a place of mind

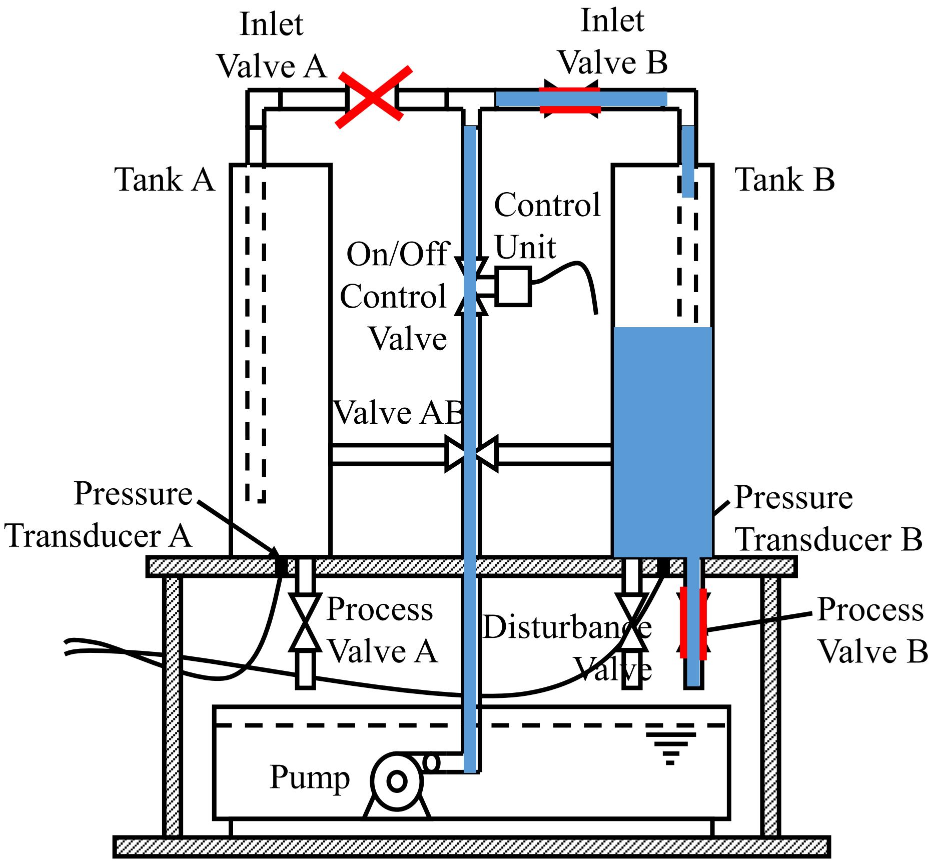
ELEC 341: Systems and Control

Lecture 8

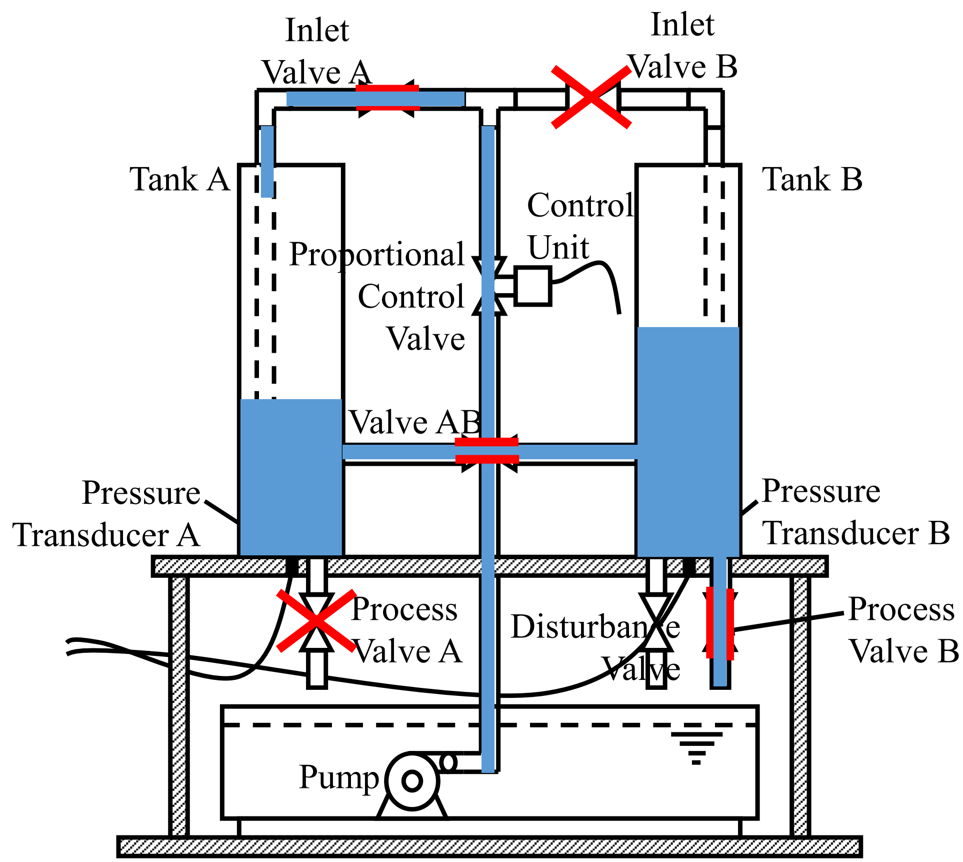
Time-domain specifications and steady-state error

Water tank level control

- Requirement: Maintain the level of Tank B at a desired level by controlling the control valve.

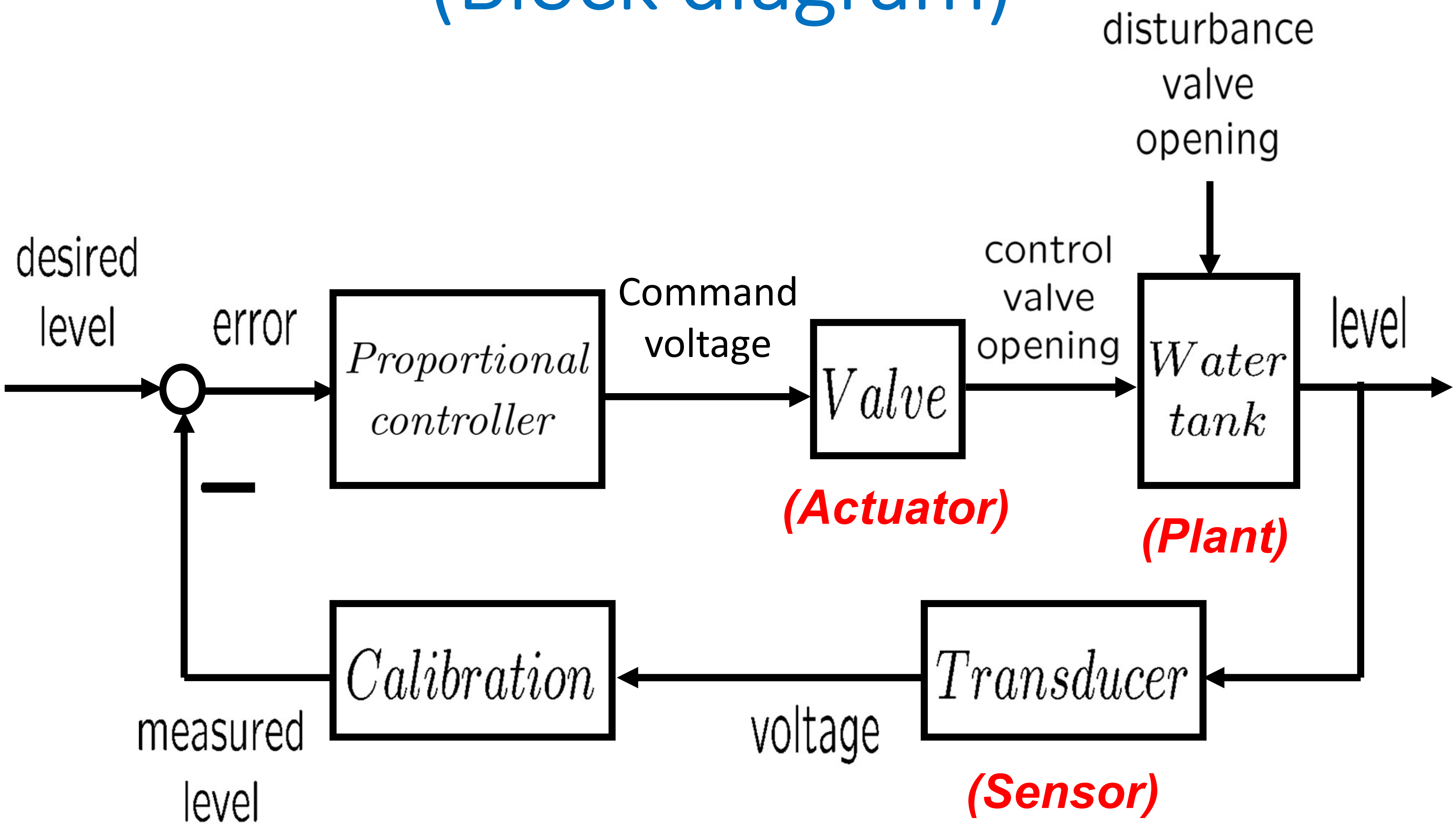


Single tank case



Two tank case

Water tank level control (Block diagram)

