B(s)}{s+1} + \frac{C(s)}{s+2}$$

Let $s = 0$, then $A = \frac{5}{2}$

Find B:

Multiply both sides by the denominator of $\frac{B}{s+1}$, i.e., $s+1$:

$$\frac{-s^2 - s + 5}{s(s+2)} = \frac{A(s+1)}{s} + B + \frac{C(s+1)}{s+2}$$

Let $s = -1$, then $B = -5$

© Siamak Najarian 2025

Example 1 (cont'd)

Find C:

Multiply both sides by the denominator of $\frac{C}{s+2}$, i.e., $s+2$:

$$\frac{-s^2-s+5}{s(s+1)} = \frac{A(s+2)}{s} + \frac{B(s+2)}{s+1} + C$$

Let $s=-2$, then $C=\frac{3}{2}$

$$\Rightarrow \quad \begin{cases} A = \dfrac{5}{2} \\ B = -5 \\ C = \dfrac{3}{2} \end{cases}$$

© Siamak Najarian 2025

Example 1 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

(*You may omit u(t).*)

$$\Rightarrow \boxed{y(t) = \left(\underbrace{\frac{5}{2}}_{A} + \underbrace{(-5)}_{B} e^{-t} + \underbrace{\frac{3}{2}}_{C} e^{-2t} \right) u(t)}$$

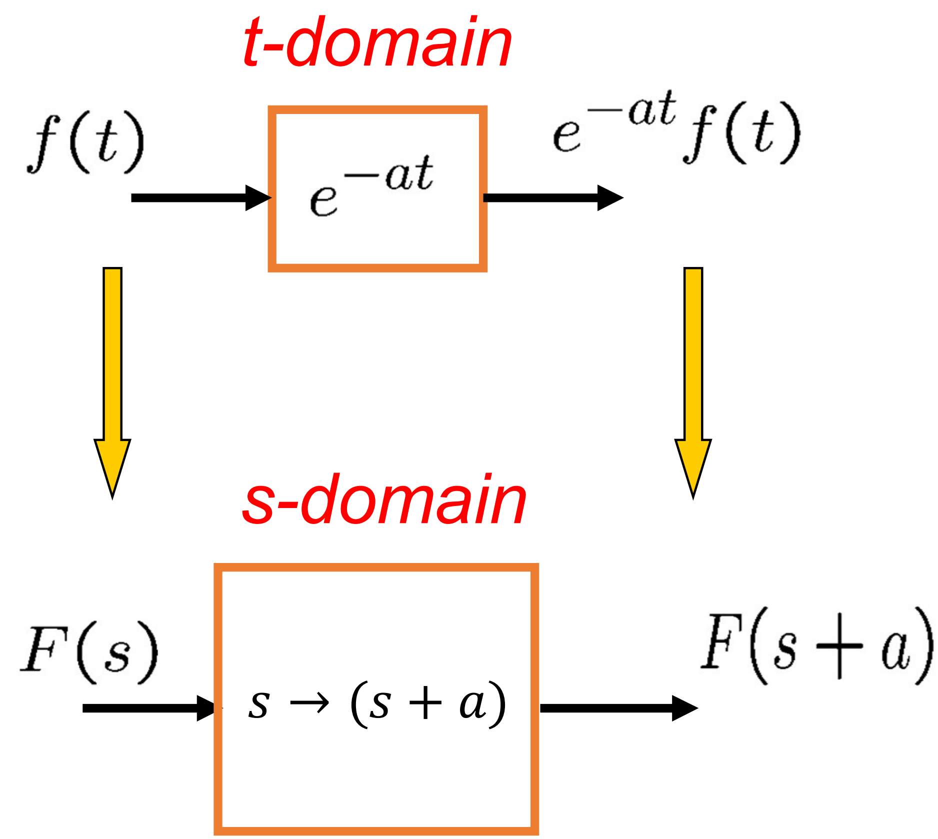
If we are interested in only the final value of $y(t)$, apply the Final Value Theorem, without explicitly computing $y(t)$:

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} \Rightarrow \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{-s^2 - s + 5}{(s+1)(s+2)} = \frac{5}{2}$$

© Siamak Najarian 2025

Properties of Laplace transform
Frequency shift theorem (review)

$$\boxed{\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a)}$$



Proof.

