a place of mind

ELEC 341: Systems and Control

Lecture 7

Routh-Hurwitz stability criterion: Examples

Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- Models for systems
  - ✓ Electrical
  - ✓ Electromechanical
  - ✓ Mechanical
- ✓ Linearization, delay

Analysis

- ✓ Stability
  - Routh-Hurwitz
  - Nyquist
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot

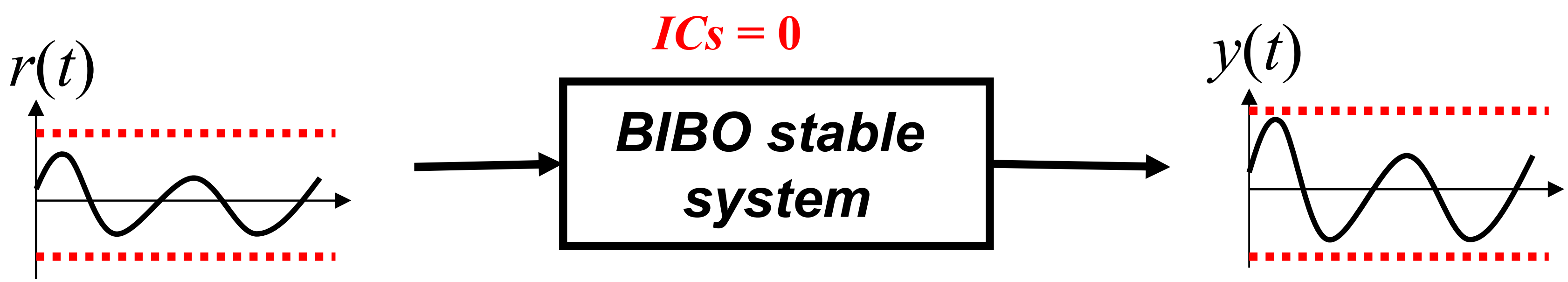
Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

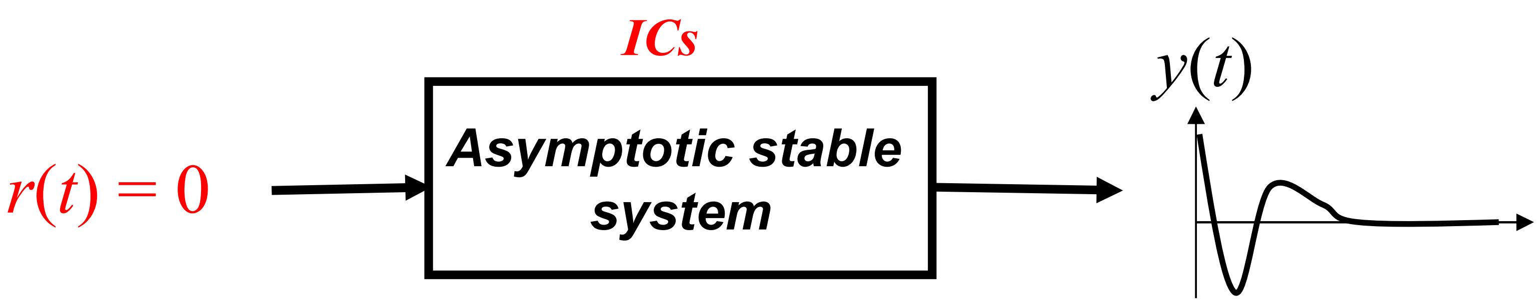
Matlab simulations

Definitions of stability (review)

- **BIBO (Bounded-Input-Bounded-Output) stability**
  Any bounded input generates a bounded output.



- **Asymptotic stability**
  Any ICs generates $y(t)$ converging to zero.



