a place of mind

ELEC 341: Systems and Control

Lecture 6

Stability:
Routh-Hurwitz stability criterion

Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- Models for systems
  - ✓ Electrical
  - ✓ Electromechanical
  - ✓ Mechanical
- ✓ Linearization, delay

Analysis

- Stability
  - Routh-Hurwitz
  - Nyquist
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot

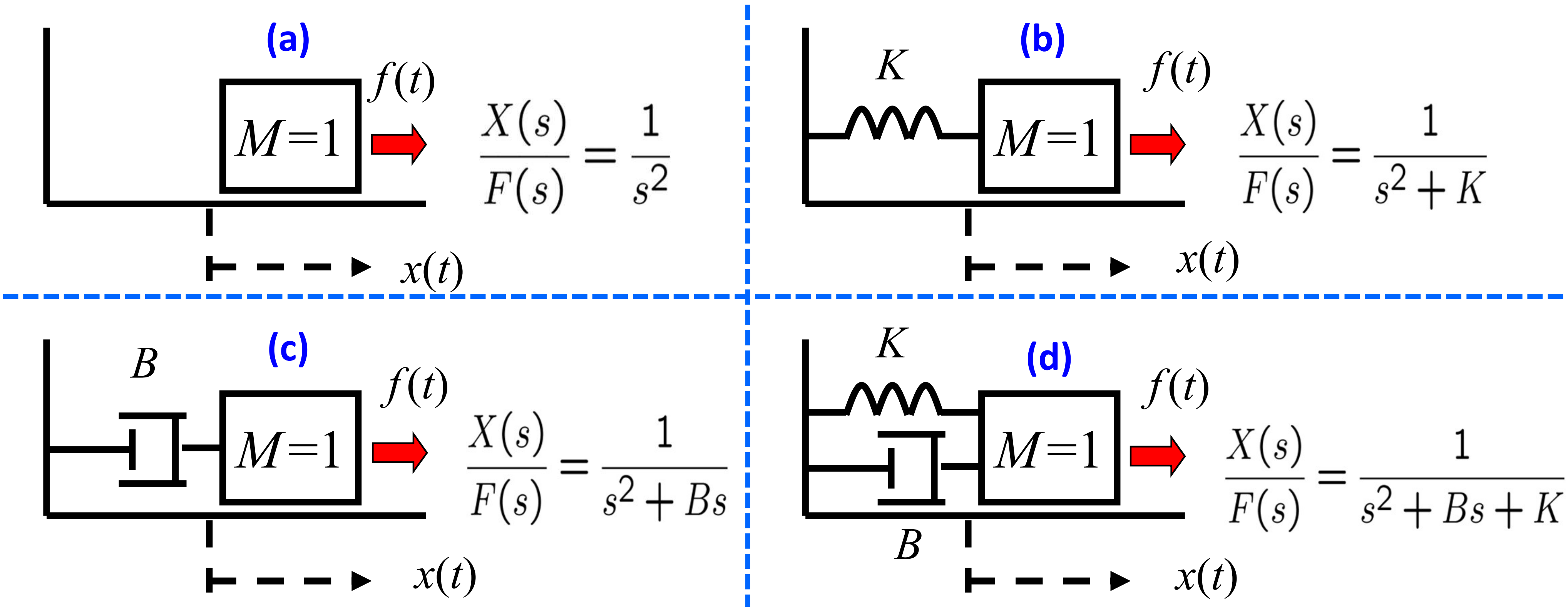
Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

Matlab simulations

Example 1: Characterizing System Behavior

- We want the mass to stay at $x = 0$, but wind causes the mass to move. What will happen?



- How to characterize different behaviors with TF (transfer function)? That is, how to investigate the **stability** of the system? We will be revisiting this example again in this lecture.

Stability

- Utmost important specification in control design!
- Unstable closed-loop systems are useless.
- Unstable systems might be stabilized by feedback.
- What if a system is unstable? ("out-of-control")
  - It may hit electrical/mechanical "stops".
  - It may break down or burn out.
  - Signals diverge.
- Examples of unstable systems
  - Tacoma Narrows Bridge collapse in 1940
  - SAAB Gripen JAS-39 prototype accident in 1989
  - Wind turbine explosion in Denmark in 2008

Stability

What happens if a system becomes unstable?