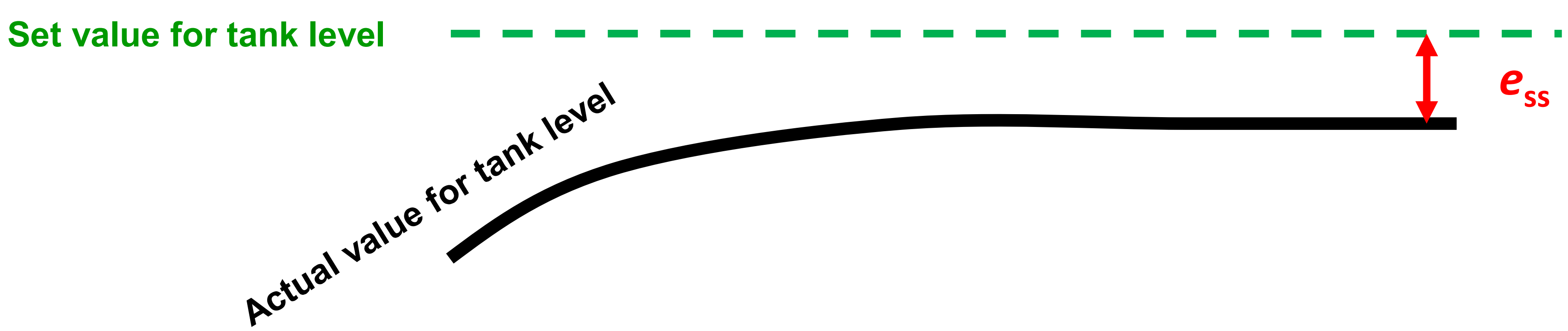


Proportional control (P-control)

- Set value (desired level)
- Controller gain K_P (design parameter)
- Command voltage $V(t)$ ($e(t)$ is an error)

$$V(t) = \begin{cases} V_{max} & \text{if } K_P e(t) > V_{max} \\ K_P e(t) & \text{if } K_P e(t) \in [V_{min}, V_{max}] \\ V_{min} & \text{if } K_P e(t) < V_{min} \end{cases}$$

- Note that there is a steady state error.



Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- Models for systems
  - ✓ Electrical
  - ✓ Electromechanical
  - ✓ Mechanical
- ✓ Linearization, delay

Analysis

- ✓ Stability
  - ✓ Routh-Hurwitz
  - Nyquist
- ➤ Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot

Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

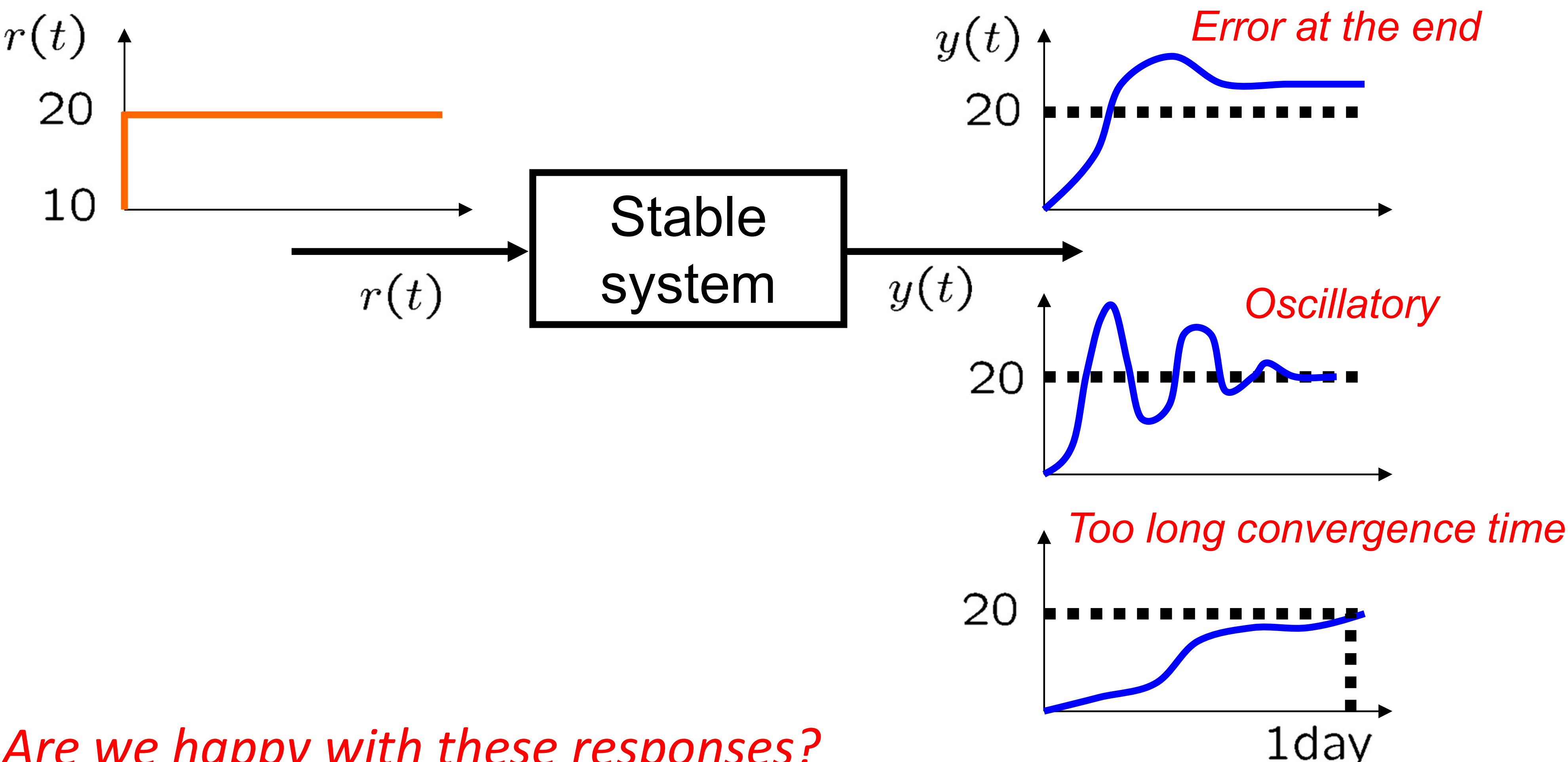
Matlab simulations

Review and next topics

- We have learned about stability.
  - Definition in time domain (*t*-domain)
  - Condition in complex domain (*s*-domain)
  - Routh-Hurwitz criterion to check the condition
- Stability is a necessary requirement, but not sufficient in most control systems. (next slide)
- Specifications other than stability:
  - How to evaluate an engineering system's specifications quantitatively in *t*-domain?
  - How to give **design specifications** in *t*-domain?
  - What are the corresponding conditions in *s*-domain?

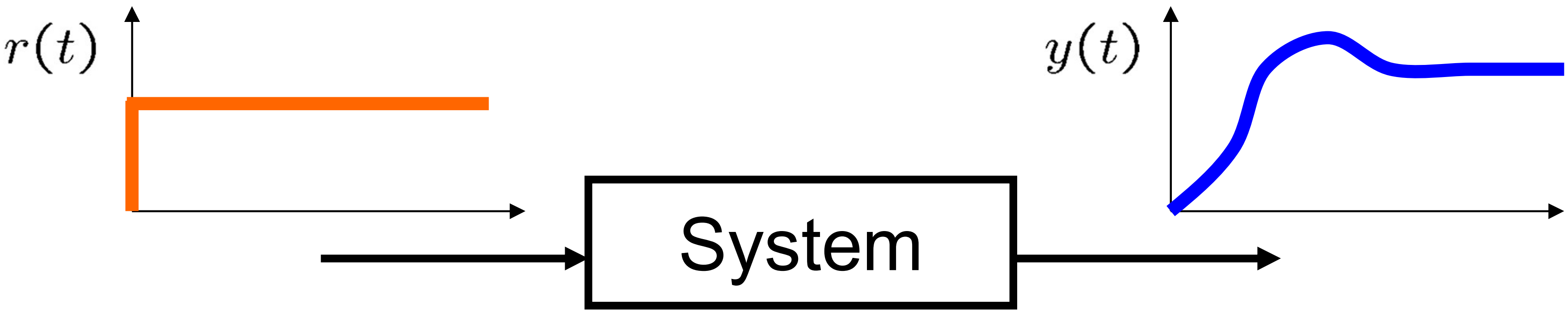
Temperature control example

- We want to change the room temperature from 10 °C to 20 °C.



Are we happy with these responses?

