# ELEC 341: Systems and Control

## Lecture 9

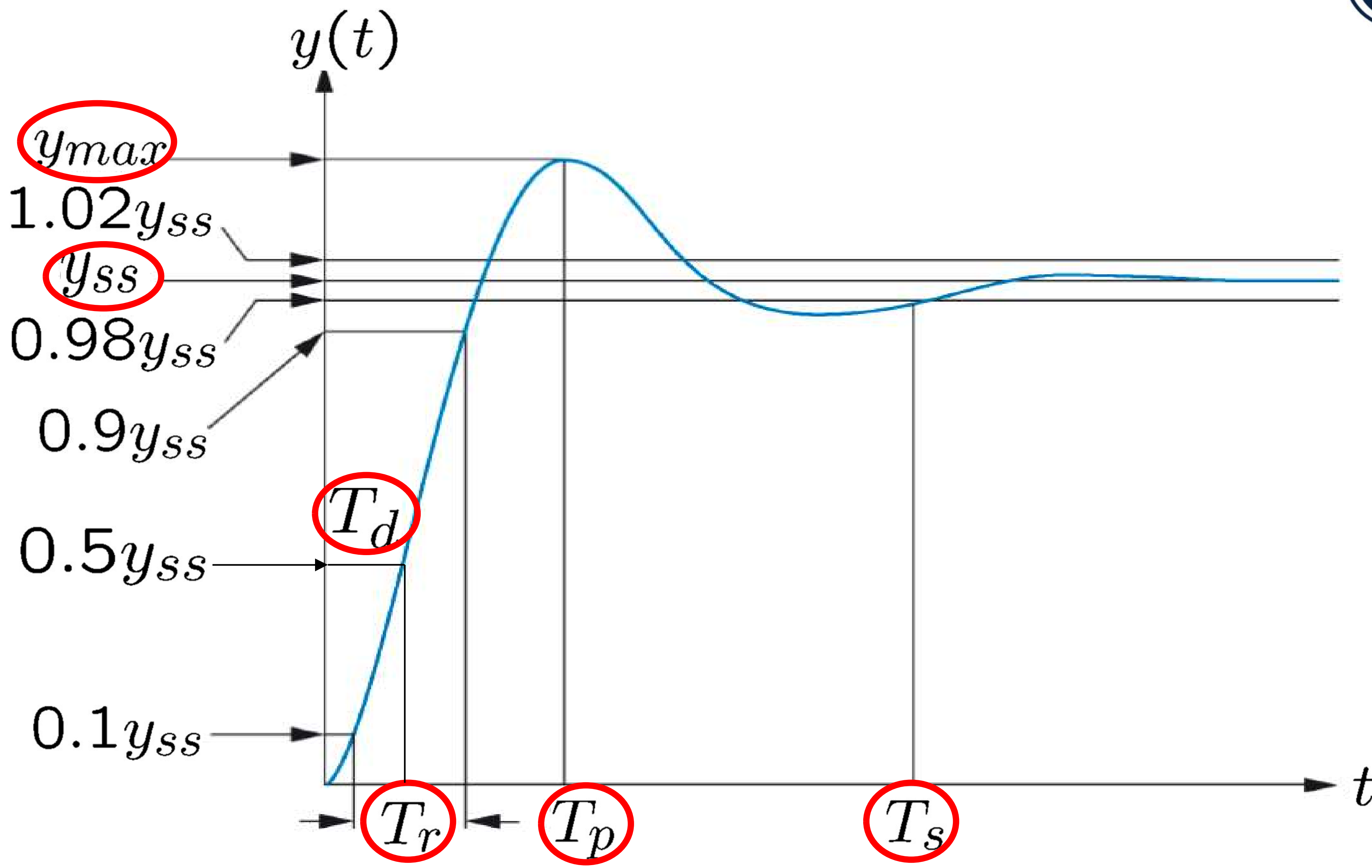
## Step responses of 1st and 2nd order systems

# Course roadmap

| *Modeling* | *Analysis* | *Design* |
| --- | --- | --- |
| ✓ Laplace transform | ✓ Stability | Design specs |
| ✓ Transfer function | ✓ • Routh-Hurwitz | Root locus |
| Models for systems | • Nyquist | Frequency domain |
| ✓ • Electrical | ✓ Time response | PID & Lead-lag |
| ✓ • Electromechanical | ➤ • Transient | Design examples |
| ✓ • Mechanical | ✓ • Steady state | |
| ✓ Linearization, delay | Frequency response | |
| | • Bode plot | |

*Matlab simulations*

# Typical step response (review)



# Performance measures

- Transient response
  - Peak value
  - Peak time
  - Percent overshoot
  - Delay time
  - Rise time
  - Settling time
- Steady state response
  - Steady state error

(Today's lecture)

***Next, we will connect these performance measures with s-domain.***

(Done)

# Today's topics

- Characterization of step responses (**performance measures**) for 1st-order and 2nd-order systems in terms of **(1) system parameters** and **(2) pole locations**:

  - 1st-order system:

$$G(s) = \frac{K}{Ts+1}$$

  - 2nd-order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

  - System parameters are: $(K,\ T)$, $(\zeta,\ \omega_n)$

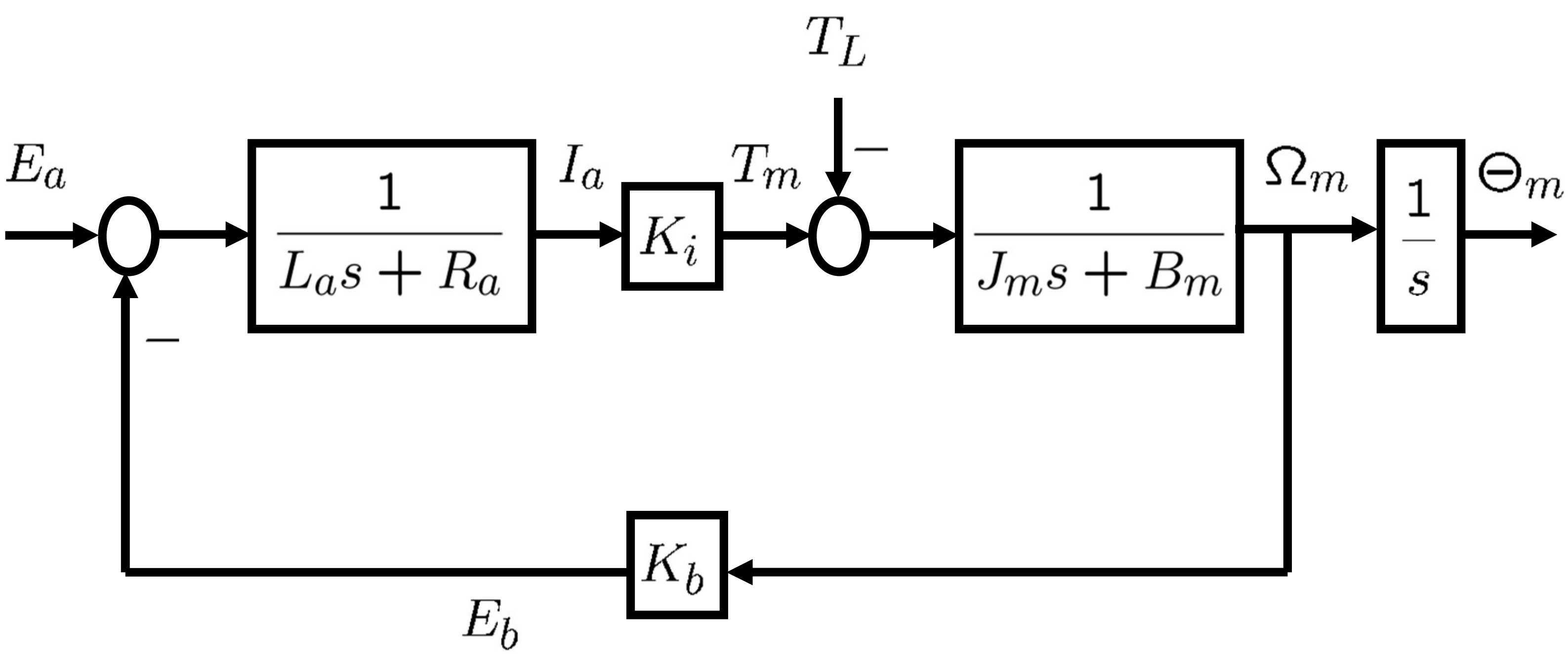
**Note:** For second-order systems, it is not necessary to have $\omega_n^2$ in the numerator of $G(s)$.