Stability summary (review)

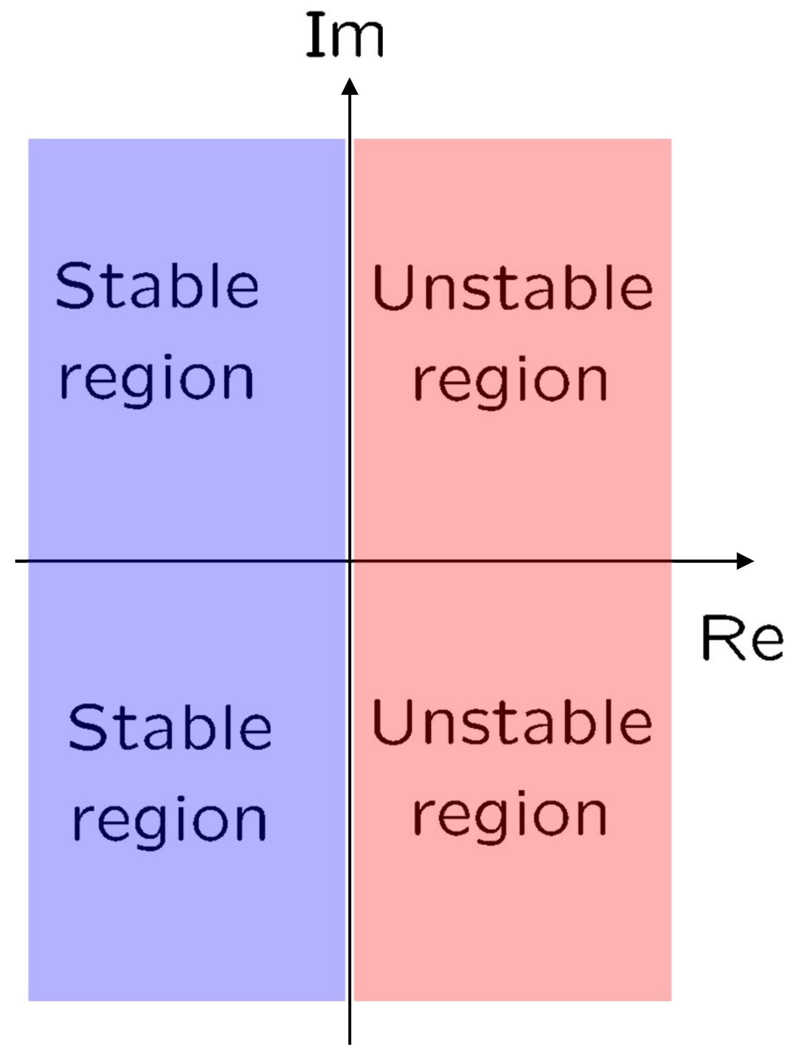
Let s_i be poles of $G(s)$.
Then, $G(s)$ is ...

- stable if

  $\mathrm{Re}(s_i) < 0$ for all i.

- marginally stable if
  - $\mathrm{Re}(s_i) \leq 0$ for all i, and
  - at least one simple pole for $\mathrm{Re}(s_i) = 0$
  - no multiple pole on $j\omega$-axis

- unstable if it is neither stable nor marginally stable.



Routh-Hurwitz criterion (review)

- This is for LTI systems with a *polynomial* denominator (without sine, cosine, exponential, etc.)
- It determines if all the roots of a polynomial
  - lie in the open LHP (left half-plane),
  - or equivalently, have negative real parts.
- It also determines the number of roots of a polynomial in the open RHP (right half-plane).
- It **does NOT** explicitly compute the roots numerical values.

Routh array (review)

$$
\begin{array}{l|lllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

From the given polynomial (first two rows)

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

Routh array
(How to compute the third row)

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

$$b_1 = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}}$$

$$b_3 = \frac{a_{n-6}a_{n-1} - a_n a_{n-7}}{a_{n-1}}$$

$$\vdots$$

Routh array (How to compute the fourth row)

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

$$c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1}$$

$$c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1}$$

$$c_3 = \frac{a_{n-7}\, b_1 - a_{n-1} b_4}{b_1}$$

$$\vdots$$

Routh-Hurwitz criterion

$$
\begin{array}{l|lllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

*The number of roots in the open right half-plane is equal to the number of sign changes in the **first column** of Routh array.*