- When a system becomes unstable, its response can grow without bounds—like a motor spinning faster and faster or a signal increasing uncontrollably. In real-world systems, though, this growth cannot continue forever. Eventually, the system will **hit physical or electrical limits**, often referred to as **"stops."**
- In a **mechanical system**, these might be physical barriers, maximum extension limits, or hard end-stops (e.g., a piston reaching the end of a cylinder).
- In an **electrical system**, limits could be maximum voltage, current saturation, or thermal shutdown.

Once these limits are reached, the system may:

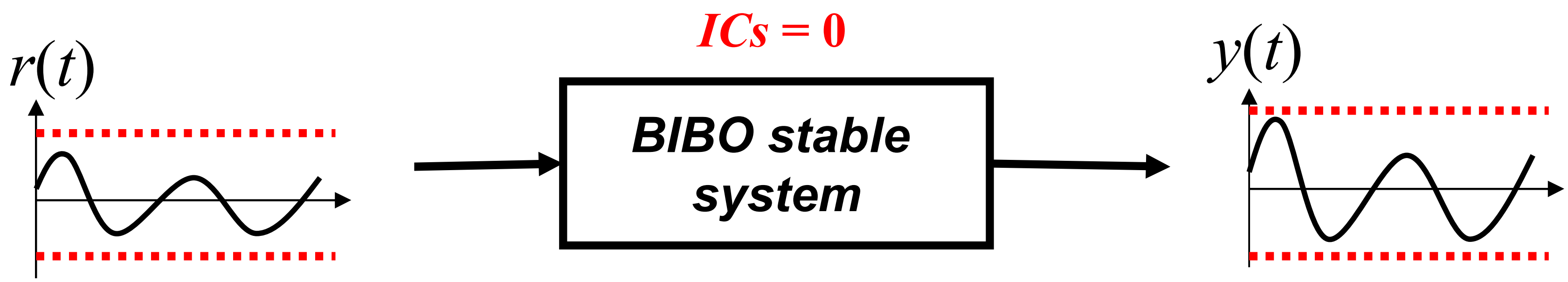
  - Become damaged
  - Shut down abruptly
  - Enter a non-functional or unpredictable state

- So, even though the math says the output might grow indefinitely, in practice, **an unstable system will usually hit a point where something breaks or trips**—which is why preventing instability is so critical in control system design.

Definitions of stability

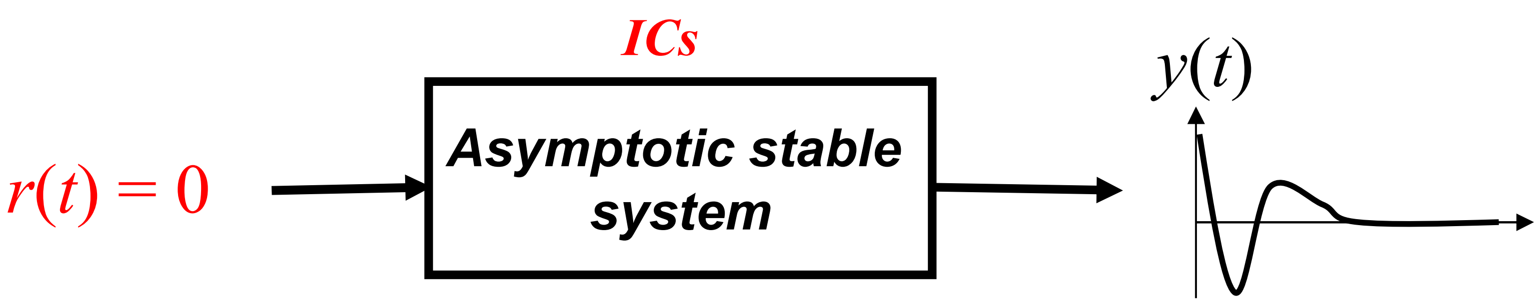
- **BIBO (Bounded-Input-Bounded-Output) stability**

  Any bounded input generates a bounded output.



- **Asymptotic stability**

  Any ICs *generates* $y(t)$ *converging to zero.*



Some terminologies

$$G(s) = \frac{n(s)}{d(s)}$$

Ex. $$G(s) = \frac{(s-1)(s+1)}{(s+2)(s^2+1)}$$

- **Zero:** roots of $n(s)$ — $(\text{Zeros of } G) = \pm 1$
- **Pole:** roots of $d(s)$ — $(\text{Poles of } G) = -2, \pm j$
- **Characteristic polynomial:** $d(s)$
- **Characteristic equation:** $d(s) = 0$

Stability condition in *s*-domain

- For a system represented by transfer function $G(s)$:

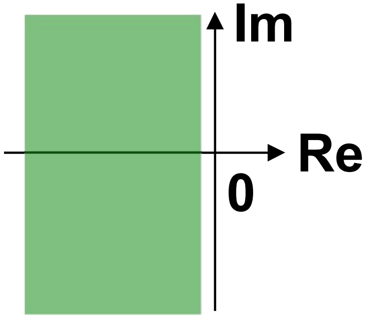
System is BIBO stable

⇕

All the poles of $G(s)$ are in the open left half of the complex plane.