# First-order system

- A standard form of the **first-order system**:

$$G(s) = \frac{K}{Ts+1}$$

- DC motor example (See L5)



# DC motor example (cont'd)

- If $L_a s << R_a$, we can obtain a 1st-order system:

$$\frac{\Omega_m(s)}{E_a(s)} = \frac{K_i}{(L_a s + R_a)(J_m s + B_m) + K_b K_i} \approx \frac{K_i}{R_a(J_m s + B_m) + K_b K_i}$$

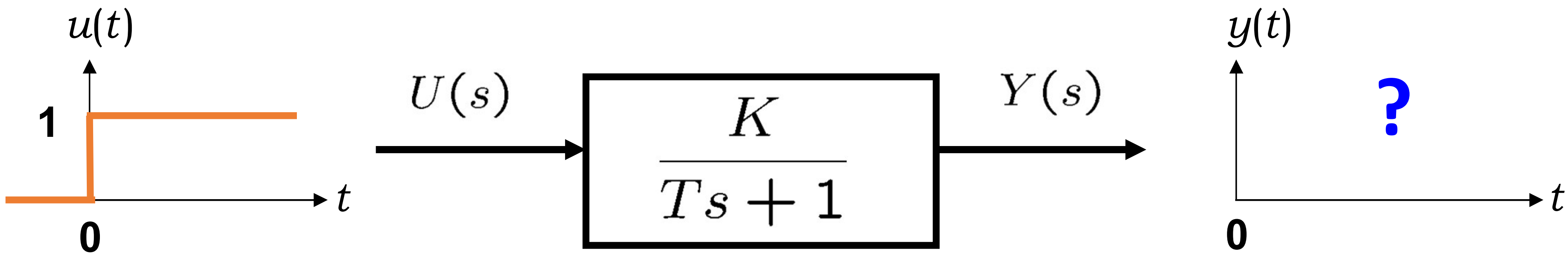
$$= \frac{K}{Ts+1} \quad \left( K = \frac{K_i}{R_a B_m + K_b K_i}, \; T = \frac{R_a J_m}{R_a B_m + K_b K_i} \right)$$

*2nd order system* → *1st order system*

- Remember that:
  - TF from motor voltage ($E_a$) to motor speed ($\Omega_m$) is 1st-order (after using the approximation)
  - TF from motor voltage ($E_a$) to motor position ($\Theta_m$) is 2nd-order

# Step response of 1$^{st}$-order system

- Input a unit step function to a first-order system. What is the output?




$$
\begin{aligned}
Y(s) &= G(s)U(s) \\
&= \frac{K/T}{s+1/T} \cdot \frac{1}{s} \\
&= \frac{K}{s} + \frac{-K}{s+1/T}
\end{aligned}
$$

(Partial fraction expansion)

$\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$\boxed{y(t) = K(1 - e^{-t/T})}$$

$$(t > 0)$$

# Meaning of $K$ and $T$

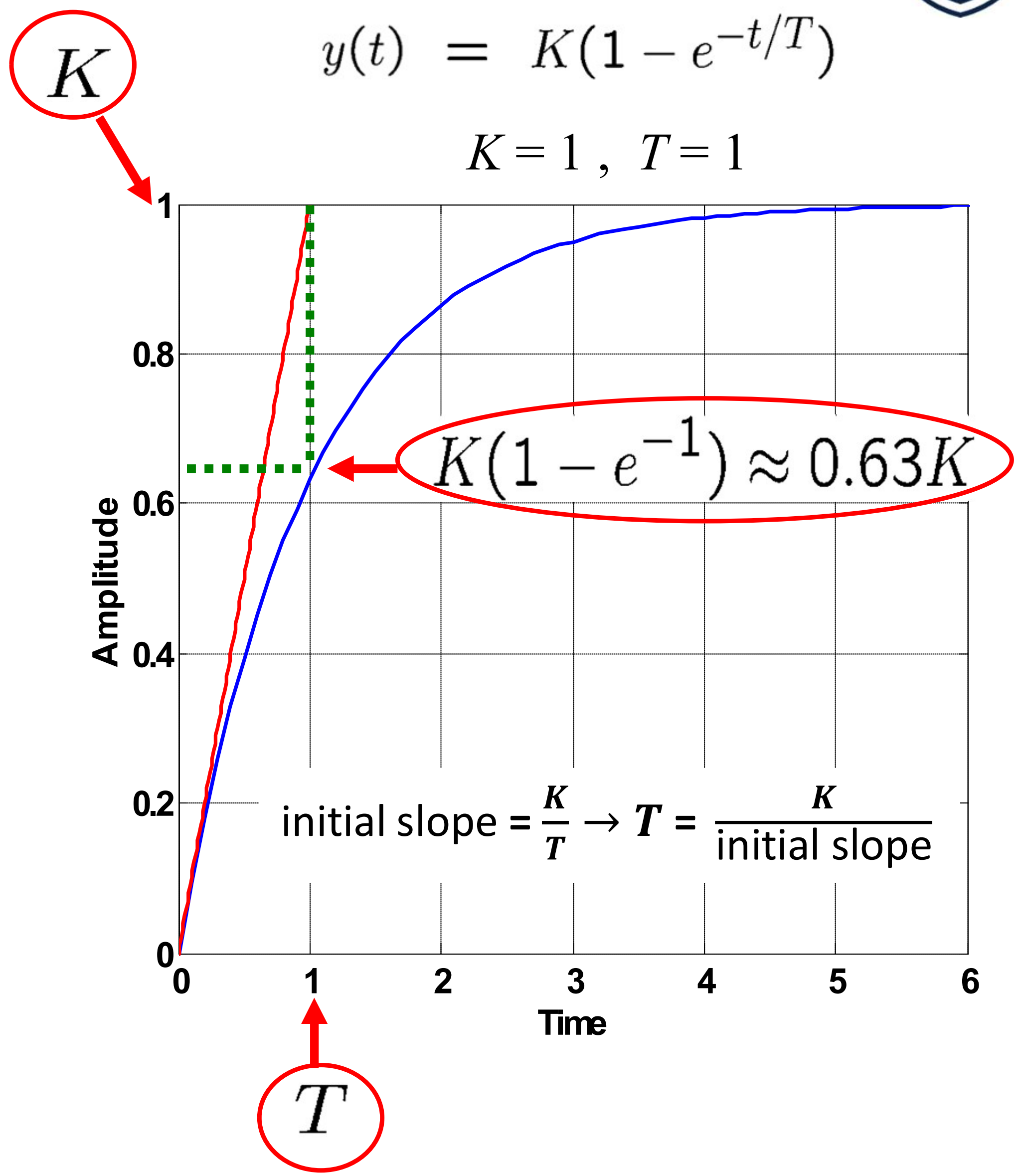
- $K$ : DC gain (next slide)
  - Final (steady-state) value

$$\lim_{t\to\infty} y(t) = K = y_{ss}$$

- $T$ : Time constant
  - Time when response rises to 63% of final value
  - Indication of speed of response (convergence)
  - Response is faster as $T$ (also shown by $\tau$) becomes smaller.

$$y(t) = K(1 - e^{-t/T})$$

$$K = 1\,,\ T = 1$$



# DC gain for a stable system

- DC gain: Final value of a unit step response
  - For a stable system $G$, DC gain is $G(0)$.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s)\frac{1}{s} = G(0)$$

*Final value theorem*

- Example:

$$G(s) = \frac{3}{2s+5} \longrightarrow G(0) = \frac{3}{5}$$

**Note:** The formula for the DC gain can also be used for systems of any other order. For example, if you have a 2nd order system, you can use the same formula to find the DC gain.