Time response



- We would like to **analyze** our stable system's property by applying a *test input* $r(t)$ and observing its time response $y(t)$.
- Time response is divided into two components as below:

$$y(t) = \underbrace{y_t(t)}_{\text{Transient (natural) response}} + \underbrace{y_{ss}(t)}_{\text{Steady-state (forced) response}}$$

Transient (natural) response

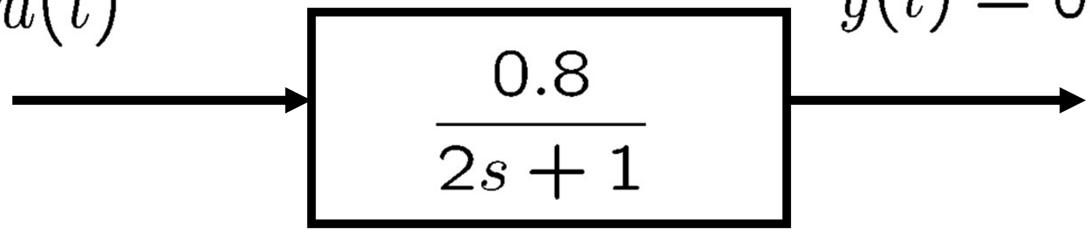
$$\lim_{t \to \infty} y_t(t) = 0$$

Steady-state (forced) response

(after y_t dies out)

Example 1: Transient & steady-state responses

$$r(t) = u(t) \qquad\qquad y(t) = 0.8 - 0.8e^{-\frac{t}{2}}$$

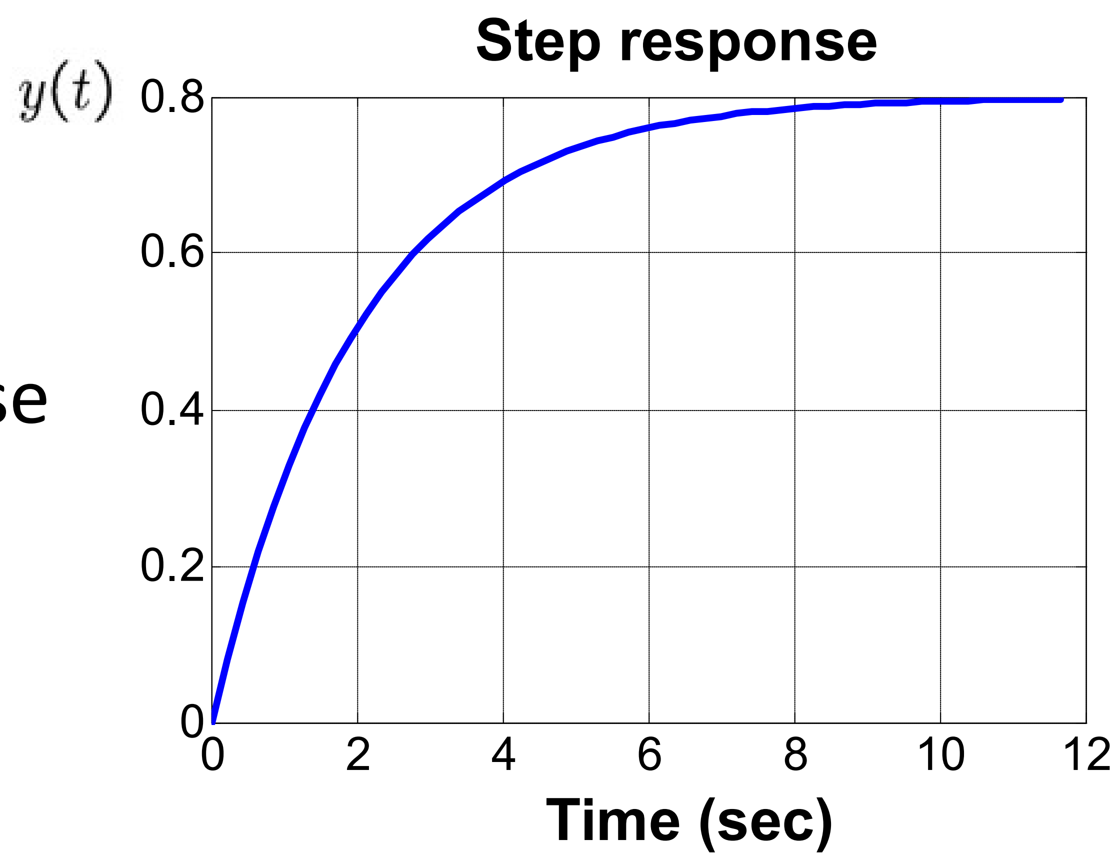


- Transient response

$$y_t(t) = -0.8e^{-\frac{t}{2}}$$

- Steady-state response

$$y_{ss}(t) = 0.8$$

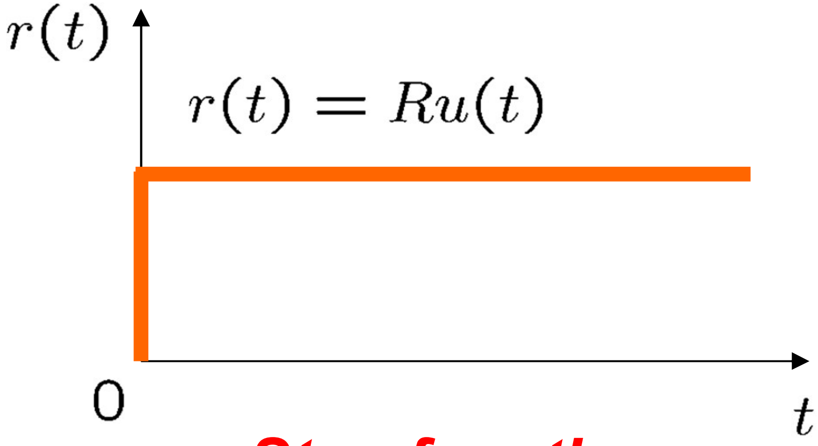


Usage of time responses

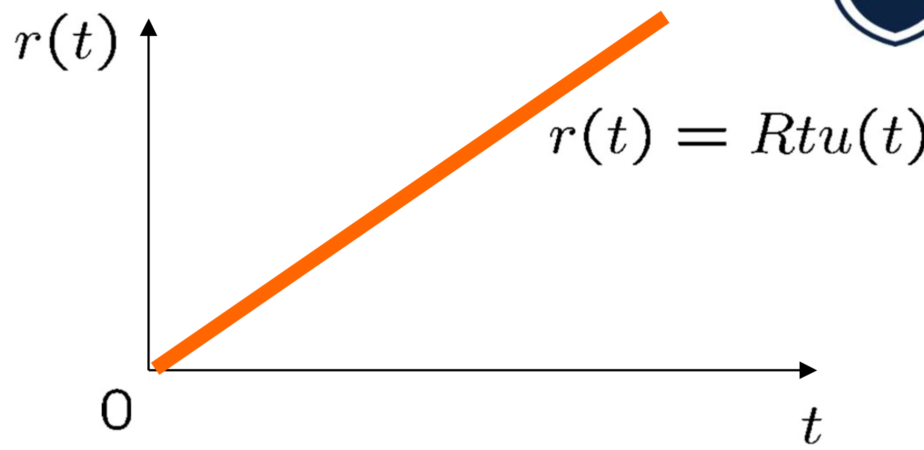
- **Modeling**
  - Some parameters in the system may be estimated by time responses.
- **Analysis**
  - A system can be evaluated by seeing transient and steady-state responses. (Satisfactory or not?)
- **Design**
  - Given design specs in terms of transient and steady-state responses, controllers are designed to satisfy all the design specs.

Note: Although in the course roadmap, time response is placed in the analysis box (since it is commonly employed at this stage), it can also be placed in the modeling and the design boxes.

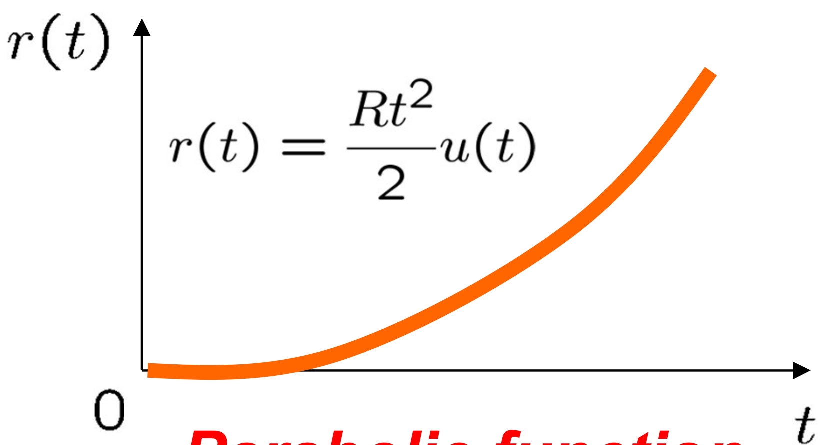
Typical test inputs



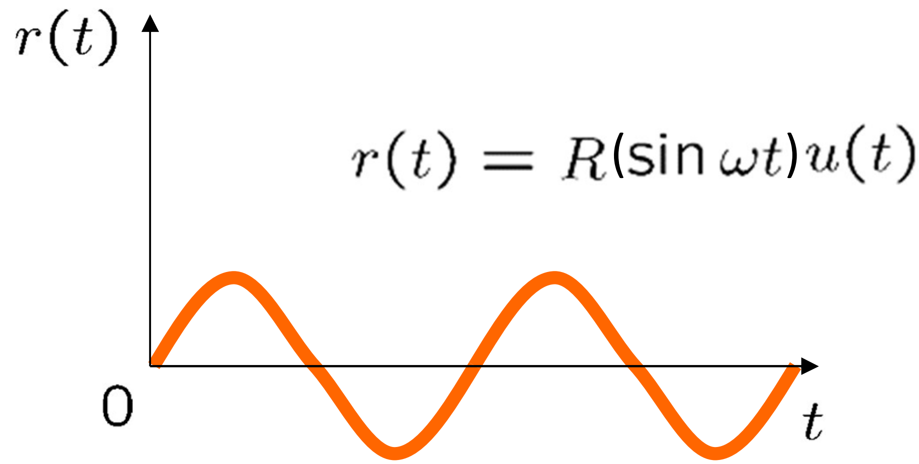
Step function
(Most popular)