⇕

System is asymptotically stable



- In control theory, **characterizing system behavior** with transfer function is performed by **investigating the stability status** (or **stability condition**) of a control system.

Example 2: Idea of Stability Condition

- Example: $y'(t) + \alpha y(t) = r(t)$

$$\Rightarrow \quad sY(s) - y(0) + \alpha Y(s) = R(s)$$

$$\Rightarrow \quad Y(s) = \frac{1}{s+\alpha}(R(s) + y(0))$$

Asymptotic Stability: $(r(t) = R(s) = 0)$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+\alpha}y(0)\right\} = e^{-\alpha t}y(0) \to 0 \Leftrightarrow \quad (-\alpha) < 0$$

BIBO Stability: $(y(0) = 0)$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)R(s)\} = \int_0^t g(\tau)r(t-\tau)d\tau = \int_0^t e^{-\alpha\tau}r(t-\tau)d\tau$$

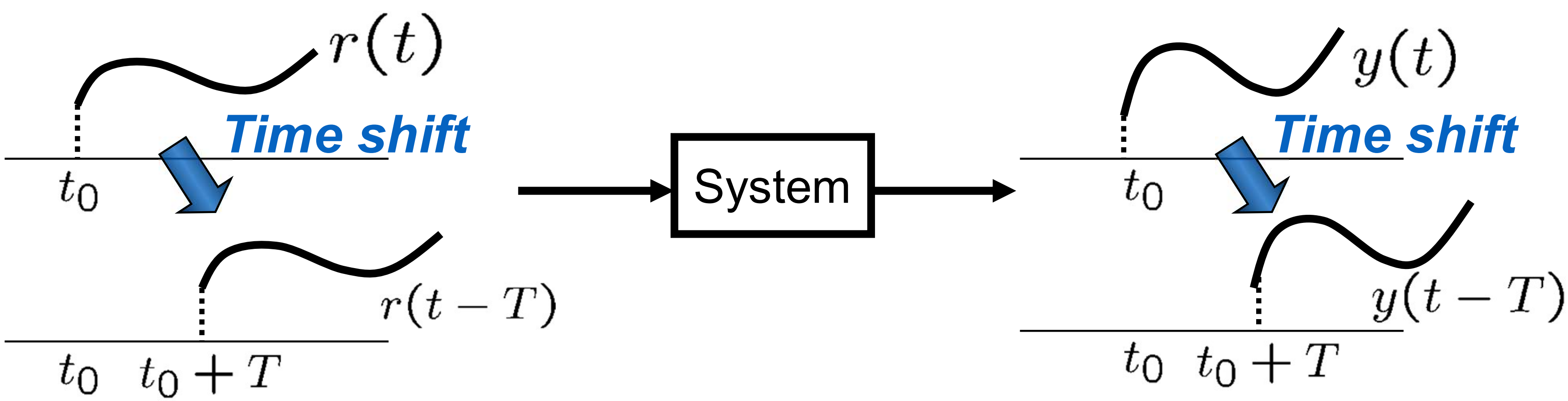
$$|y(t)| \leq \int_0^t |e^{-\alpha\tau}||r(t-\tau)|d\tau \leq \int_0^t |e^{-\alpha\tau}|d\tau \cdot r_{max} \quad \Rightarrow$$

$$\lim_{t\to\infty} |y(t)| \leq \frac{r_{\max}}{\alpha}$$

Bounded if $(-\alpha) < 0$

Time-invariant & time-varying

- A system is called *time-invariant* if system parameters do not change in time. If they do, it is called *time-varying*.
- Examples:
  - $\boldsymbol{M}\,\ddot{x}(t) = f(t)$ (time-invariant)
  - $\boldsymbol{M(t)}\,\ddot{x}(t) = f(t)$ (time-varying)
- For time-invariant systems:



- This course deals with time-invariant systems.