# Settling time of 1st-order systems

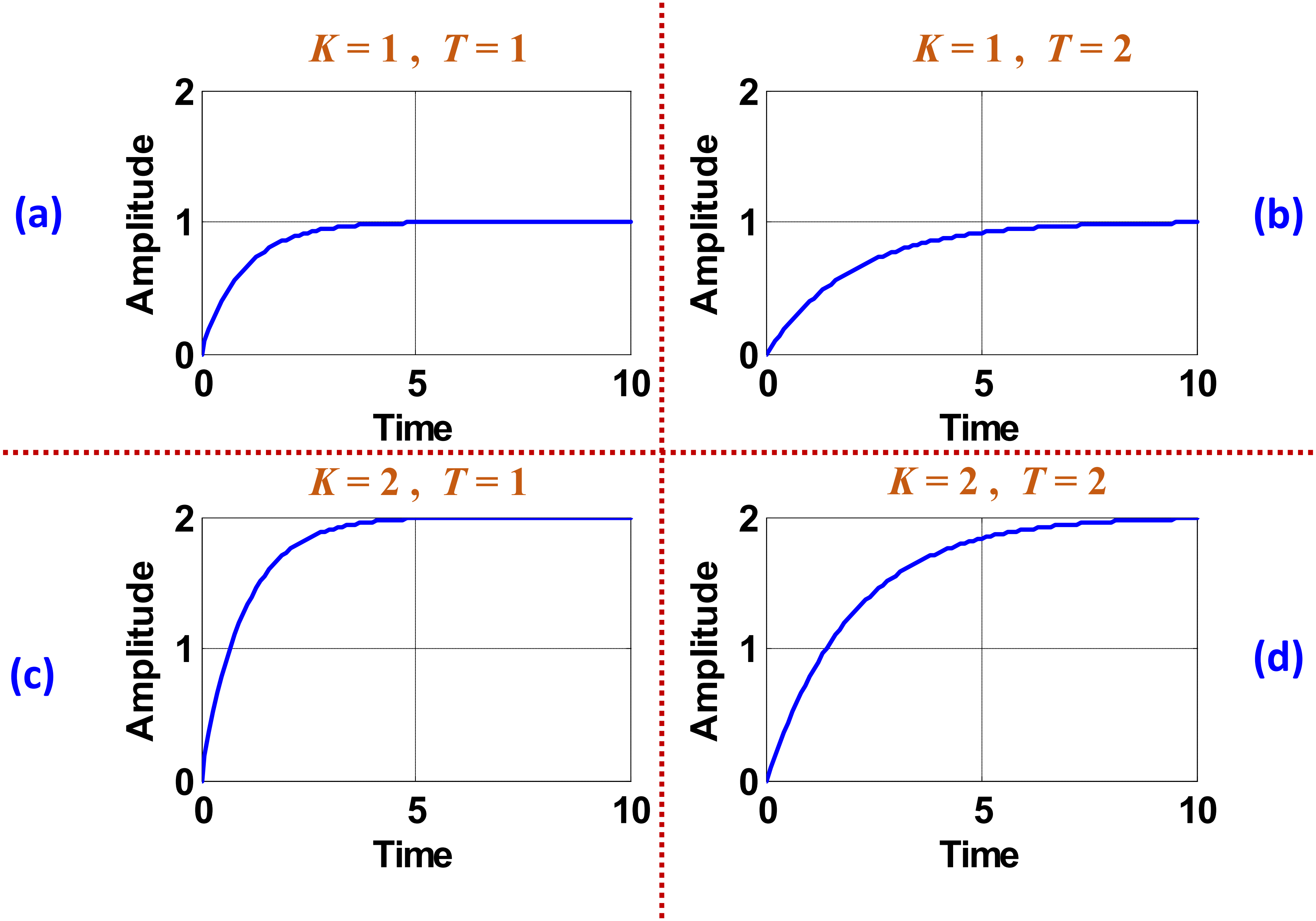
$$y(t) = K(1 - e^{-t/T}) \quad \text{or} \quad y(t) = y_{ss}(1 - e^{-t/T})$$

- Relation between time and exponential decay ($K = y_{ss}$):

| $t$ | $e^{-\frac{t}{T}}$ | $y(t)$ |
|---|---|---|
| 0 | 1 | 0 |
| $T$ | 0.3679 | $0.6321y_{ss}$ |
| $2T$ | 0.1353 | $0.8647y_{ss}$ |
| **$3T$** | $0.0498 \approx$ **5%** | $0.9502y_{ss} \approx$ **$95\%y_{ss}$** |
| **$4T$** | $0.0183 \approx$ **2%** | $0.98171y_{ss} \approx$ **$98\%y_{ss}$** |
| $5T$ | 0.0067 | $0.9933y_{ss}$ |

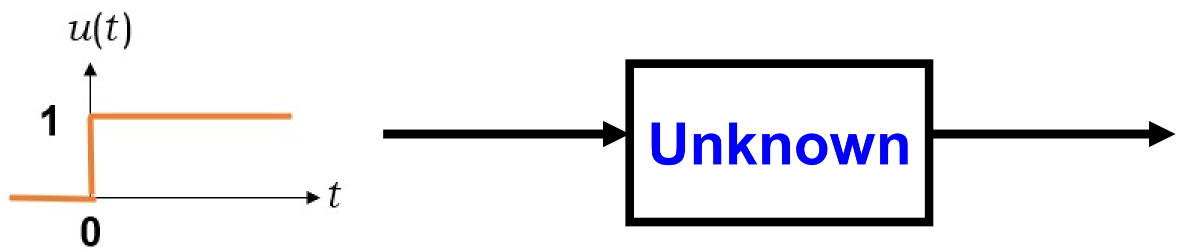
**5% settling time is about $3T$**
**2% settling time is about $4T$**

# Step response for some $K$ & $T$

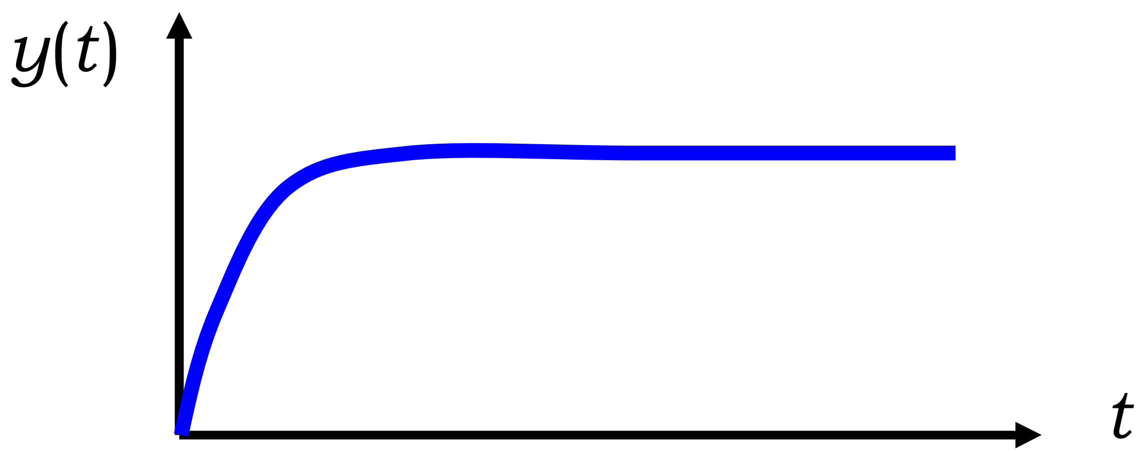


# System identification

- Suppose that we have a “black-box” system:




- Obtain step response experimentally:



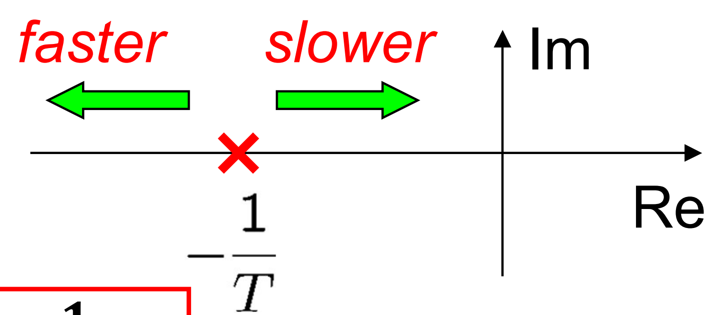

- Can you obtain a transfer function? How?

# Summary: Step response of 1st order systems

$$G(s) = \frac{K}{Ts + 1}$$

- For 1st order systems, step responses have:
  - Steady-state value: $\boldsymbol{K}$
  - Peak value, peak time, percent overshoot: **undefined**
  - Delay time: **0.7**$\boldsymbol{T}$
  - Rise time: **2.2**$\boldsymbol{T}$
  - Settling time:
    - 2%: **4**$\boldsymbol{T}$
    - 5%: **3**$\boldsymbol{T}$
  - Characterization in terms of poles:

× is used to show **poles**




$$Ts + 1 = 0 \rightarrow s = -\frac{1}{T} \rightarrow |\text{pole}| = \frac{1}{T} \rightarrow \boxed{\boldsymbol{T} = \frac{\mathbf{1}}{|\textbf{pole}|}}$$

# Second-order systems

- A standard form of the **second-order system**:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \begin{cases} \zeta : & \text{damping ratio} \\ \omega_n : & \text{undamped natural frequency} \end{cases}$$

**Note 1:**

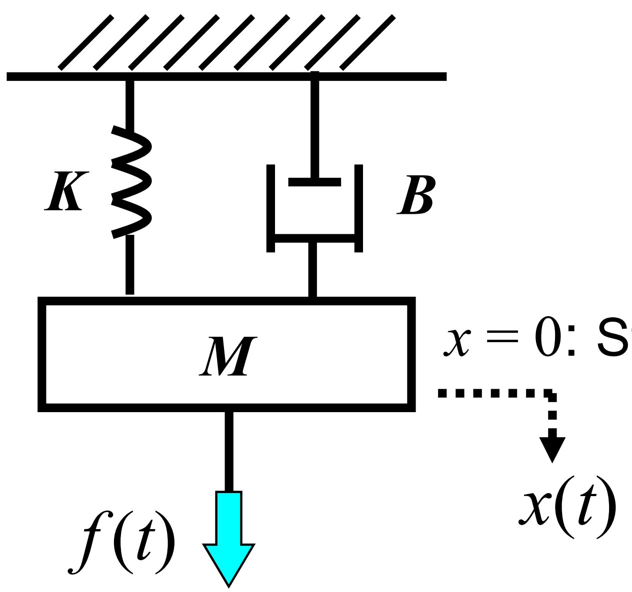
For second-order systems, it is not necessary to have $\omega_n^2$ in the numerator of $G(s)$.