Example 1 (The Epsilon Method)

Investigate the stability.

$$Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

Routh array

s^5	1	2	11
s^4	2	4	10
s^3	~~0~~ ε	6	
s^2	$\frac{-12}{\varepsilon}$	10	
s^1	≈ 6		
s^0	10		

If 0 appears in the first column of a ***nonzero row*** in Routh array, replace it with a small positive number (ε).

$$\lim_{\varepsilon \to 0} \frac{4\varepsilon - 12}{\varepsilon} = \frac{-12}{\varepsilon}$$

Important Note:
First, take the limit of the entry that contains ε and then place it in the table.

$$\varepsilon \to \underbrace{\frac{-12}{\varepsilon}}_{<0} \to 6$$

Two sign changes in the first column. ⇒ Two roots in RHP.

In this case, Q has some roots in RHP.

⇓

Unstable

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Example 2 (The ROZ Method)

Investigate the stability. $Q(s) = s^4 + s^3 + 3s^2 + 2s + 2$

If a **row of zero** (**ROZ**) appears in Routh array (here, s^1 row), Q has roots either on the imaginary axis or in RHP.

Routh array

s^4	1	3	2
s^3	1	2	
s^2	1	2	
s^1	~~0~~ **2**		
s^0	2		

No sign changes in the first column. → No roots in RHP. → But some roots are on imaginary axis.

S^1 is a ROZ = row of zero

How to write $P(s)$: Go one row above the row of zero and begin with the power of ***s*** that starts the row and then skip every other power until you reach the end of the row.

$P(s) = (1)s^2 + (2)s^0 \longrightarrow P(s) = s^2 + 2$ **(auxiliary polynomial)**

Take the derivative of the **auxiliary polynomial** (which is a factor of $Q(s)$):

$\longrightarrow P' = \frac{dP}{ds} = \mathbf{2}s$ $\longrightarrow$ Replace the row of zero with the coefficients of the derivative polynomial in the dP/ds equation.

→ **Marginally Stable**

What does ROZ signify?

If a row of zero (ROZ) appears in Routh array, then $Q(s)$ has roots either on the imaginary axis or in RHP.

- If $Q(s)$ has roots in RHP → **Unstable**
- If $Q(s)$ has roots on imaginary axis → **Unstable** / **Marginally Stable**

Example 3

Investigate the stability.

$$Q(s) = s^3 + s^2 + s + 1 \;\; = (s+1)(s^2+1)$$

Routh array

$P(s) = s^2 + 1$ **(auxiliary polynomial)**

s^3	1	1
s^2	1	1
ROZ → s^1	~~0~~ **2**	
s^0	1	

Derivative of auxiliary polynomial:

$$\acute{P}(s) = \; (s^2+1)' = 2s$$

Auxiliary polynomial is a factor of $Q(s)$.

No sign changes in the first column. ⇒ No roots in OPEN RHP. ⇒ **Marginally Stable**

Note: Based on $s^2 + 1 = 0$, we have at least one simple pole on the imaginary axis and also no repeating poles on the imaginary axis . So, it is **marginally stable**.

Example 4

Investigate the stability.

$$Q(s) = s^5+s^4+2s^3+2s^2+s+1 \;\; = (s+1)(s^2+1)^2$$

Routh array

	s^5	1	2	1
	s^4	1	2	1
ROZ →	s^3	~~0~~ 4	~~0~~ 4	
	s^2	1	1	
ROZ →	s^1	~~0~~ 2		
	s^0	1		

Derivative of auxiliary polynomial:

$$(s^4+2s^2+1)' = 4s^3+4s$$

$$(s^2+1)' = 2s$$

No sign changes in the first column. No roots in OPEN RHP. Unstable

Note: Based on $s^4+2s^2+1=(s^2+1)^2=0$, we have repeating poles on imaginary axis. So, it is **unstable**.