Ramp function



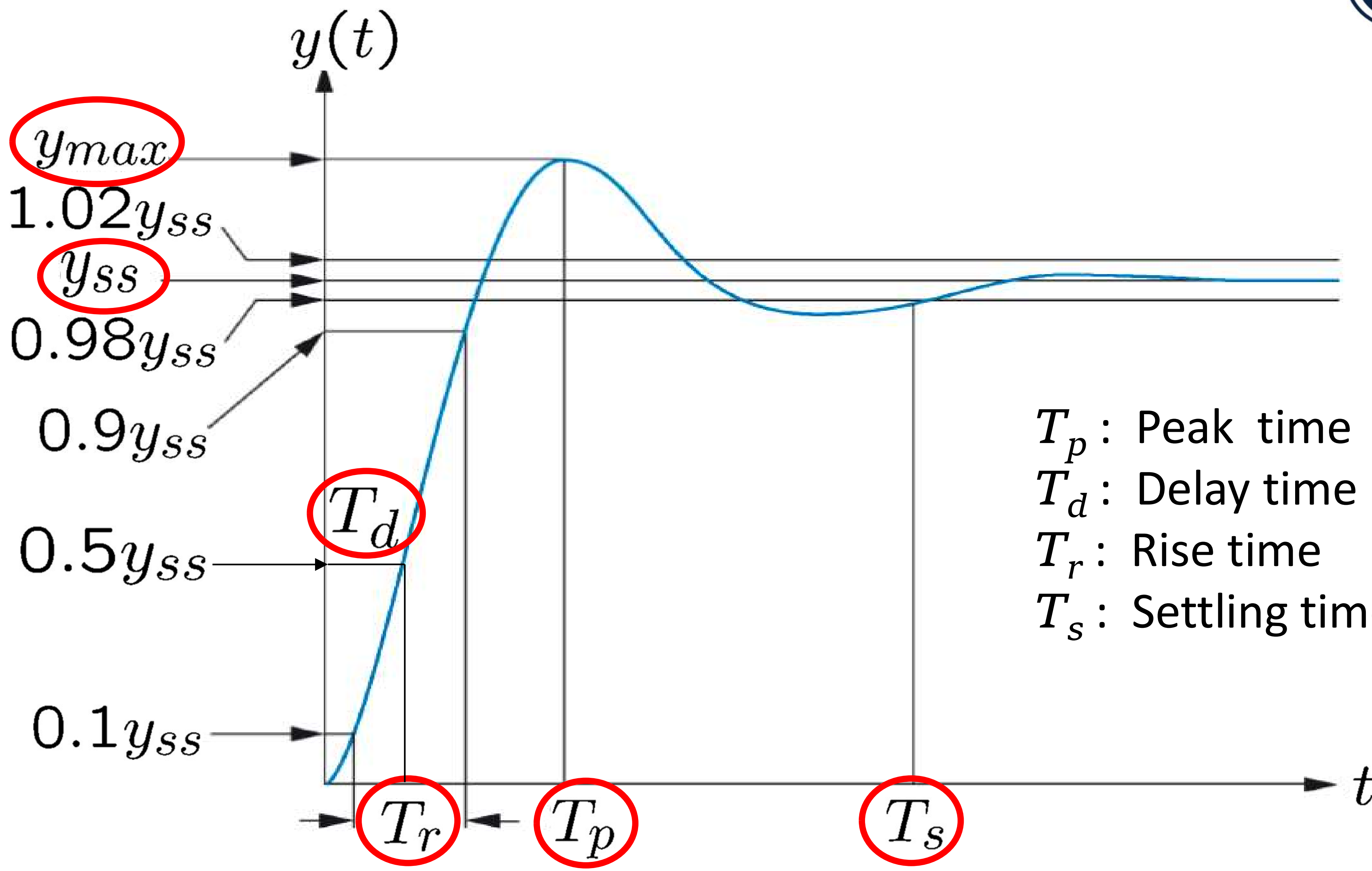
Parabolic function



Sinusoidal function

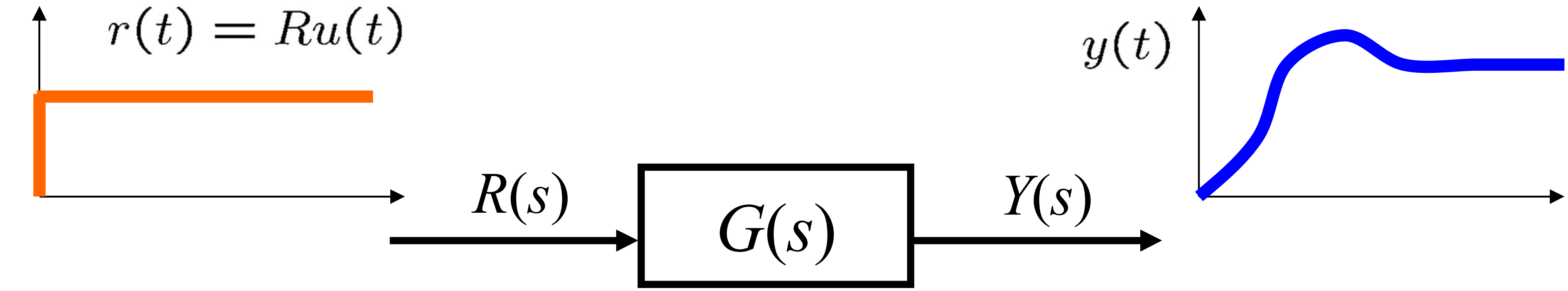
Note: Sinusoidal function will be dealt with later.

Typical step response



T_p : Peak time
T_d : Delay time
T_r : Rise time
T_s : Settling time

Steady-state value for step input



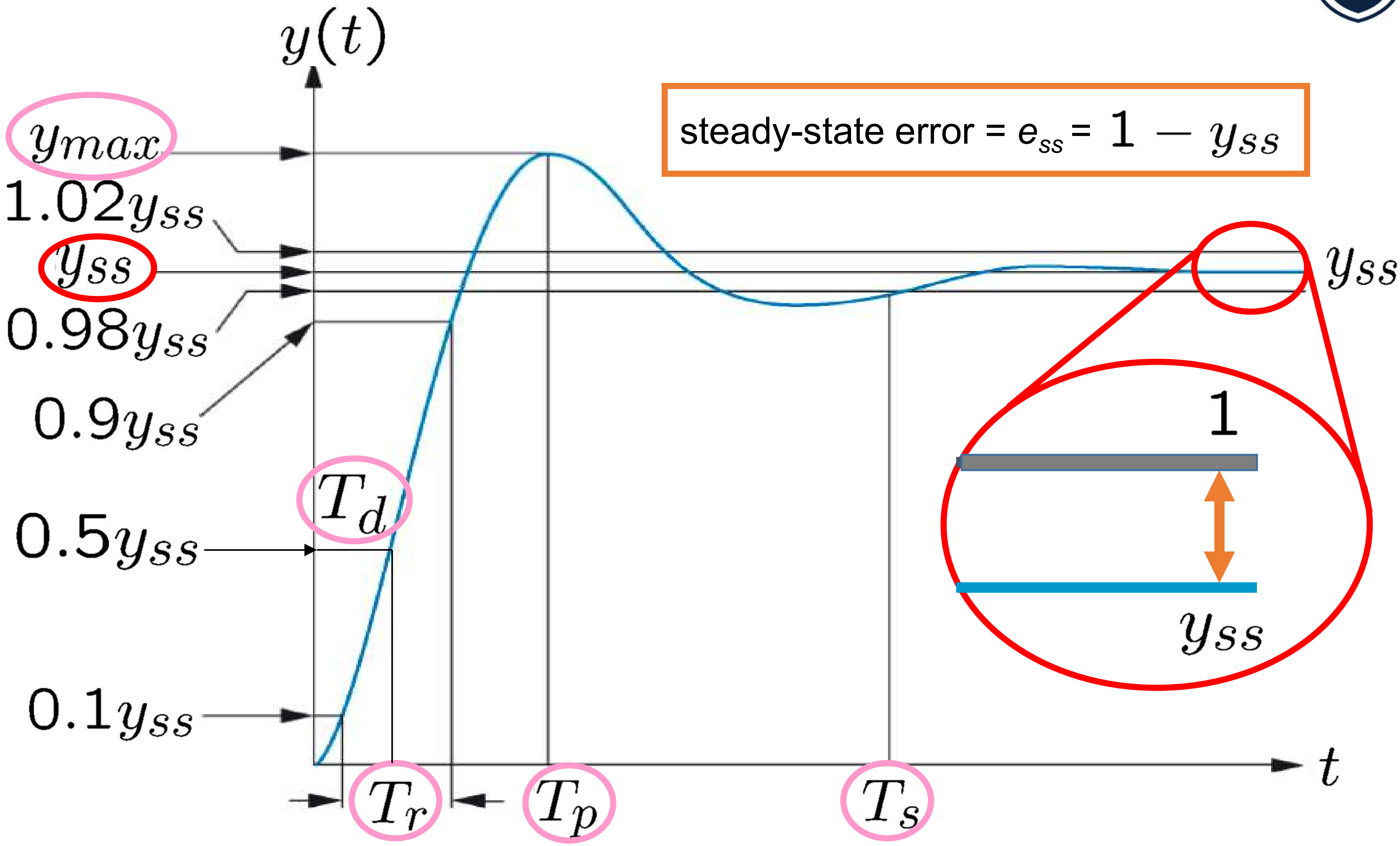
- Suppose that $G(s)$ is stable.
- By the final value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\cdot Y(s) = \lim_{s\to 0} s\cdot G(s)\cdot R(s) = \lim_{s\to 0} s\cdot G(s)\cdot U(s) = \lim_{s\to 0} s\cdot G(s)\cdot \frac{R}{s}$$

$$\longrightarrow \quad \lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s)\frac{R}{s} = RG(0) = y_{ss} \quad \longrightarrow \quad \boxed{y_{ss} = RG(0)}$$

- Step response converges to some finite value, called *steady-state value*, and is shown by y_{ss}.