**Note 2:**

Other names for $\zeta$ are **damping factor** and **damping coefficient**.

# Example 1:
# Second-order system (a mechanical system)

- Mass spring damper system (L4):



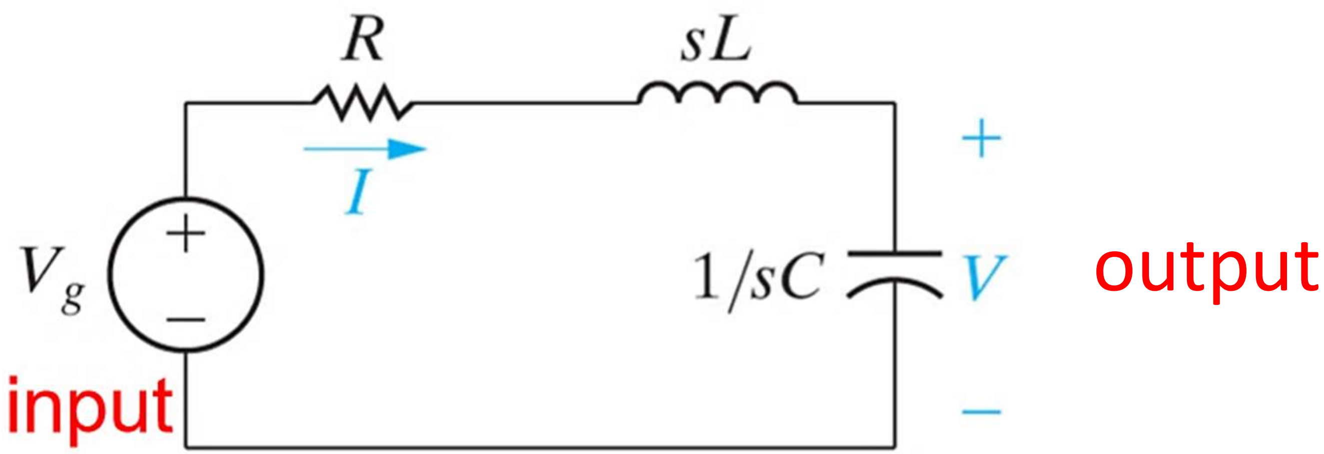

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$
$$= \frac{1}{K} \cdot \frac{1}{(M/K)s^2 + (B/K)s + 1}$$
$$= \frac{1}{K} \cdot \frac{(K/M)}{s^2 + (B/M)s + (K/M)}$$
$$= \frac{1}{K} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\Rightarrow \begin{cases} \omega_n^2 = K/M \\ 2\zeta\omega_n = B/M \end{cases} \Rightarrow \boxed{\zeta = \frac{B}{2\sqrt{KM}}, \quad \omega_n = \sqrt{\frac{K}{M}}}$$

**Note:** Presenting $\frac{X(s)}{F(s)}$ in the form of $\frac{\left(\frac{1}{M}\right)}{s^2+\left(\frac{B}{M}\right)s+\left(\frac{K}{M}\right)}$ is also acceptable.

# Example 2:
# Second-order system (an electrical system)

- Series RLC circuit system:



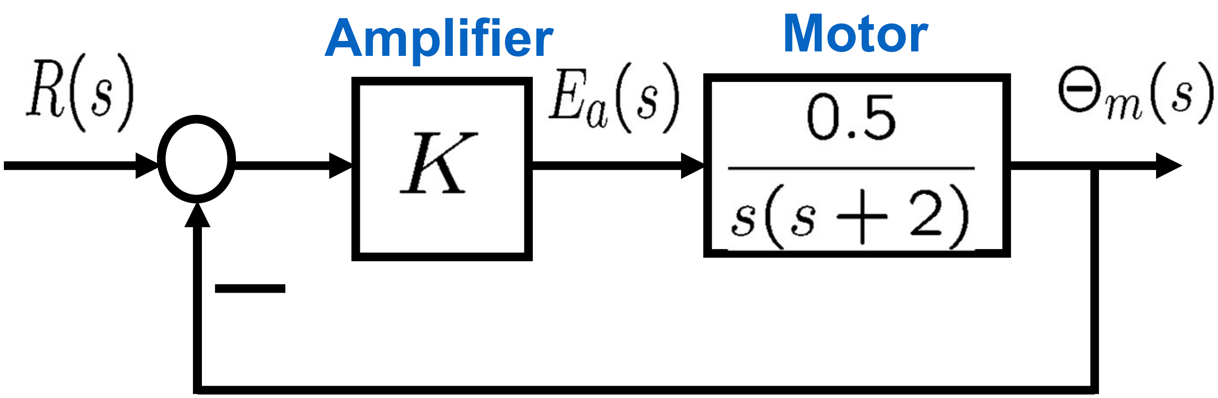

- If the output is the capacitor voltage ($V$):

$$\frac{V}{V_g} = \frac{(sC)^{-1}}{R + sL + (sC)^{-1}} = \frac{1}{s^2 LC + sRC + 1}$$

$$\omega_n = ?; \;\; \zeta = ?$$

# Example 3:
# Second-order system (an electromechanical system)

- DC motor position control:




**Closed-loop TF (CLTF):**

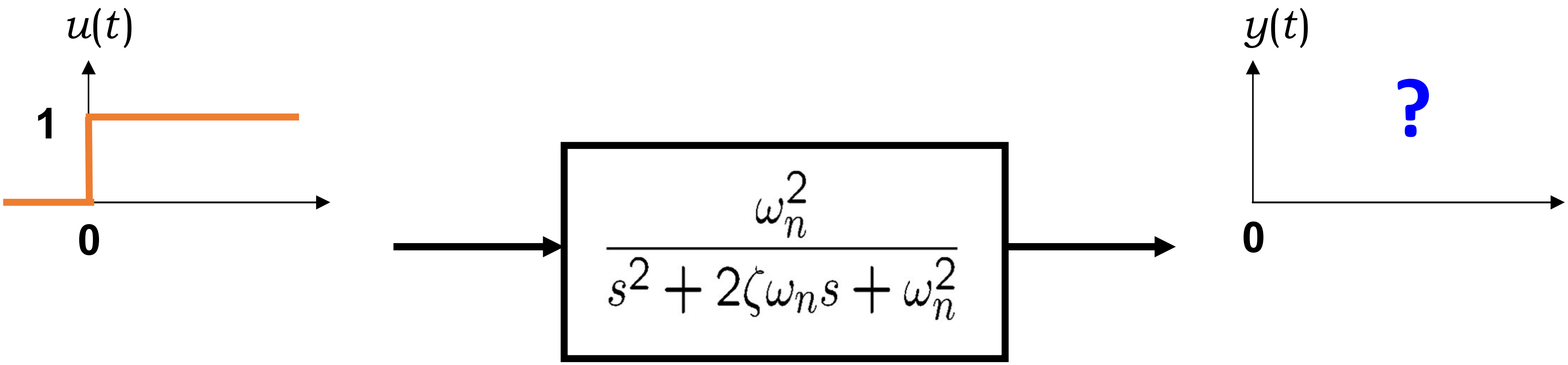
$$\frac{\Theta_m(s)}{R(s)} = \frac{0.5K}{s^2 + 2s + 0.5K}$$

- Standard form of the second-order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \Rightarrow \quad \boxed{\omega_n = \sqrt{0.5K} \;;\; \zeta = \frac{1}{\sqrt{0.5K}}}$$

# Step response of $2^{nd}$-order system

- Input a unit step function to a 2nd-order system. What is the output?