Example 5

Investigate the stability.

$$Q(s) = s^4 - 1 \;\; = (s+1)(s-1)(s^2+1)$$

Routh array

s^4	1	0	-1
ROZ → s^3	~~0~~ **4**	~~0~~ **0**	
s^2	~~0~~ ε	-1	
s^1	$4/\varepsilon$		
s^0	-1		

Derivative of auxiliary polynomial:

$$(s^4 - 1)' = 4s^3$$

One sign changes in the first column. → One root in OPEN RHP. → **Unstable**

Summary for stability when ROZ appears

Summary of procedure for determining if ROZ will lead to unstable or marginally stable systems:

- If there is a sign change in the first column, then we have poles in the open RHP and the system is **unstable**.
- If there is no sign change in the first column, there is no poles in the open RHP, but we will have poles on the Im-axis and we should see whether the system is marginally stable or unstable.
  - If all the poles on the Im-axis are simple poles (i.e., with the multiplicity of 1), then the system is **marginally stable**.
  - If the poles on the Im-axis have multiplicity of more than 1, then the system is **unstable**.

Notes on Routh-Hurwitz criterion

- **Advantages:**
  - No need to explicitly compute roots of the polynomial.
    - High order $Q(s)$ can be handled by hand calculations.
  - Polynomials including undetermined parameters (plant and/or controller parameters in feedback systems) can be dealt with.
    - Root computation does not work in such cases!
- **Disadvantage:**
  - Exponential functions (delay) cannot be dealt with.
    - Example:

$$Q(s) = e^{-s} + s^2 + s + 1$$

We will study Nyquist stability criterion later to deal with these cases!

Example 6

$$Q(s) = s^3 + 3Ks^2 + (K+2)s + 4$$

Find the range of K so that $Q(s)$ has all roots in the left half plane. Here, K is a **design parameter**.

Routh array

s^3	1	$K+2$
s^2	$3K$	4
s^1	$\frac{3K(K+2)-4}{3K}$	
s^0	4	

In order to have no sign changes in the first column:

$$\Rightarrow \begin{cases} 3K > 0 \Rightarrow K > 0 \\ 3K(K+2) - 4 > 0 \end{cases}$$

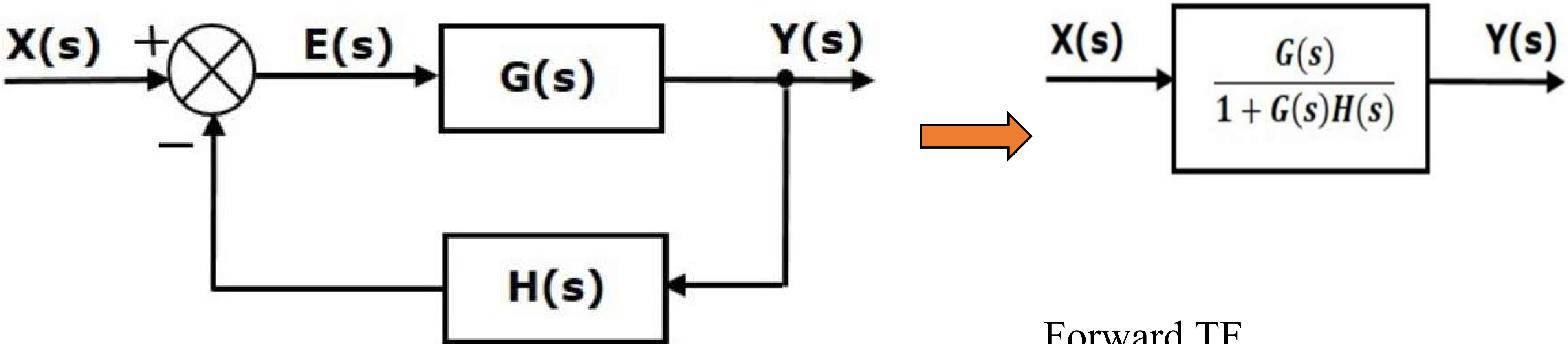
$$\Rightarrow K > -1 + \frac{\sqrt{21}}{3}$$

$$\Rightarrow \boxed{K > 0.527}$$

Note: In this course, we assume that the design parameters are always **positive**.