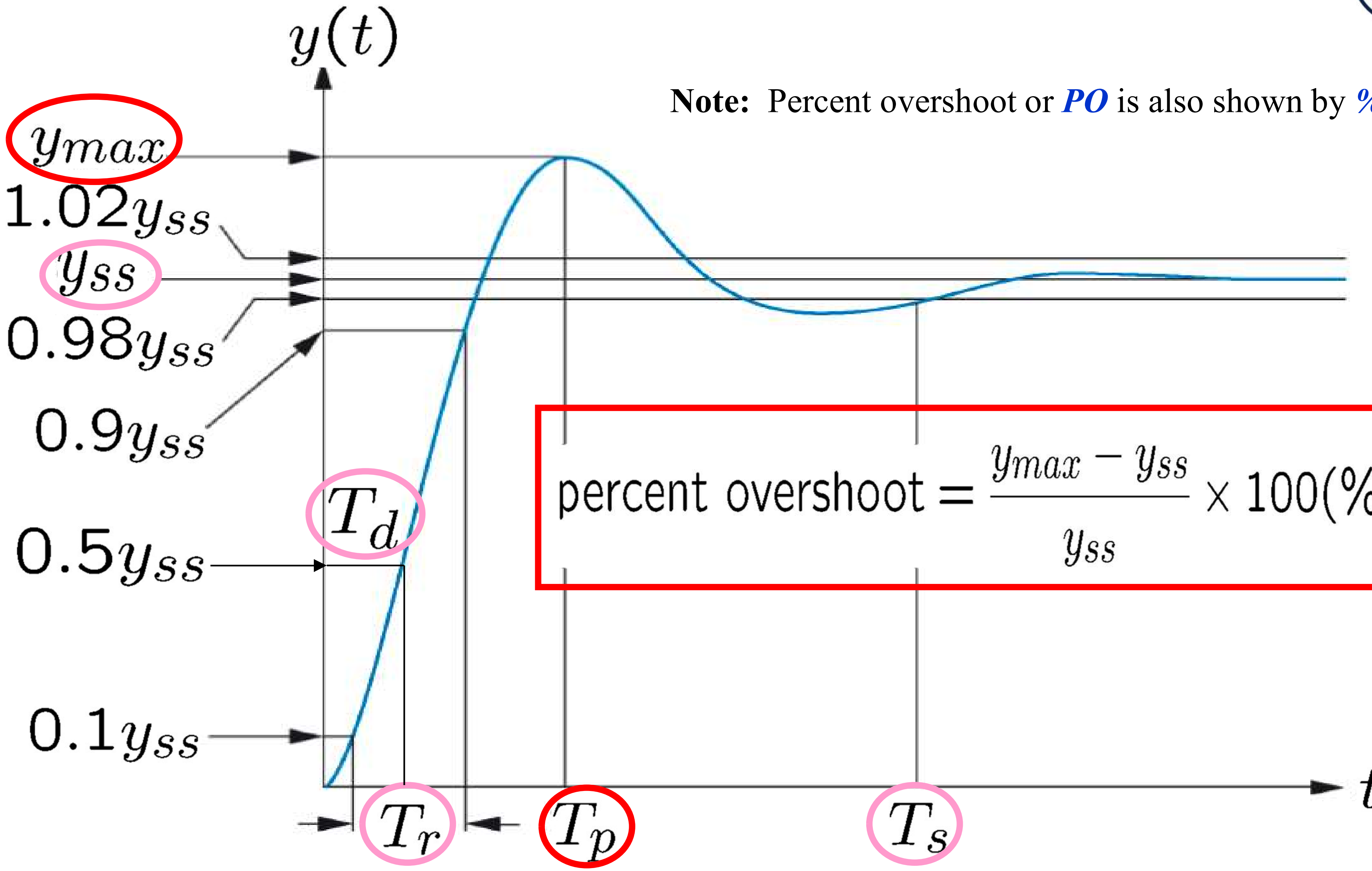
Steady-state error for a unit step input $u(t)$



Note: For a step input of size R, we have: $e_{ss} = R - y_{ss}$

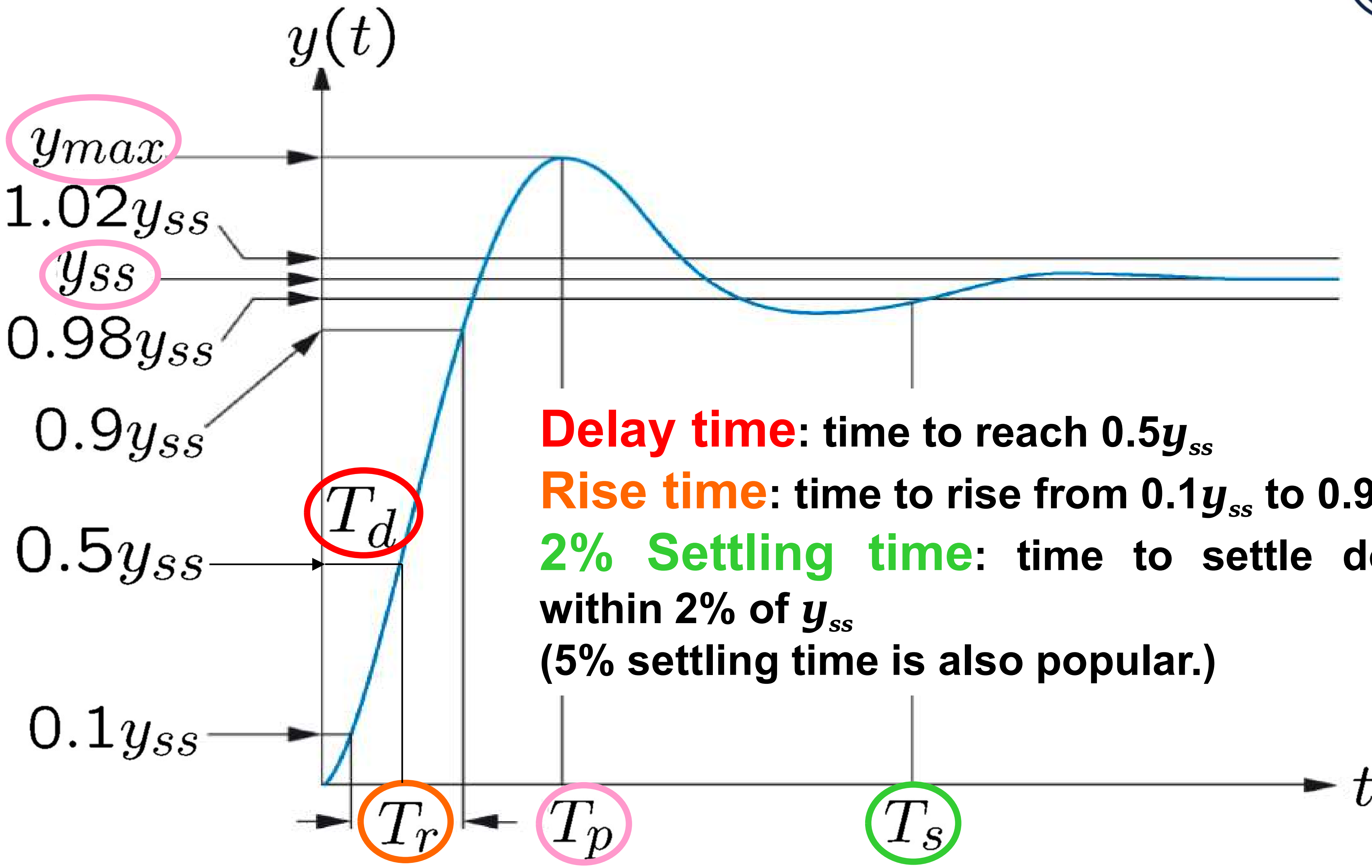
Peak value (y_{max}), peak time (T_p), and percent overshoot (PO)

Note: Percent overshoot or ***PO*** is also shown by ***%OS***.