$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sG(s)\frac{1}{s} = G(0)$$

*Final value theorem*

**DC gain**

$$\lim_{t\to\infty} y(t) = G(0) = 1; \text{if } G \text{ is stable}$$

# Step response of 2nd-order system for various damping ratios

- Undamped

$$\zeta = 0$$

- Underdamped

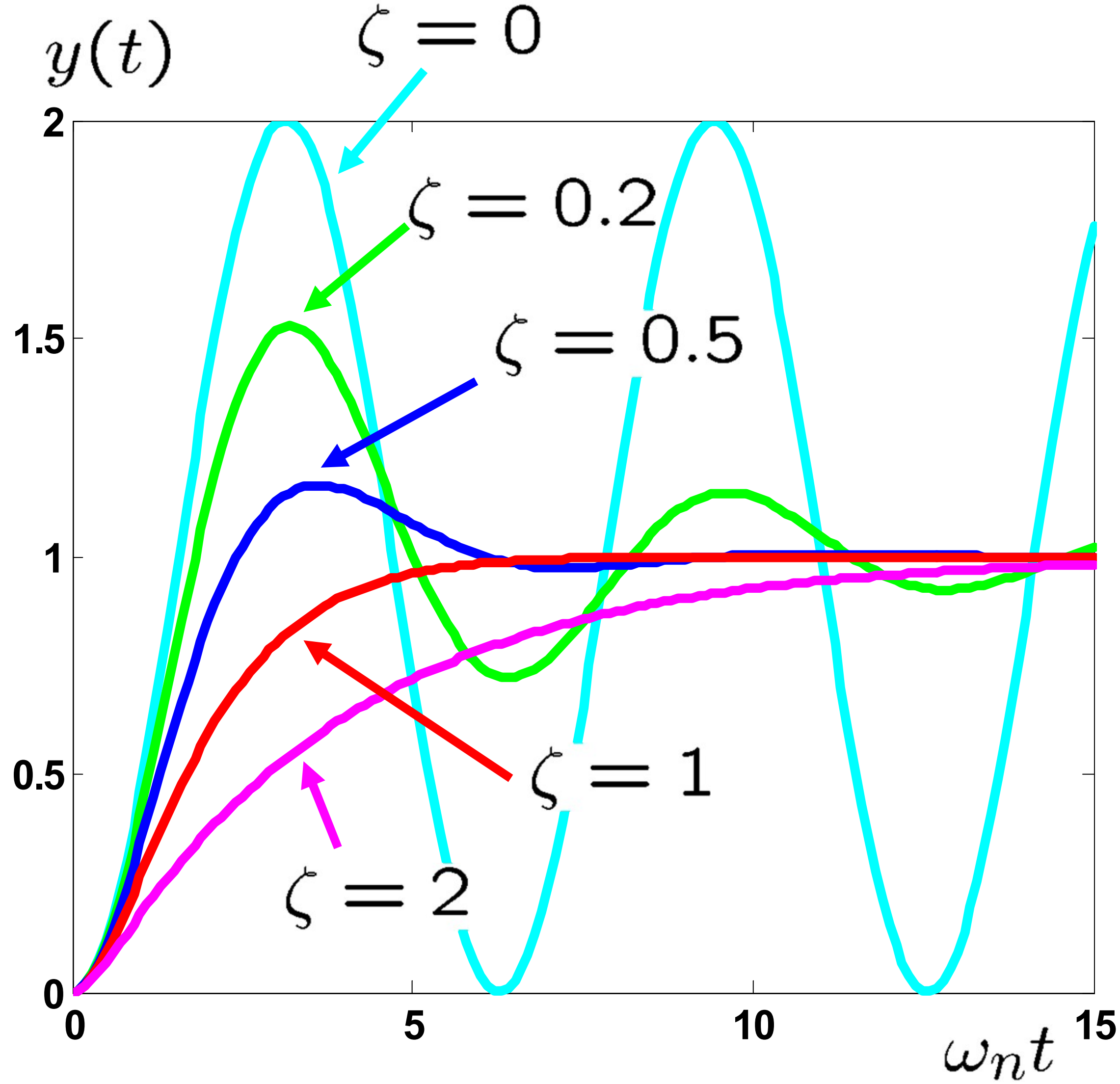
$$0 < \zeta < 1$$

- Critically damped

$$\zeta = 1$$

- Overdamped

$$\zeta > 1$$



# Step response of 2nd-order system: Underdamped case

- Math expression of $y(t)$ for underdamped case, i.e., for $0 < \zeta < 1$:

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad \boxed{y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)}$$

*Damped natural frequency* $\longrightarrow$ $\omega_d = \omega_n\sqrt{1-\zeta^2}$

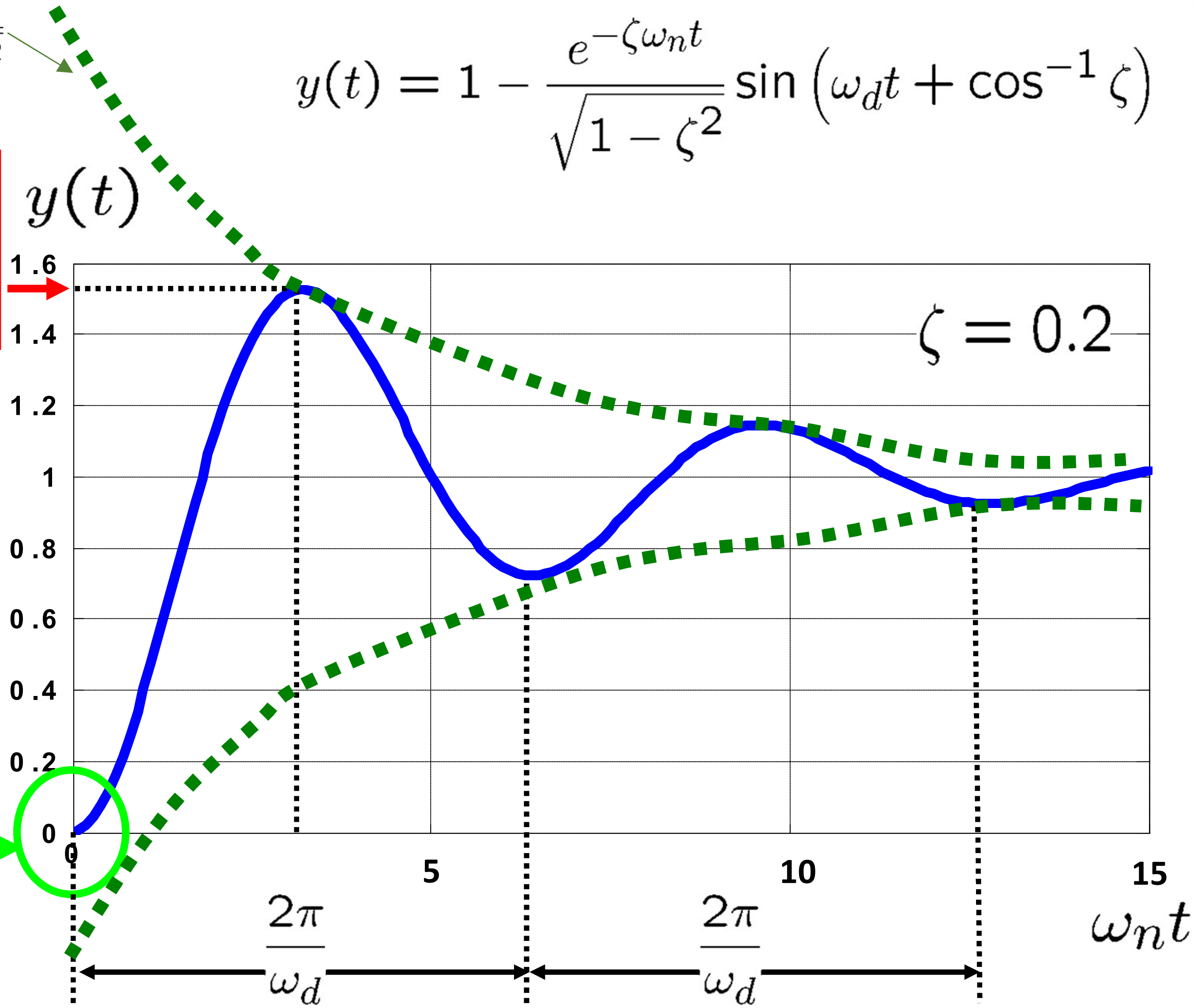
# Peak value and peak time: Underdamped case

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)$$

$$\boxed{y_{max} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}$$

$$\boxed{T_p = \frac{\pi}{\omega_d}}$$

$$\boxed{\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned}}$$



# Properties of underdamped 2nd-order system in terms of $\zeta$ and $\omega_n$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad 0 < \zeta < 1$$

| | (5%) (2%) |
|---|---|
| Settling time, $T_s$ | $\approx \frac{3}{\zeta\omega_n}$ or $\frac{4}{\zeta\omega_n}$ |
| Peak time, $T_P$ | $\frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ |
| Peak value, $y_{max}$ | $1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$ |
| Percent overshoot, $PO$ or $\%OS$ | $100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$ |