Characteristic Equation

- The following figure shows a negative feedback control system. Here, two blocks having transfer functions $G(s)$ and $H(s)$ form a closed loop.



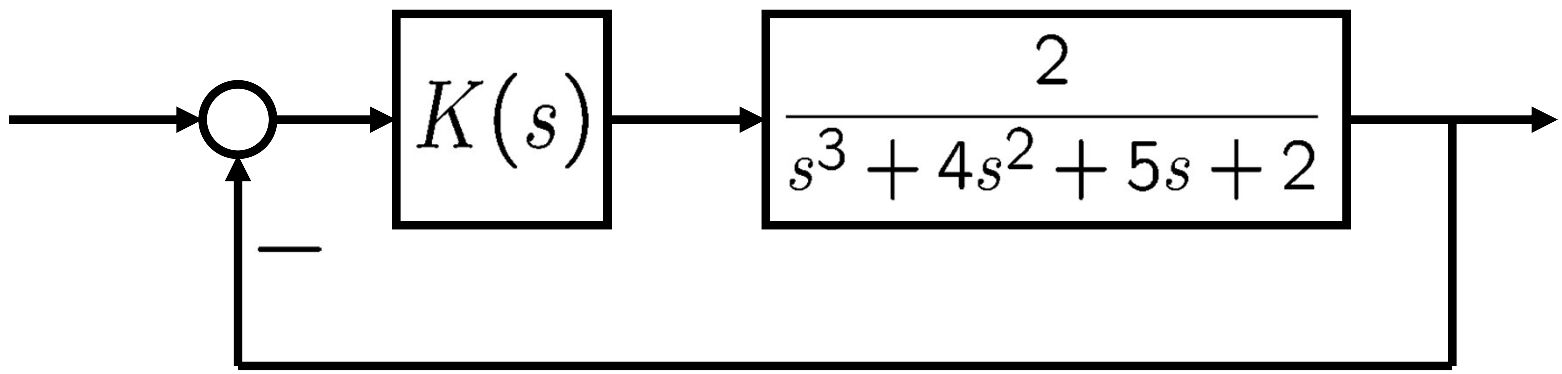
$$\text{CLTF} = \frac{\text{Forward TF}}{1 + \text{OLTF}}$$

$$\text{CLTF} = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Characteristic Equation: $1 + \text{OLTF} = 0$.

- $1 + G(s)H(s) = 0$ is called the **Characteristic Equation**. It is the left hand side of this equation that we have shown by $Q(s)$ in the previous slides. That is, $Q(s) = 1 + G(s)H(s)$. Here, $Q(s)$ is called the **characteristic polynomial**.

Example 7



- Design $K(s)$ that stabilizes the closed-loop system for the following cases:
  - **(a)** $K(s) = K$ (constant, P or Proportional controller)
  - **(b)** $K(s) = K_P + \dfrac{K_I}{s}$ (PI or Proportional-Integral controller)

Example 7 (cont'd): $K(s) = K$

(a)

- Characteristic equation:

$$1 + K\frac{2}{s^3 + 4s^2 + 5s + 2} = 0$$

$$\Rightarrow \quad s^3 + 4s^2 + 5s + 2 + 2K = 0$$

- Routh array:

s^3	1	5
s^2	4	$2 + 2K$
s^1	$\frac{18-2K}{4}$	
s^0	$2 + 2K$	

$\Rightarrow \quad -1 < K < 9$

$\Rightarrow \quad 0 < K < 9$

Example 7 (cont'd): $K(s) = K_P + \frac{K_I}{s}$

(b)