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = \int_0^\infty e^{-at}f(t)e^{-st}dt$$

$$= \int_0^\infty f(t)e^{-(s+a)t}dt = F(s+a)$$

Ex.

$$\mathcal{L}\left\{te^{-2t}\right\} = \frac{1}{(s+2)^2}$$

© Siamak Najarian 2025

Properties of Laplace transform
Frequency shift theorem

$f(t)$	$F(s)$
$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
$e^{-\alpha t} \sin \omega t$	$\dfrac{\omega}{(s+\alpha)^2 + \omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
$e^{-\alpha t} \cos \omega t$	$\dfrac{s+\alpha}{(s+\alpha)^2 + \omega^2}$

Frequency shift theorem

$$\boxed{\mathcal{L}\left\{e^{-at} f(t)\right\} = F(s+a)}$$

© Siamak Najarian 2025

Example 2 (repeated roots)

ODE with zero initial conditions (ICs):

$$\frac{d^3y(t)}{dt^3}+5\frac{d^2y(t)}{dt^2}+8\frac{dy(t)}{dt}+4y(t)=2\delta(t),\ y(0)=y'(0)=y''(0)=0$$

1. Laplace transform

$$\begin{aligned}&s^3Y(s)-s^2y(0)-sy'(0)-y''(0) && \longleftarrow \mathcal{L}\left\{y'''(t)\right\}\\ &+5\left\{s^2Y(s)-sy(0)-y'(0)\right\} && \longleftarrow 5\mathcal{L}\left\{y''(t)\right\}\\ &+8\left\{sY(s)-y(0)\right\}+4Y(s)\\ &=2\end{aligned}$$

$$\Rightarrow \quad Y(s)=\frac{2}{(s+1)(s+2)^2}$$

Repeated roots

© Siamak Najarian 2025

Example 2 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

Multiply both sides by $(s+1)(s+2)^2$

$$2 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

Compare coefficients:

$$\begin{aligned} s^2\text{-term} &: \quad 0 = A + B \\ s^1\text{-term} &: \quad 0 = 4A + 3B + C \\ s^0\text{-term} &: \quad 2 = 4A + 2B + C \end{aligned} \quad \Rightarrow \quad \begin{cases} A = 2 \\ B = -2 \\ C = -2 \end{cases}$$

© Siamak Najarian 2025

Example 2 (cont'd)

Note: We can also use a modified version of **Residue Method** for finding the coefficients.

$$\frac{2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

- **For A:** Multiply both sides by $(s+1)$:

$$\frac{2}{(s+2)^2} = A + \frac{B(s+1)}{s+2} + \frac{C(s+1)}{(s+2)^2}$$

Let $s=-1$, then $A=2$

- **For C:** Multiply both sides by $(s+2)^2$:

$$\frac{2}{s+1} = \frac{A(s+2)^2}{s+1} + B(s+2) + C$$

Let $s=-2$, then $C=-2$

© Siamak Najarian 2025

Example 2 (cont'd)

- **For B:** Multiply both sides by $(s+2)$:

$$\frac{2}{(s+1)(s+2)} = \frac{A(s+2)}{s+1} + B + \frac{C}{s+2}$$

Let $s=-3$ (an arbitrary number other than $s=-1$ or $s=-2$), then $B=-2$

$$\Rightarrow \begin{cases} A = 2 \\ B = -2 \\ C = -2 \end{cases}$$

© Siamak Najarian 2025

Example 2 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

($u(t)$ omitted.)

$$\Rightarrow \boxed{y(t) = \underbrace{2}_{A} e^{-t} + \underbrace{(-2)}_{B} e^{-2t} + \underbrace{(-2)}_{C} te^{-2t}}$$

If we are interested in only the final value of $y(t)$, apply the Final Value Theorem, without explicitly computing $y(t)$:

$$Y(s) = \frac{2}{(s+1)(s+2)^2} \Rightarrow \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{2s}{(s+1)(s+2)^2} = 0$$