Time constant = $T = \frac{1}{\zeta\omega_n}$

$PO = 100e^{-\pi/\tan\theta}$

Delay time = $T_d = \frac{1 + 0.7\zeta}{\omega_n}$

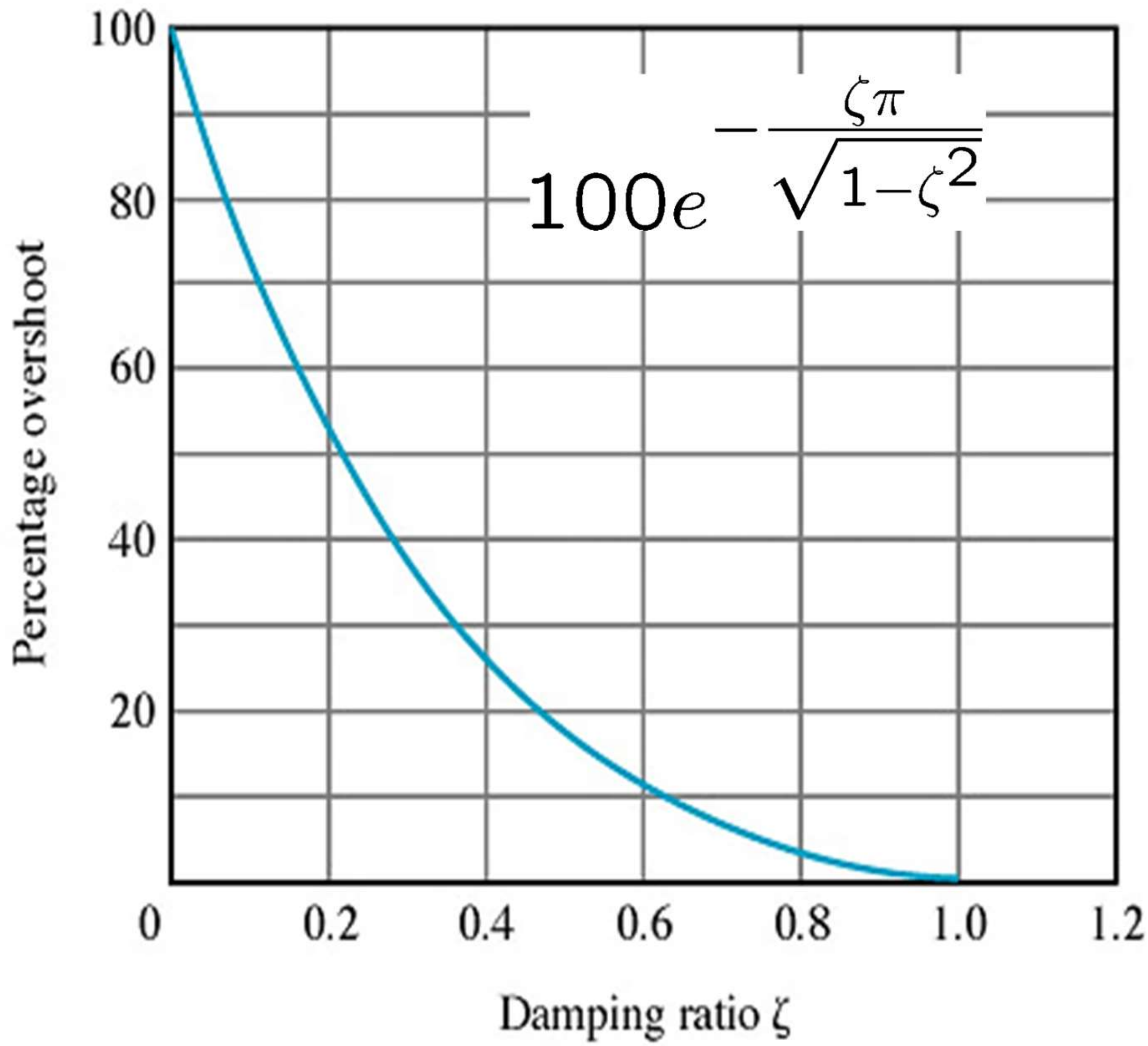
# Remarks for underdamped case

- **Time constant** is $T = \frac{1}{\zeta\omega_n}$, indicating convergence speed.
- Percent overshoot depends on $\zeta$, but NOT $\omega_n$. (See the next slide.)
- For $\zeta > 1$ (overdamped case), we cannot define peak time, peak value, and percent overshoot ($PO$).
- For the 2nd-order transfer function, we can use the following formula for the **rise time**:

$$T_r = \frac{1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1}{\omega_n} \qquad 0 < \zeta < 1$$

- It has been shown that the rise time is mainly affected by the **dominant poles** (i.e., the poles closest to the imaginary axis).

# PO vs. damping ratio



# Step response properties of underdamped 2nd order system in terms of pole locations

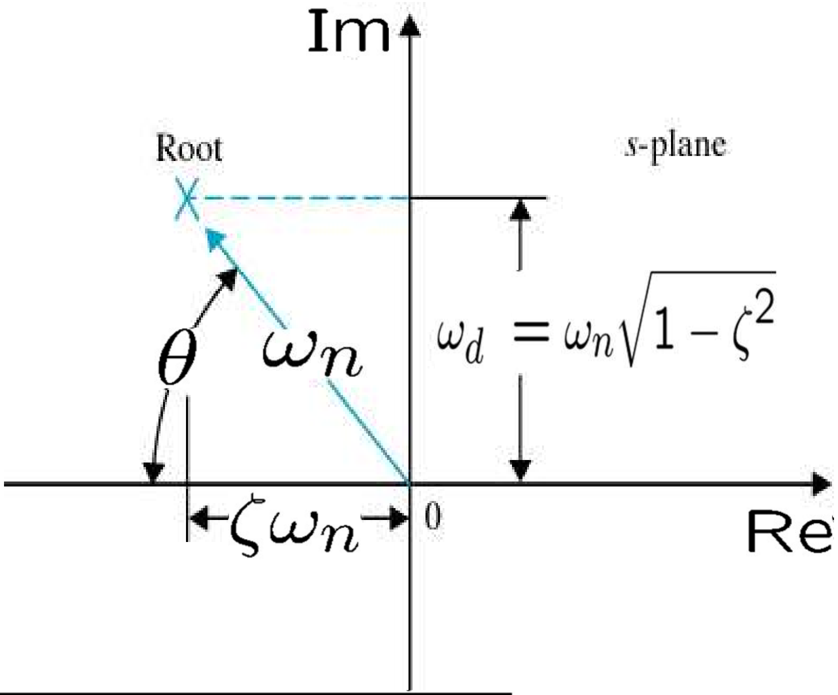
- Poles ($0 < \zeta < 1$)

$$s = -\zeta\omega_n \pm j\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$$

$$\zeta = \cos\theta$$

$$T = \text{Time Constant} = 1/\zeta\omega_n$$

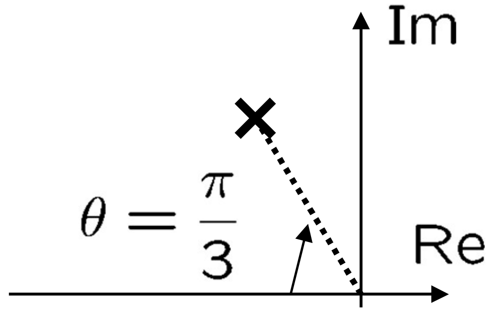
$$5\% T_s = 3T = \frac{3}{\zeta\omega_n} \quad ; \quad 2\% T_s = 4T = \frac{4}{\zeta\omega_n}$$



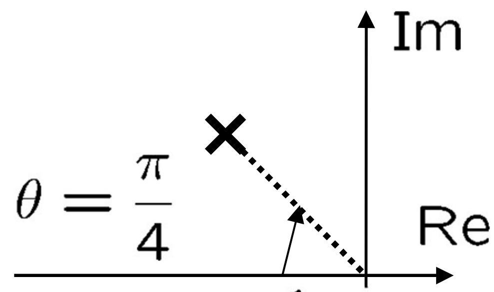

| Pole | | | Performance |
|---|---|---|---|
| Real part | $\zeta\omega_n$ | determines | $T_s = \frac{3}{\zeta\omega_n}, \frac{4}{\zeta\omega_n}$ $\boxed{= \frac{3}{\lvert\text{Re}\rvert}, \frac{4}{\lvert\text{Re}\rvert}}$ |
| Imag. part | $\omega_d$ | determines | $T_p = \frac{\pi}{\omega_d}$ $\boxed{= \frac{\pi}{\lvert\text{Im}\rvert}}$ |
| Angle | $\theta$ | determines | overshoot |

$$\cos\theta = \frac{\zeta\omega_n}{\omega_n} = \zeta \quad ; \quad \tan\theta = \frac{\omega_d}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta} \Rightarrow PO = 100e^{-\pi/\tan\theta}$$