- Characteristic equation:

$$1 + \left(K_P + \frac{K_I}{s}\right)\frac{2}{s^3 + 4s^2 + 5s + 2} = 0$$

$$\Rightarrow \quad s^4 + 4s^3 + 5s^2 + (2 + 2K_P)s + 2K_I = 0$$

- Routh array:

s^4	1	5	$2K_I$
s^3	4	$2+2K_P$	
s^2	$\frac{18-2K_P}{4}$	$2K_I$	
s^1	$(*)$		
s^0	$2K_I$		

$$(*) = \frac{(2 + 2K_P)\left(\frac{18 - 2K_P}{4}\right) - 8K_I}{\left(\frac{18 - 2K_P}{4}\right)}$$

s^2 row $\rightarrow K_P < 9$

s^0 row $\rightarrow K_I > 0$

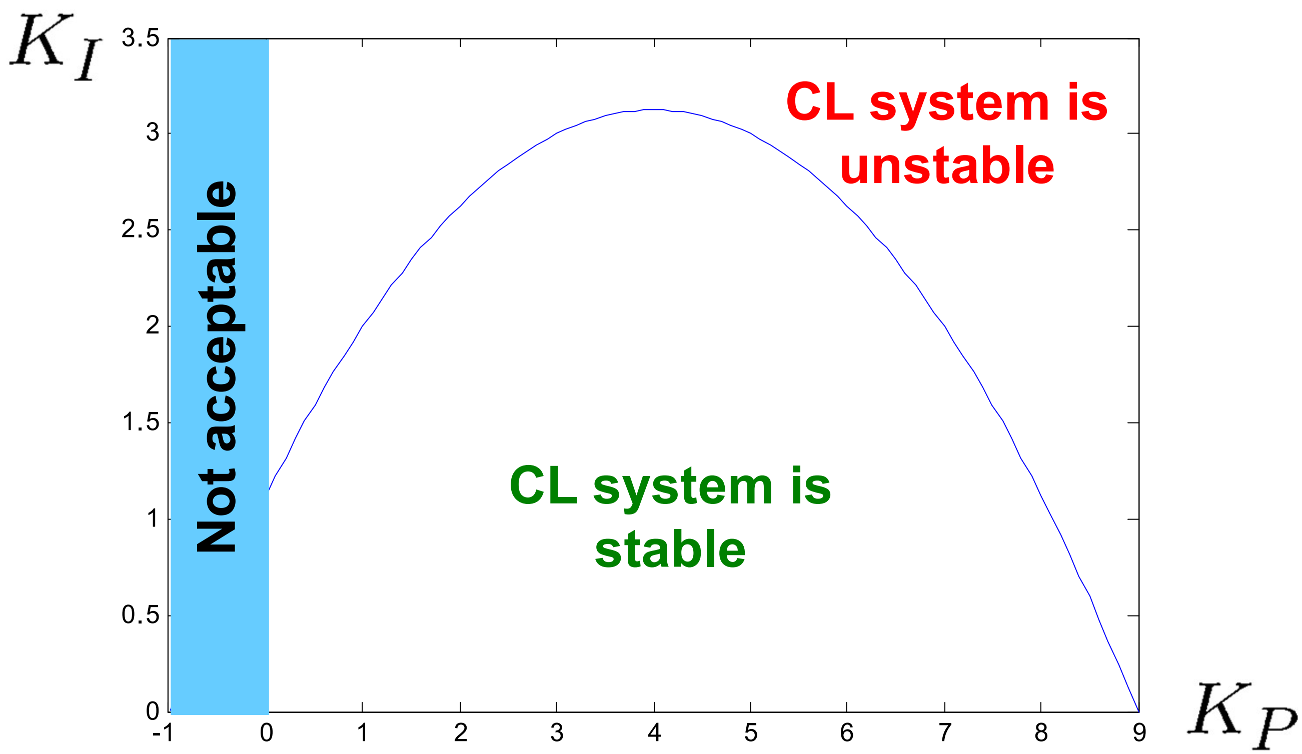
Example 7 (cont'd): Range of (K_P, K_I)