$$\text{percent overshoot} = \frac{y_{max} - y_{ss}}{y_{ss}} \times 100(\%)$$

Delay, rise, and settling times



Delay time: time to reach $0.5y_{ss}$

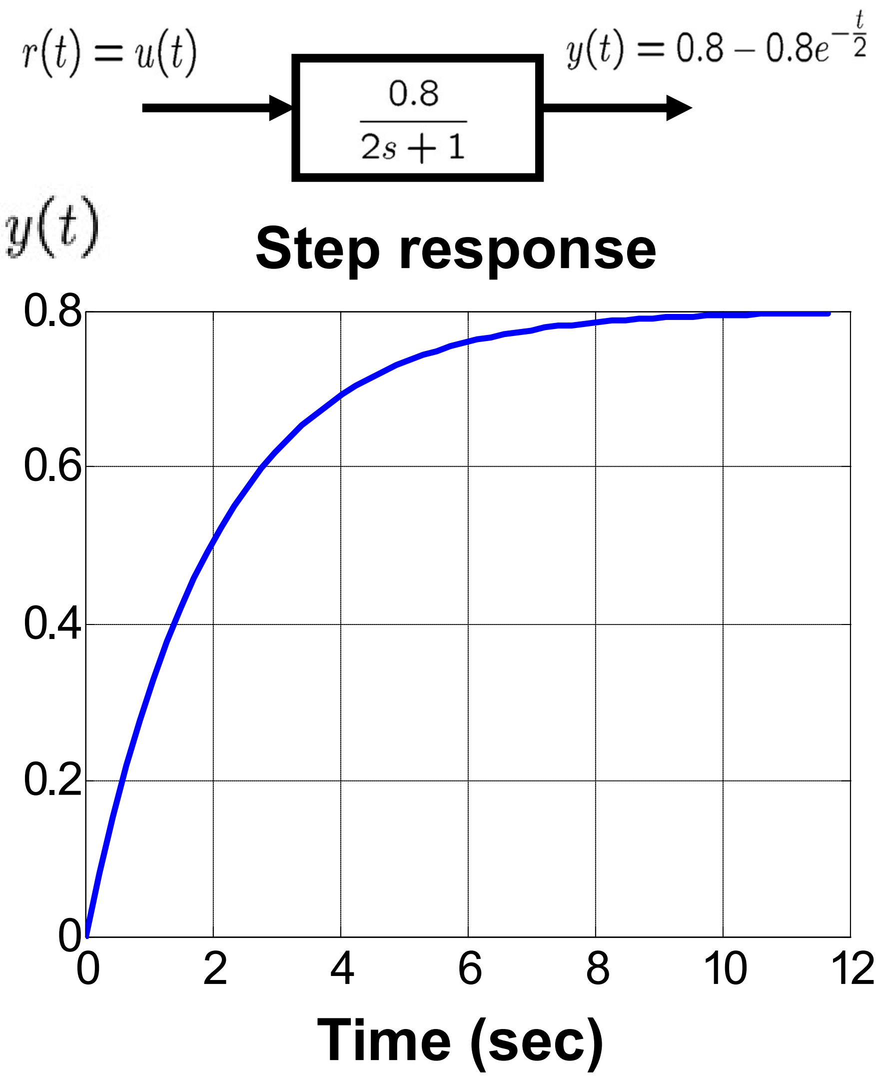
Rise time: time to rise from $0.1y_{ss}$ to $0.9y_{ss}$

2% Settling time: time to settle down within 2% of y_{ss}

(5% settling time is also popular.)

Example 2: (revisiting Example 1)

- For the previous example shown here again:
  - Steady-state error: 1 - 0.8 = 0.2
  - Delay time around 1.4 sec
  - Rise time around 5 sec
  - Settling time around 8 sec
- Remark: There is no peak in this case, so the following are undefined:
  - peak value
  - peak time
  - percent overshoot



Remarks on time responses

- Speed of response is measured by …
  - Rise time, delay time, peak time, and settling time
- Relative stability is measured by …
  - Percent overshoot
- Typically …
  - Fast response (short rise time, short peak time)

  → Large percent overshoot → Small stability margin
- In controller design, we normally face a trade-off between response speed and stability.

  (“*No-free-lunch theorem*” in Control Engineering!)

Performance measures

- Transient response ← (Next lecture)
  - Peak value
  - Peak time
  - Percent overshoot
  - Delay time
  - Rise time
  - Settling time
- Steady state response
  - Steady state error ← (Today's lecture)

Next, we will connect these performance measures with s-domain.

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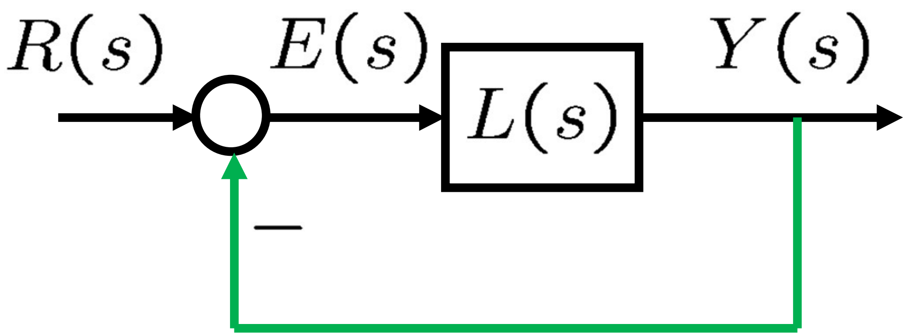
Matlab simulations

Steady-state error of a unity feedback system



Assumptions:

- $L(s)$ = Controller(s)×Plant(s)×…
- Unity feedback (no block on feedback path)
- CL system is stable

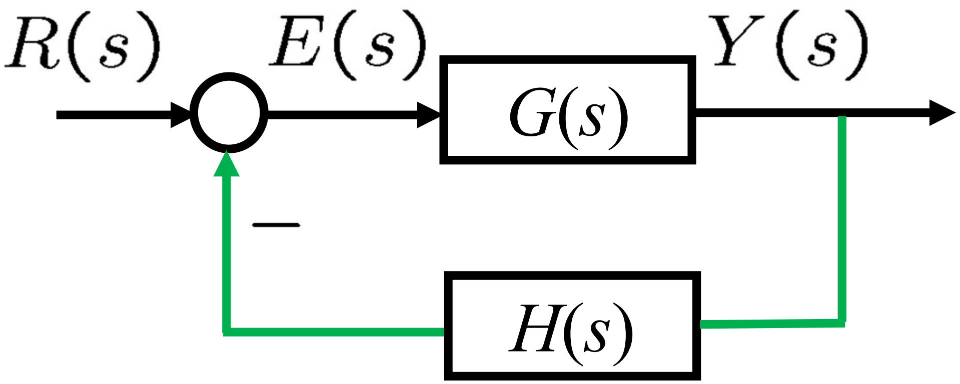
- Suppose that we want output $y(t)$ to track $r(t)$.
- Error $e(t) = r(t) - y(t)$
- Steady-state error:

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+L(s)}R(s)$$

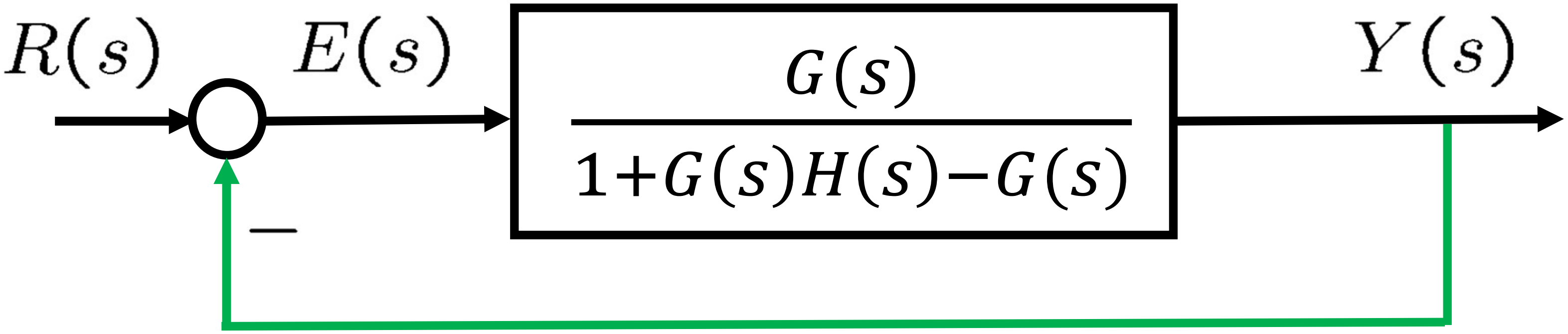
Final value theorem
(Suppose CL system is stable!)

$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+L(s)}R(s)$$

Steady-state error of a non-unity feedback system



Convert to unity feedback system



$$L(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)}$$

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s\frac{1}{1 + L(s)}R(s)$$

Error constants

- **Error constant** reflects the CL system's ability to reduce steady-state error e_{ss}.
  - "Large error constant" means "large ability".
- Three error constants (also called **static error constants**) are:
  - Step-error (**p**osition-error) constant:

$$K_p = \lim_{s \to 0} L(s)$$

  - Ramp-error (**v**elocity-error) constant:

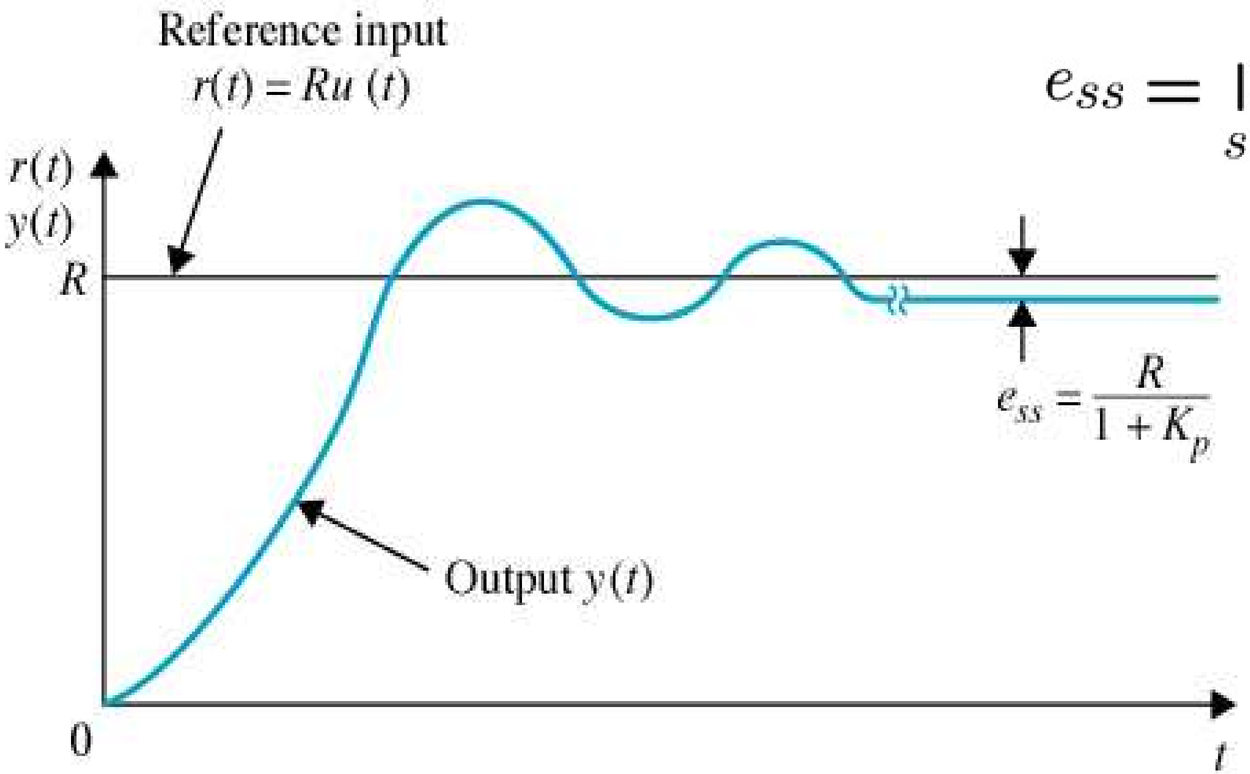
$$K_v = \lim_{s \to 0} sL(s)$$

  - Parabolic-error (**a**cceleration-error) constant:

$$K_a = \lim_{s \to 0} s^2 L(s)$$

Steady-state error for step *r*(*t*)

$$r(t) = Ru(t) \Rightarrow \boxed{e_{ss} = \frac{R}{1+K_p}}$$

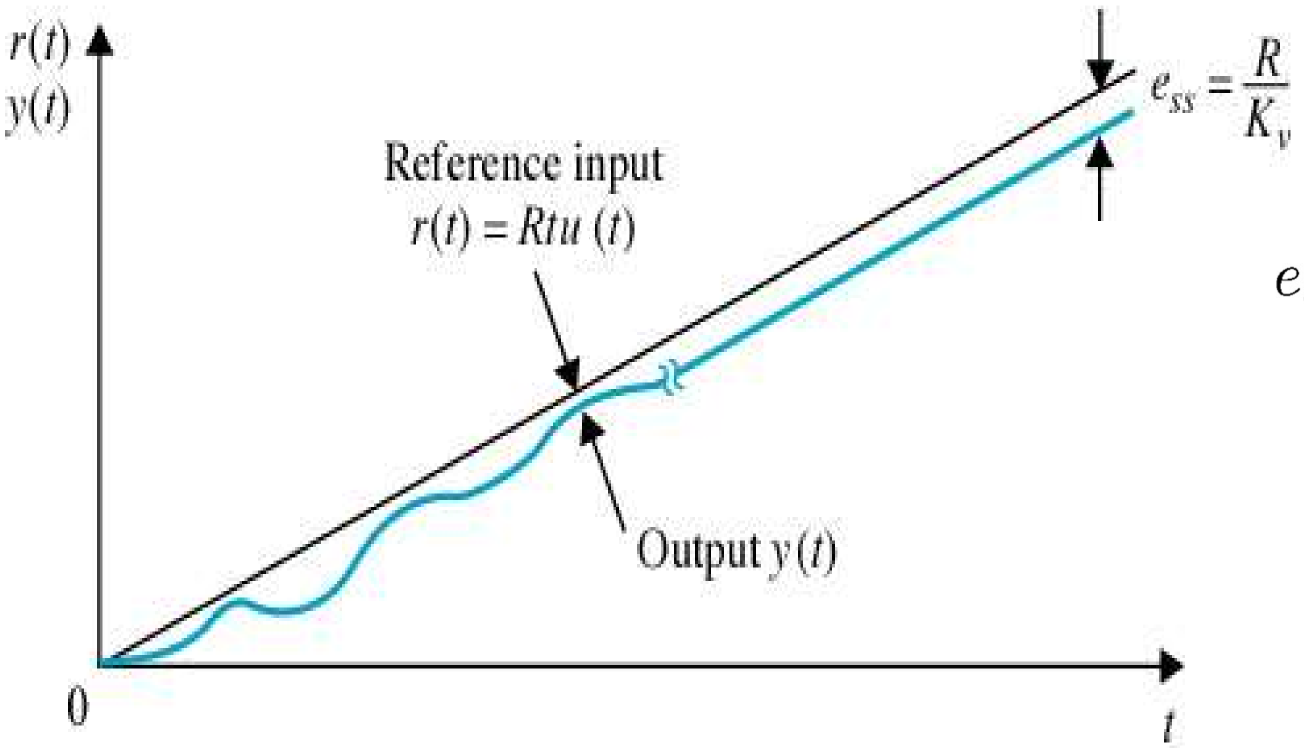


$$e_{ss} = \lim_{s\to 0} s\frac{1}{1+L(s)}R(s) \longrightarrow$$

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s\frac{1}{1+L(s)}\cdot\frac{R}{s} \\ &= \frac{R}{1+\underbrace{L(0)}_{K_p}} \end{aligned}$$

Steady-state error for ramp $r(t)$

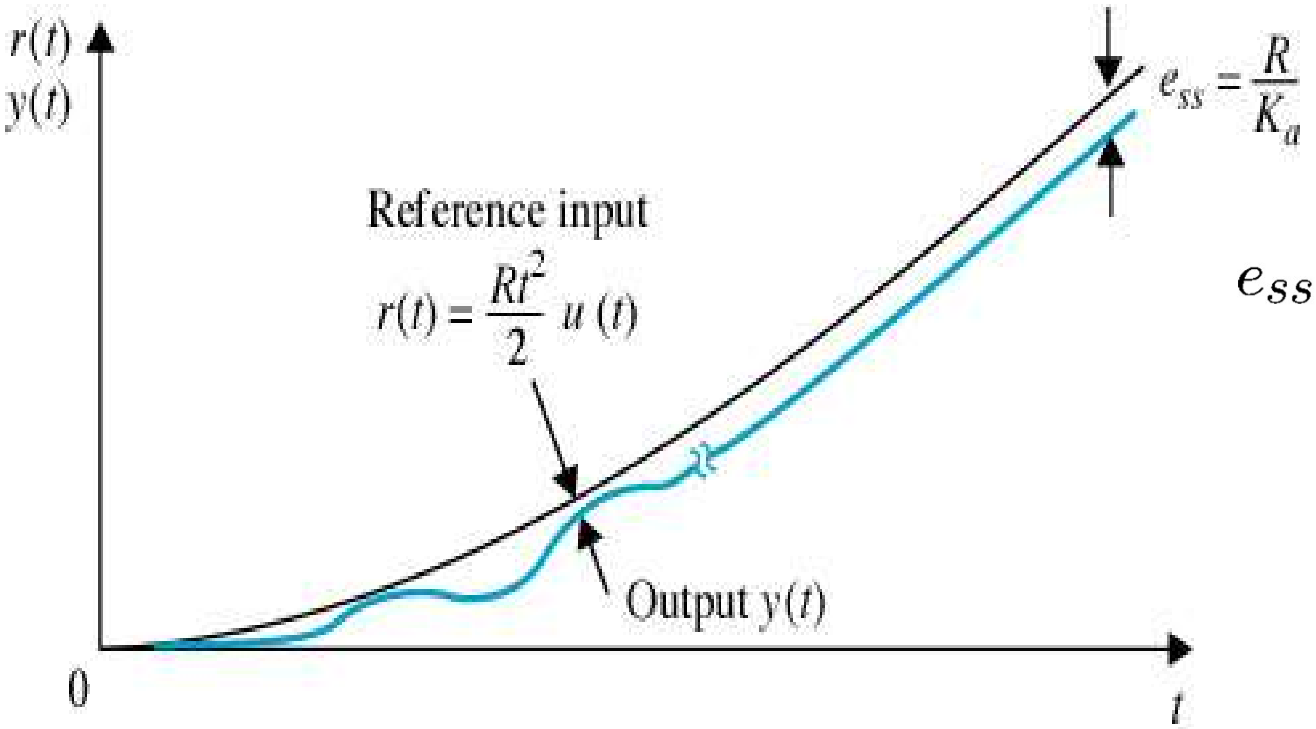
$$r(t) = Rtu(t) \Rightarrow \boxed{e_{ss} = \frac{R}{K_v}}$$



$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s \frac{1}{1+L(s)} \cdot \frac{R}{s^2} \\ &= \frac{R}{\underbrace{\lim_{s\to 0} sL(s)}_{K_v}} \end{aligned}$$