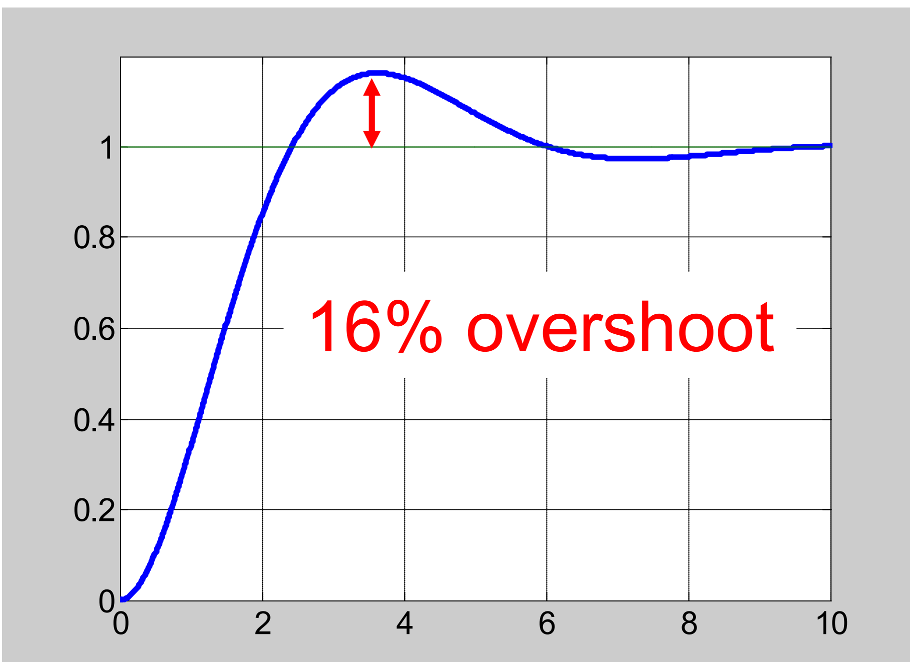
# Angle θ and the overshoot

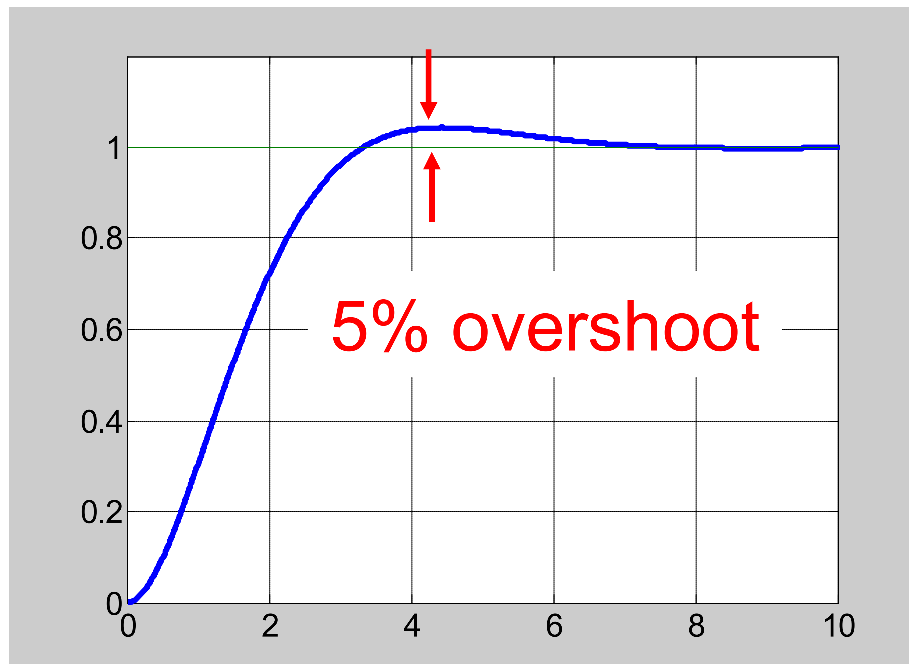



$$\zeta = \cos\theta = 0.5$$



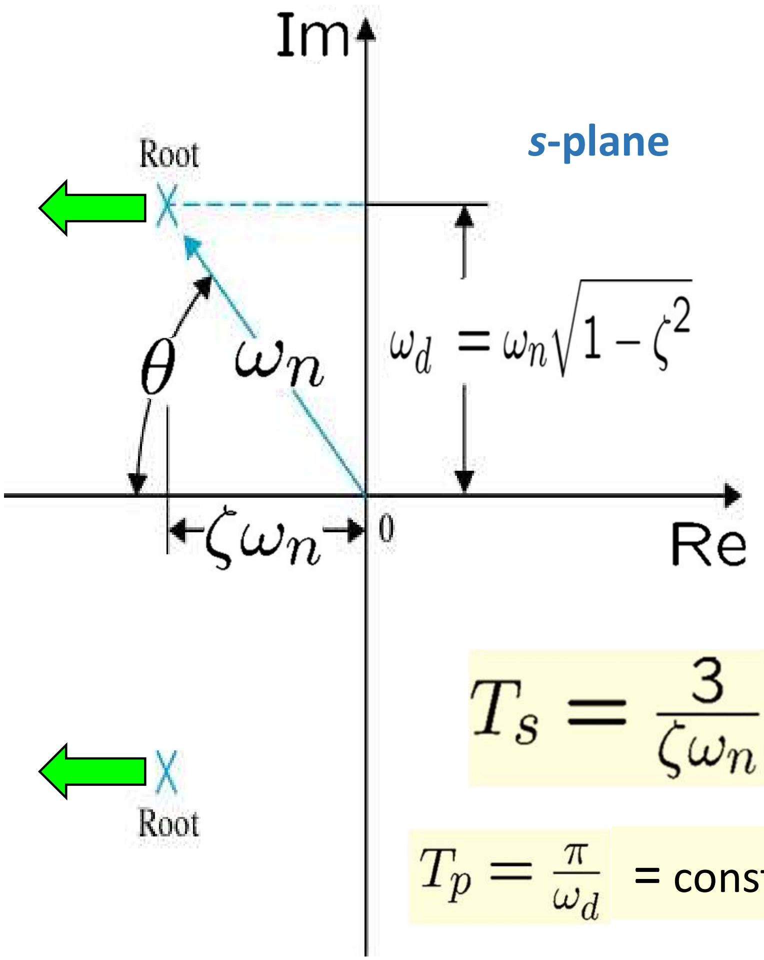

$$\zeta = \cos\theta = \frac{1}{\sqrt{2}} \approx 0.707$$



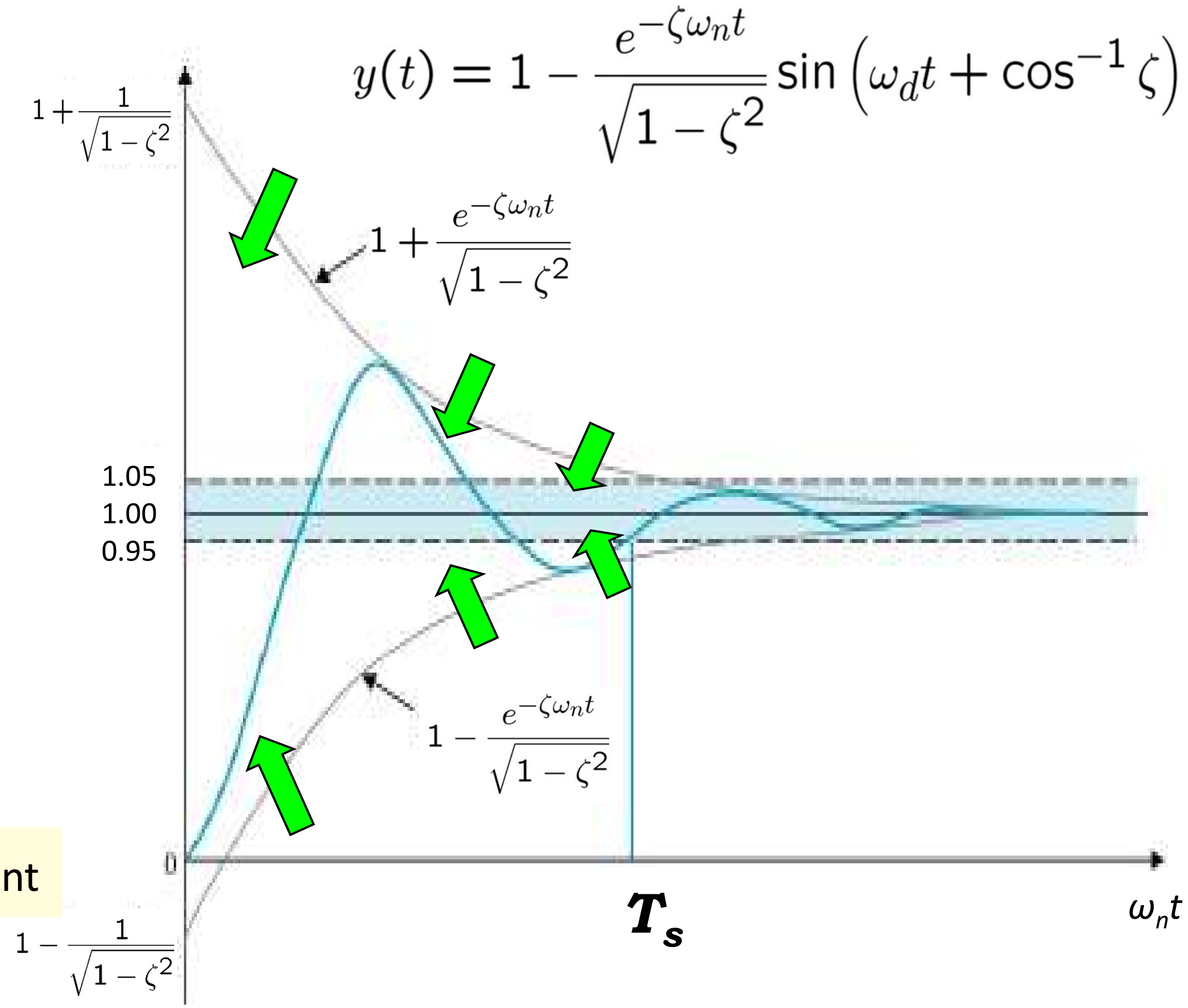