© Siamak Najarian 2025

Example 3 (complex roots)

ODE with zero initial conditions (ICs):

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+5y(t)=3u(t),\ \ y(0)=0,\ \ y'(0)=0$$

1. Laplace transform

$$s^2Y(s)+2sY(s)+5Y(s)=\frac{3}{s}$$

$$\Rightarrow\ Y(s)=\frac{3}{s(s^2+2s+5)}$$

Complex roots

Example 3 (complex roots)

a place of mind UBC

A Note on Partial Fraction Decomposition:

If the denominator is a polynomial of order 2 or more, the partial fraction numerator will be in the form of

- $(a_0 s + a_1)$, for degree 2
- $(a_0 s^2 + a_1 s + a_2)$, for degree 3, etc.

This is only true if the denominator polynomial does **not** have any **repetitive roots**.

Examples:

$$\frac{3}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

$$\frac{3}{s(s^3+5s^2+6s+3)} = \frac{A}{s} + \frac{Bs^2+Cs+D}{s^3+5s^2+6s+3}$$

Note: The degree of the numerator is always **one less** than the degree of the denominator polynomial in each term.

© Siamak Najarian 2025

Example 3 (cont'd)

unknowns

2. Partial fraction expansion

$$Y(s) = \frac{3}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

Multiply both sides by $s(s^2+2s+5)$

$$3 = A(s^2+2s+5) + s(Bs+C)$$

Compare coefficients:

$$\begin{aligned} s^2\text{-term} &: \quad 0 = A + B \\ s^1\text{-term} &: \quad 0 = 2A + C \\ s^0\text{-term} &: \quad 3 = 5A \end{aligned} \quad \Rightarrow \quad \begin{cases} A = \dfrac{3}{5} \\ B = -\dfrac{3}{5} \\ C = -\dfrac{6}{5} \end{cases}$$

© Siamak Najarian 2025

Example 3 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s} + \boxed{\frac{Bs+C}{s^2+2s+5}}$$

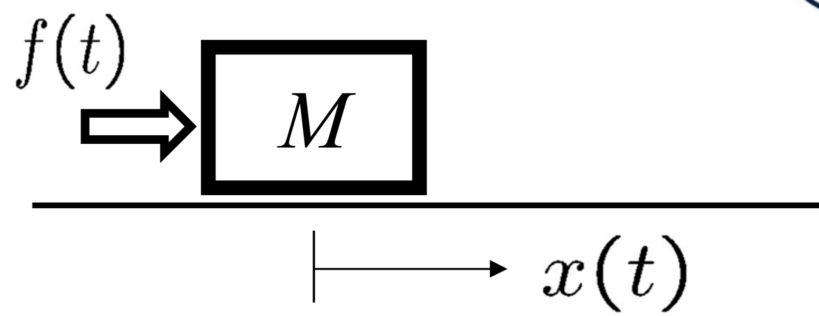
$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{Bs+C}{s^2+2s+5}\right\} &= \mathcal{L}^{-1}\left\{\frac{B(s+1)+C-B}{(s+1)^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{B(s+1)}{(s+1)^2+4} + \frac{C-B}{(s+1)^2+4}\right\} \\ &= B\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} + \frac{C-B}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} \\ &= Be^{-t}\cos 2t + \frac{C-B}{2}e^{-t}\sin 2t\end{aligned}$$

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)}$$

© Siamak Najarian 2025

Example 4: Newton's law

$$M\frac{d^2x(t)}{dt^2} = f(t)$$



Want to know position $x(t)$ when force $f(t)$ is applied.