Remarks on stability

- For general systems (nonlinear, time-varying), BIBO stability condition and asymptotic stability condition are different (beyond the scope of this course).
- For **linear time-invariant (LTI)** systems (for which we can use Laplace transform and we can obtain transfer functions), these two stability conditions happen to be the same.
- In this course, since we are interested in only LTI systems, we simply use "stable" to mean both BIBO and asymptotic stability.

Remarks on stability (cont'd)

- Marginally stable if
  - **Step 1:** $G(s)$ has no pole in the open RHP (Right Half Plane), and
  - **Step 2:** $G(s)$ has at least one simple pole on $j\omega$-axis, and
  - **Step 3:** $G(s)$ has no multiple pole on $j\omega$-axis.

$$G(s) = \frac{1}{s(s^2+4)(s+1)^2}$$

Marginally stable

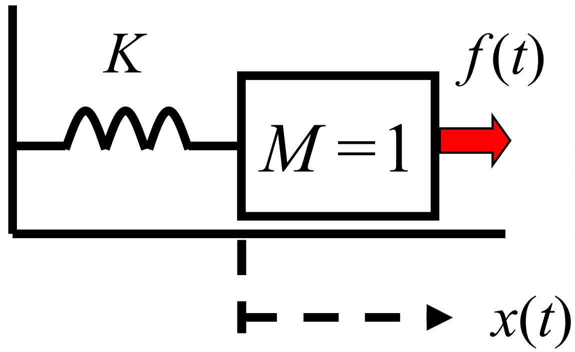
$$G(s) = \frac{1}{s(s^2+4)^2(s+1)^2}$$

NOT marginally stable

- Unstable if a system is neither stable nor marginally stable. In this context, ***NOT marginally stable*** means the system is **unstable**.

Note: A **simple pole** is a pole of order one (i.e., a **non-repeating pole**).

"Marginally stable" in *t*-domain



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + K} = G(s)$$

- For any bounded input, except only special sinusoidal (bounded) inputs, the output is bounded.
  - In the example above, the special inputs are in the form of:

$$f(t) = \alpha \sin \sqrt{K} t + \beta \cos \sqrt{K} t \quad \Rightarrow \quad x(t) \rightarrow \pm\infty$$

- If a system is marginally stable, it means that for **some inputs**, the system behaves in a stable way (output remains bounded), but for **other inputs**, the system becomes unstable (output grows unbounded).

Stability summary

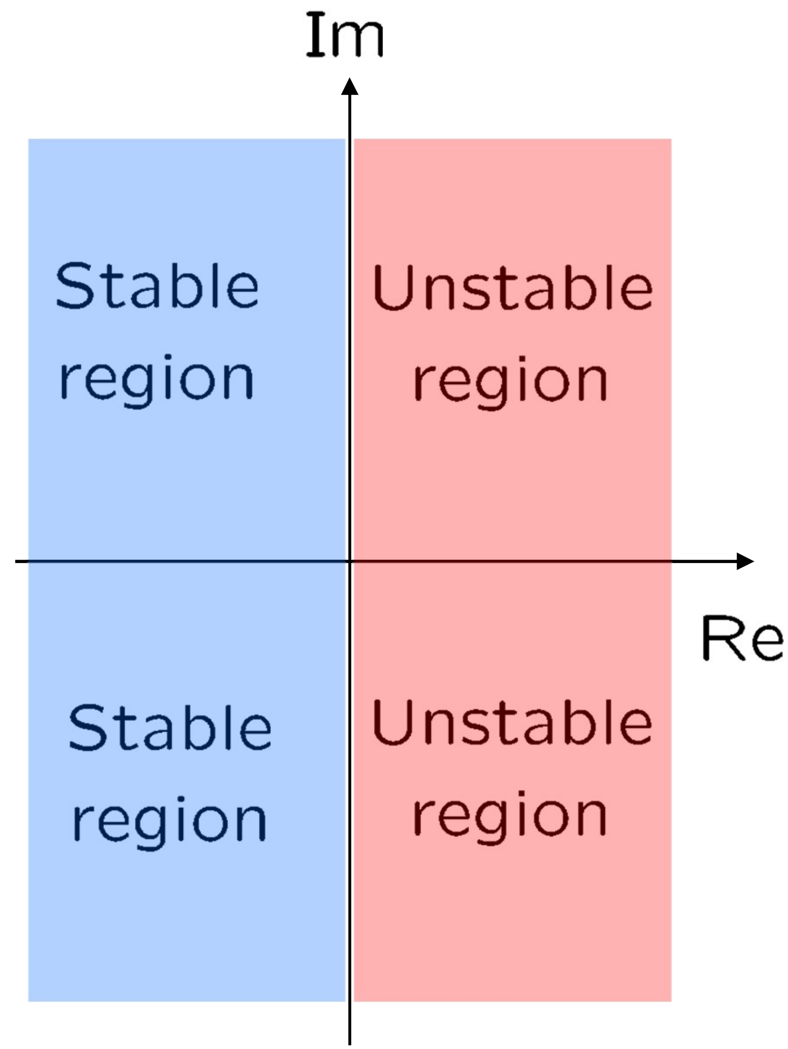
Let s_i be poles of $G(s)$.
Then, $G(s)$ is …