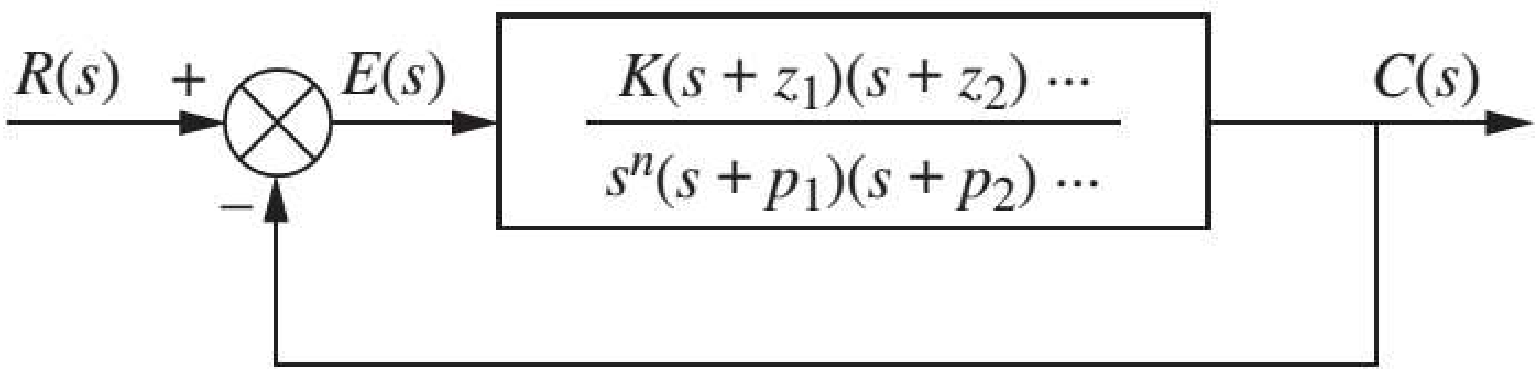
Steady-state error for parabolic $r(t)$

$$r(t) = \frac{Rt^2}{2}u(t) \Rightarrow \boxed{e_{ss} = \frac{R}{K_a}}$$



$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s\frac{1}{1+L(s)}\cdot\frac{R}{s^3} \\ &= \frac{R}{\underbrace{\lim_{s\to 0} s^2 L(s)}_{K_a}} \end{aligned}$$

System Type



Feedback control system for defining system type

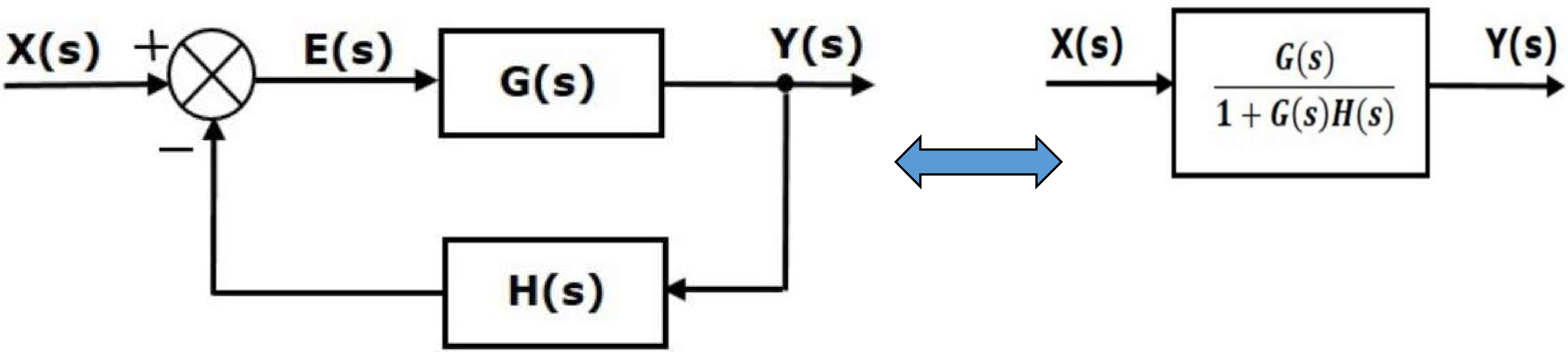
- We define system type to be the value of n in the denominator or, equivalently, the ***number of pure integrators*** in the forward path. Therefore, a system with $n = 0$ is a **Type 0** system. If $n = 1$ or $n = 2$, the corresponding system is a **Type 1** or **Type 2** system, respectively.

Zero steady-state error

- When does steady-state error become zero? (i.e., **accurate tracking**!)
- Infinite error constant!
  - For step $r(t)$: $K_p = \lim_{s\to 0} L(s) = \infty$
    - ➡ $L(s)$ must have at least 1-integrator. (Type 1 system)
  - For ramp $r(t)$: $K_v = \lim_{s\to 0} sL(s) = \infty$
    - ➡ $L(s)$ must have at least 2-integrators. (Type 2 system)
  - For parabolic $r(t)$: $K_a = \lim_{s\to 0} s^2L(s) = \infty$
    - ➡ $L(s)$ must have at least 3-integrators. (Type 3 system)

Characteristic Equation (review)

- The following figure shows a negative feedback control system. Here, two blocks having transfer functions $G(s)$ and $H(s)$ form a closed loop.



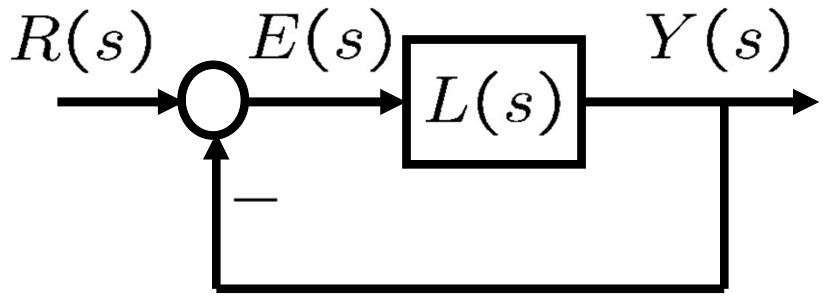
$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$1 + G(s)H(s) = 0$ is called the **Characteristic Equation**.

Example 3: Stability and e_{ss}

- $L(s)$ has 2-integrators.

$$L(s) = \frac{K}{s^2(s+12)}$$



- Characteristic equation:

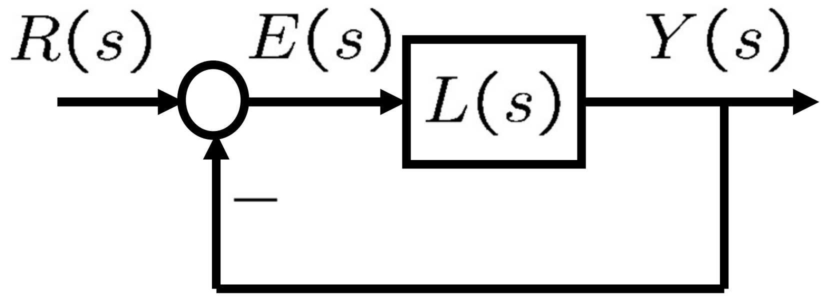
$$1+L(s)=0 \Leftrightarrow s^2(s+12)+K=0 \Leftrightarrow s^3+12s^2+K=0$$

- CL system is NOT stable for any K (use Routh-Hurwitz criterion). **Unstable.**
- $e(t)$ will not converge. (Do not use today's results if CL system is not stable!). **e_{ss} equation is not applicable.**

Example 4: Stability and e_{ss}

- $L(s)$ has 1-integrator.

$$L(s) = \frac{K(s+3.15)}{s(s+1.5)(s+0.5)}$$



- By Routh-Hurwitz criterion, CL is **stable** if $0 < K < 1.304$

- Step *r*(*t*): $K_p = \lim_{s\to 0} L(s) = \infty$ ⟹ $e_{ss} = \frac{R}{1+K_p} = 0$

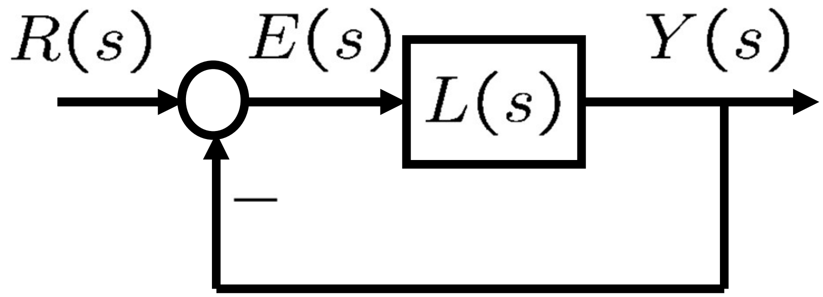
- Ramp *r*(*t*): $K_v = \lim_{s\to 0} sL(s) = \frac{3.15K}{0.75} = 4.2K$ ⟹ $e_{ss} = \frac{R}{K_v} = \frac{R}{4.2K}$

- Parabolic *r*(*t*): $K_a = \lim_{s\to 0} s^2 L(s) = 0$ ⟹ $e_{ss} = \frac{R}{K_a} = \infty$

Example 5: Stability and e_{ss}

- $L(s)$ has 2-integrators.

$$L(s) = \frac{5(s+1)}{s^2(s+12)(s+5)}$$



- By Routh-Hurwitz criterion, we can show that CL system is **stable**.
- Step $r(t)$: $K_p = \lim_{s \to 0} L(s) = \infty$ ⟹ $e_{ss} = \frac{R}{1+K_p} = 0$
- Ramp $r(t)$: $K_v = \lim_{s \to 0} sL(s) = \infty$ ⟹ $e_{ss} = \frac{R}{K_v} = 0$
- Parabolic $r(t)$: $K_a = \lim_{s \to 0} s^2 L(s) = \frac{1}{12}$ ⟹ $e_{ss} = \frac{R}{K_a} = 12R$

Integrators in $L(s)$

- Integrators in $L(s)$ (i.e., plant, controller, etc.) are very powerful to eliminate the steady-state errors.
  - Examples 4 & 5
- However, integrators in $L(s)$ might destabilize the feedback system.
  - Example 3
  - To be explained later in "Nyquist stability criterion."

("***No-free-lunch theorem***" again.)

Summary

- Time response and time domain specifications.
- Steady-state error
  - For stable unity feedback systems and for a specific type of input (step, ramp, parabolic, etc.), the number of integrators determines if the steady-state error is zero.
  - The key mathematical tool is the final value theorem.
- Next
  - Step responses of 1^{st} and 2^{nd} order systems.