- From Routh array,

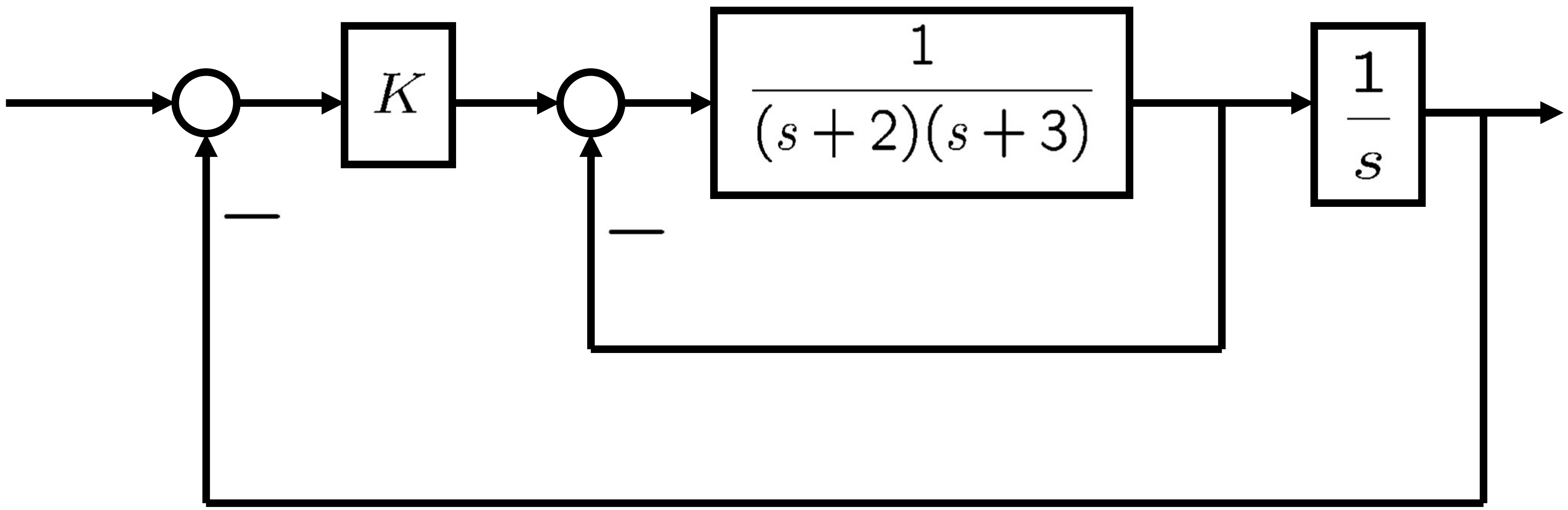
$$K_P < 9$$

$$K_I > 0$$

$$(*) \Leftrightarrow (1+K_P)(9-K_P) - 8K_I > 0$$

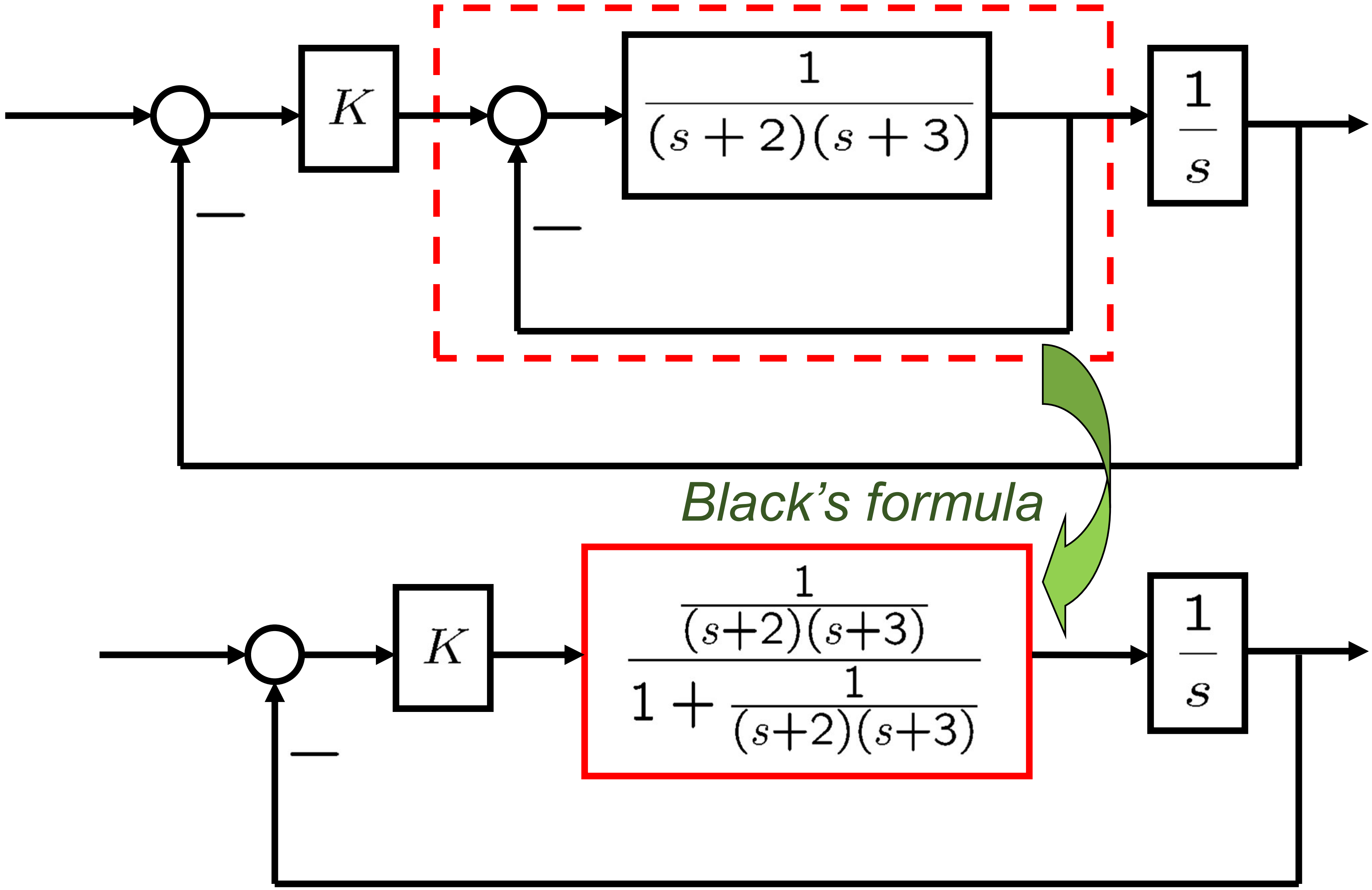


Example 8



- Determine the range of K that stabilizes the closed-loop system.

Example 8 (cont'd)



Note: We need to use the Black's formula ***twice*** if we want the closed-loop transfer function for the whole system.

Example 8 (cont'd)

- Characteristic equation:

$$1 + K\frac{\frac{1}{(s+2)(s+3)}}{1+\frac{1}{(s+2)(s+3)}}\cdot\frac{1}{s} = 0$$

$$\Rightarrow \quad 1 + K\cdot\frac{1}{s(s+2)(s+3)+s} = 0$$

$$\Rightarrow \quad s(s+2)(s+3)+s+K = 0$$

$$\Rightarrow \quad s^3+5s^2+7s+K = 0$$

Example 8 (cont'd)

- Routh array for $s^3 + 5s^2 + 7s + K = 0$

s^3	1	7
s^2	5	K
s^1	$\frac{35-K}{5}$	
s^0	K	

$$0 < K < 35$$

- If K = 35, the closed-loop system is **marginally stable**.
- We can show that for K = 35, if we apply a step input, the system response (the output) will oscillate and that the frequency of oscillation will be $\sqrt{7}$ or 2.64 rad/sec.

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Summary

- Examples for Routh-Hurwitz criterion
  - Cases when zeros appear in Routh array
  - P controller gain range for stability
  - PI controller gain range for stability
- Next
  - Time responses and steady-state errors