- stable if

  $\mathrm{Re}(s_i) < 0$ for all i.

- marginally stable if
  - $\mathrm{Re}(s_i) \le 0$ for all i, and
  - at least one simple pole for $\mathrm{Re}(s_i) = 0$
  - no multiple pole on $j\omega$-axis

- unstable if it is neither stable nor marginally stable.



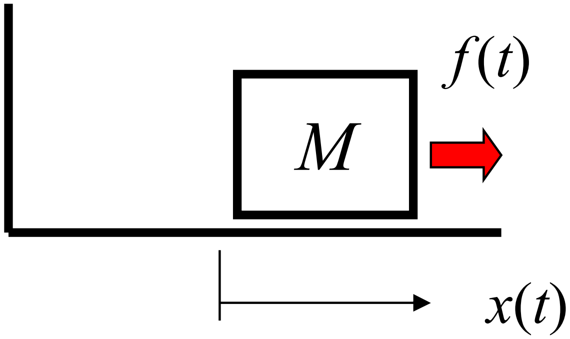
Re-axis is sometimes shown by σ or δ while **Im-axis** is shown by $j\omega$.

Example 3: Characterizing System Behavior (revisited)

- **Characterizing system behavior** with transfer function is performed by **investigating the stability status** (or stability condition) of a control system.

(Let $M = 1$, $B = 2$ and $K = 3$)

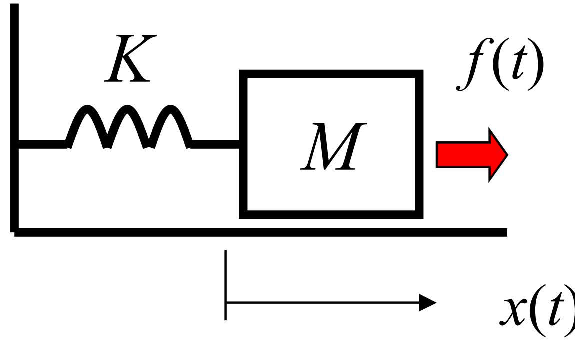
(a)



$$\frac{X(s)}{F(s)} = \frac{1}{s^2}$$

Poles ?
Stable ?

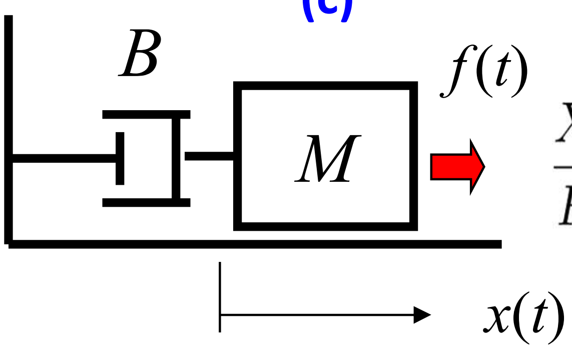
(b)



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + K}$$

Poles ?
Stable ?

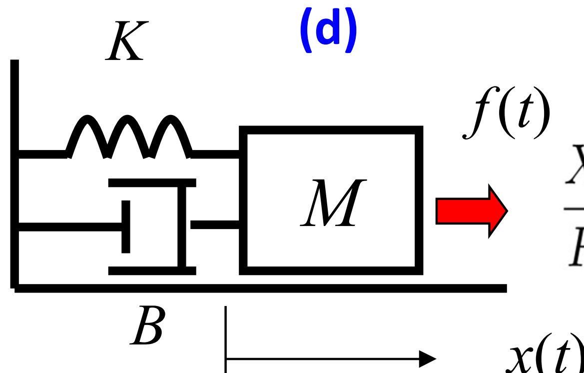
(c)



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + Bs}$$

Poles ?
Stable ?