$$M\left(s^2X(s) - sx(0) - x'(0)\right) = F(s)$$

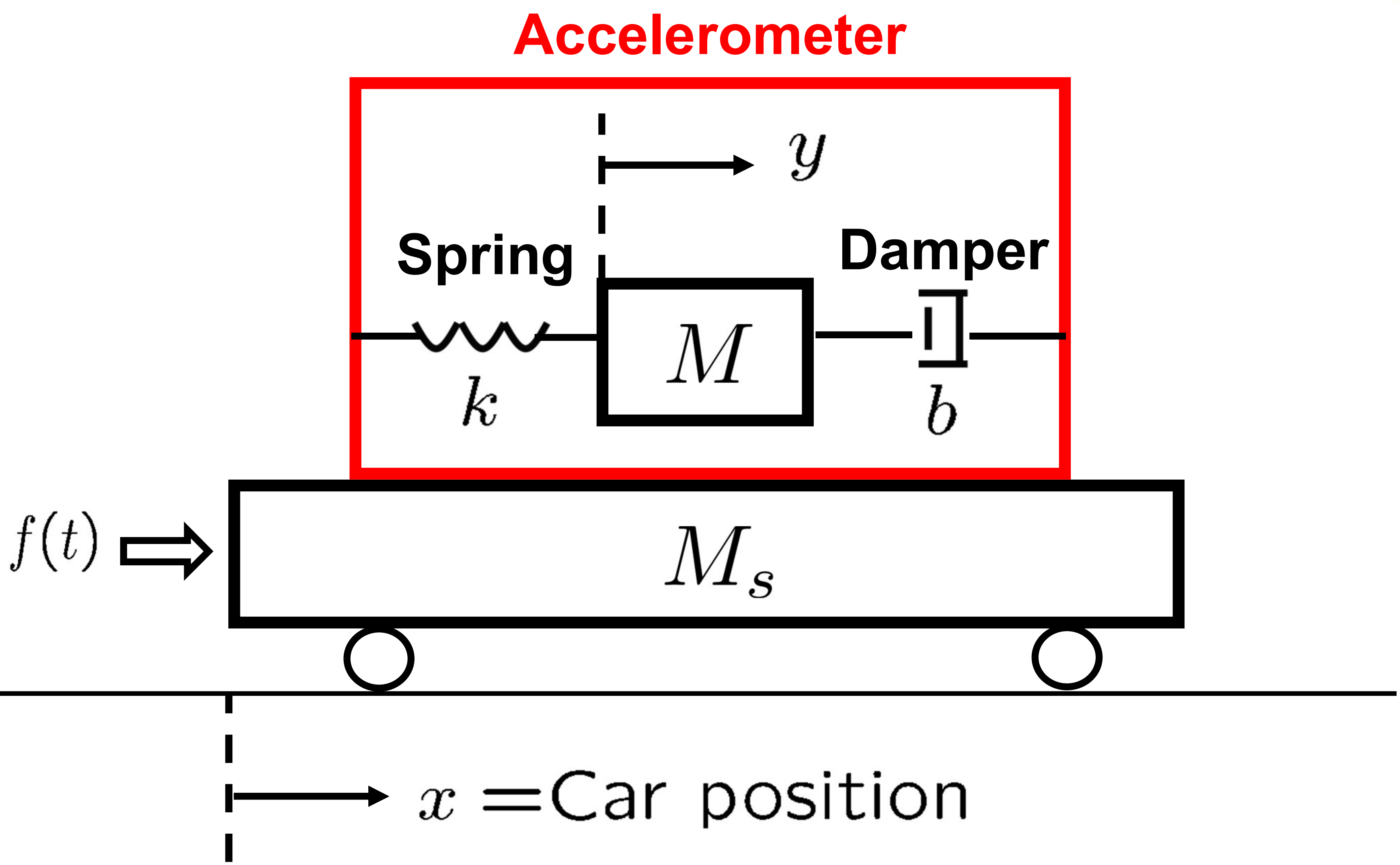
$$\Rightarrow X(s) = \underbrace{\frac{1}{Ms^2}F(s)}_{} + \underbrace{\frac{x(0)}{s} + \frac{x'(0)}{s^2}}_{}$$

(Total Response) = (Forced Response) + (IC Response)

$$\Rightarrow \boxed{x(t) = \overbrace{\mathcal{L}^{-1}\left\{\frac{1}{Ms^2}F(s)\right\}}^{} + \overbrace{x(0)u(t) + x'(0)tu(t)}^{}}$$