(d)



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + Bs + K}$$

Poles ?
Stable ?

Example 4: Stability Status

	$G(s)$	Stable/Marginally Stable/Unstable
(1)	$\frac{20}{(s+1)(s+2)(s+3)}$	?
(2)	$\frac{20(s+1)}{(s-1)(s^2+2s+3)}$	?
(3)	$\frac{1}{(s+5)(s^2+2)^2}$	?
(4)	$\frac{1}{s^4+5s^3+10s^2+3s+1}$	?

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Routh-Hurwitz criterion

- This is for LTI systems with a *polynomial* denominator (without sine, cosine, exponential, etc.)
- It determines if all the roots of a polynomial
  - lie in the open LHP (left half-plane),
  - or equivalently, have negative real parts.
- It also determines the number of roots of a polynomial in the open RHP (right half-plane).
- It does NOT explicitly compute the roots numerical values.
- Consider a polynomial

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

Routh array

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

From the given polynomial (first two rows)

Routh array
(How to compute the third row)

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

$$b_1 = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}}$$

$$b_3 = \frac{a_{n-6}a_{n-1} - a_n a_{n-7}}{a_{n-1}}$$

$$\vdots$$

Note: For every calculation, we always use the entries in the first column (in the two rows above the line of our calculation) as our anchor.

Routh array (How to compute the fourth row)

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

$$c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1}$$

$$c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1}$$

$$c_3 = \frac{a_{n-7}\,b_1 - a_{n-1}b_4}{b_1}$$

$$\vdots$$

Routh-Hurwitz criterion

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

*The number of roots in the open right half-plane is equal to the number of sign changes in the **first column** of Routh array.*

Note: If the polynomial has any roots in the open RHP, the system is unstable.

Example 5

Investigate the stability.

$$Q(s) = s^3 + s^2 + 2s + 8 = (s+2)(s^2 - s + 4)$$

Routh array

s^3	1	2
s^2	1	8
s^1	-6	
s^0	8	

$$-6 = \frac{2-8}{1}$$

$$8 = \frac{8 \times (-6) - 1 \times 0}{-6}$$

Two sign changes in the first column $1 \rightarrow -6 \rightarrow 8$ ⇒ Two roots in RHP $\frac{1}{2} \pm \frac{j\sqrt{15}}{2}$ ⇒ Unstable

Example 6

Investigate the stability.

$$Q(s) = s^3 + 3s^2 + 6s + 8 = (s+2)(s^2 + s + 4)$$

Routh array

s^3	1	6
s^2	3	8
s^1	10/3	
s^0	8	

$$\frac{6 \times 3 - 8}{3}$$

Always same!

No sign changes in the first column:

$1 \rightarrow 3 \rightarrow 10/3 \rightarrow 8$

No roots in RHP:

$$-2, -\frac{1}{2} \pm \frac{j\sqrt{15}}{2}$$

Stable

Example 7 (from slide 15)

Investigate the stability.

$$Q(s) = s^4 + 5s^3 + 10s^2 + 3s + 1$$

Routh array

s^4	1	10	1
s^3	5	3	
s^2	9.4	1	
s^1	2.468		
s^0	1		

Always same!

No sign changes in the first column. → No roots in RHP. → Stable

Simple important criteria for stability

- 1st order polynomial $Q(s) = a_1 s + a_0$

  All roots are in LHP $\Leftrightarrow$ a_1 and a_0 have the same sign

- 2nd order polynomial $Q(s) = a_2 s^2 + a_1 s + a_0$

  All roots are in LHP $\Leftrightarrow$ a_2, a_1 and a_0 have the same sign

Example 8

	$Q(s)$	All roots in open LHP?
(1)	$3s + 5$	Yes / No
(2)	$-2s^2 - 5s - 100$	Yes / No
(3)	$523s^2 - 57s + 189$	Yes / No
(4)	$s^4 + 2s^3 + s^2 - 1$	Yes / No
(5)	$s^3 + 5s^2 + 10s - 3$	Yes / No

Summary

- Stability for LTI systems
  - (BIBO, asymptotically) stable, marginally stable, unstable
  - Stability for $G(s)$ is determined by poles of $G(s)$.
- Routh-Hurwitz stability criterion
  - To determine stability without explicitly computing the poles of a system.
- Next
  - More examples on Routh-Hurwitz criterion.