© Siamak Najarian 2025

Example 5: Electromechanical accelerometer



© Siamak Najarian 2025

Example 5 (cont'd): Accelerometer

- We want to know how $y(t)$ moves when a unit step $f(t)$ is applied with zero ICs.
- By Newton's law:

$$\begin{cases} M\dfrac{d^2}{dt^2}(x(t)+y(t)) = -b\dfrac{dy(t)}{dt} - ky(t) \\ M_s\dfrac{d^2x(t)}{dt^2} = f(t) \end{cases}$$

$$\Rightarrow \quad My''(t) + by'(t) + ky(t) = -\frac{M}{M_s}f(t) \qquad \mathcal{L}\{f(t)\} = \frac{1}{s}$$

$$\xRightarrow{\mathcal{L}} \quad Y(s) = -\frac{M}{M_s}\cdot\frac{1}{Ms^2+bs+k}\cdot\frac{1}{s} = -\frac{1}{M_s}\cdot\frac{1}{s^2+(b/M)s+(k/M)}\cdot\frac{1}{s}$$

© Siamak Najarian 2025

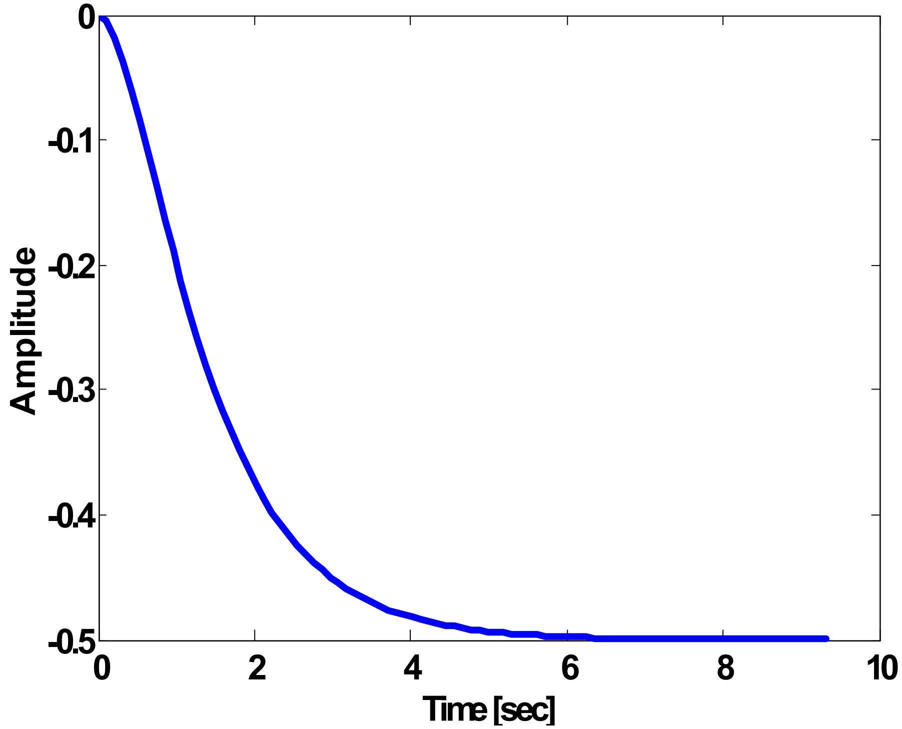
Example 5 (cont'd): Accelerometer

- Suppose that $b/M = 3$, $k/M = 2$ and $M_s = 1$.
- Partial fraction expansion

$$Y(s) = -\frac{1}{s^2+3s+2}\cdot\frac{1}{s} = -\frac{1}{2s} + \frac{1}{s+1} - \frac{1}{2(s+2)}$$

- Inverse Laplace transform

$$\boxed{y(t) = \left(-\frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t}\right)u(t)}$$



© Siamak Najarian 2025

Summary

- Solution to an ODE via Laplace transform consists of performing the following steps:
  1. Taking Laplace transform
  2. Using partial fraction expansion
  3. Taking inverse Laplace transform
- Next
  - Modeling of engineering systems in *s*-domain

© Siamak Najarian 2025