# Influence of real part of poles

- As $\zeta\omega_n$ increases, settling time $T_s$ decreases.



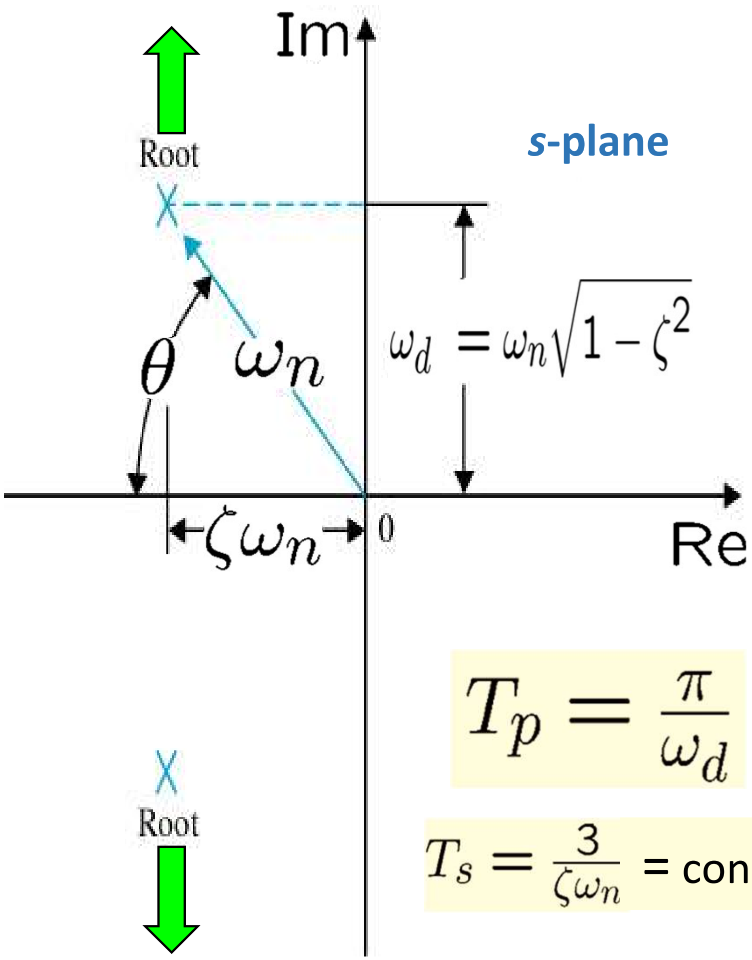

$$T_s = \frac{3}{\zeta\omega_n}$$

$T_p = \frac{\pi}{\omega_d}$ = constant



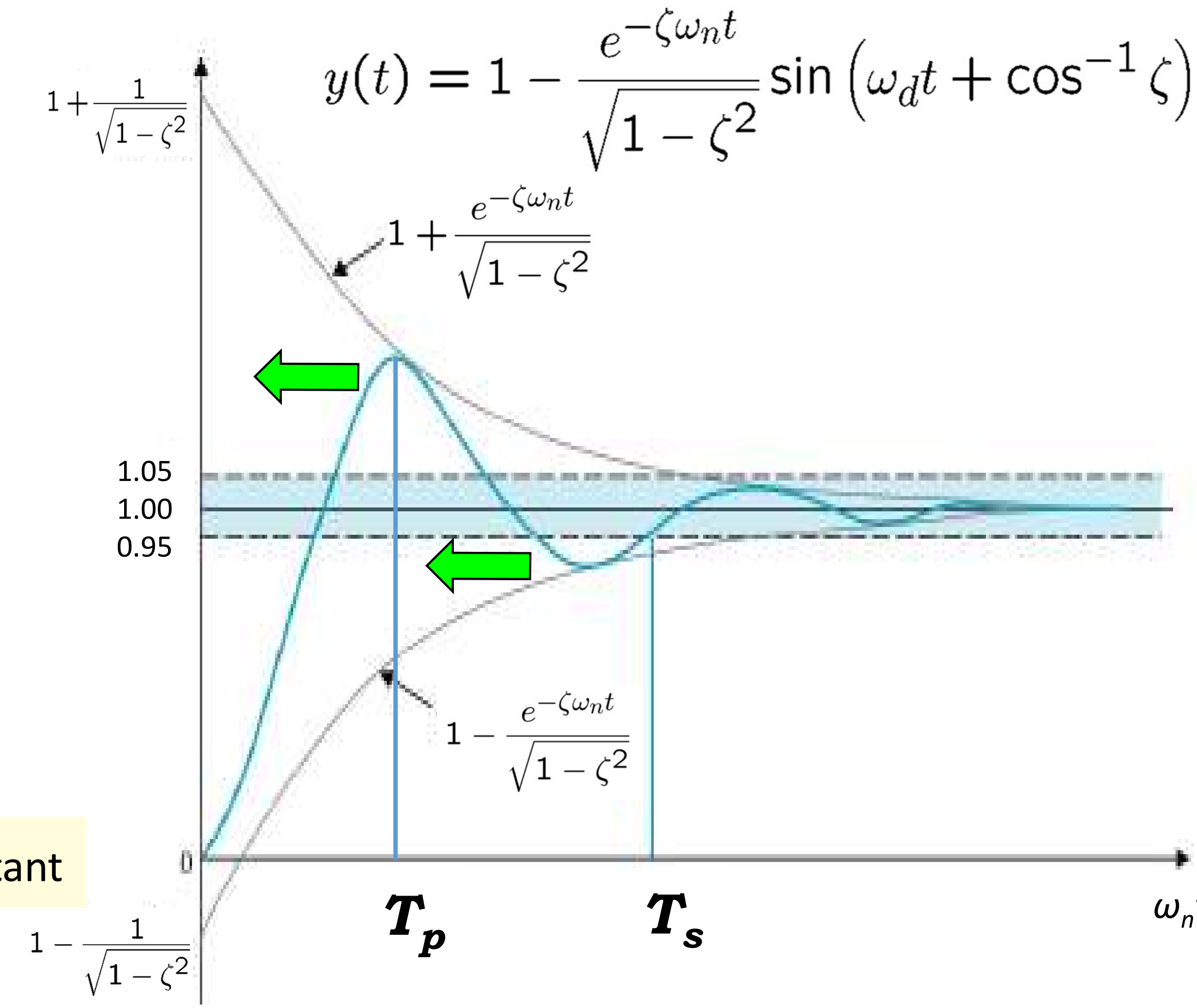
# Influence of imag. part of poles

- As damped oscillation frequency ($\omega_d$) increases, $T_P$ decreases.



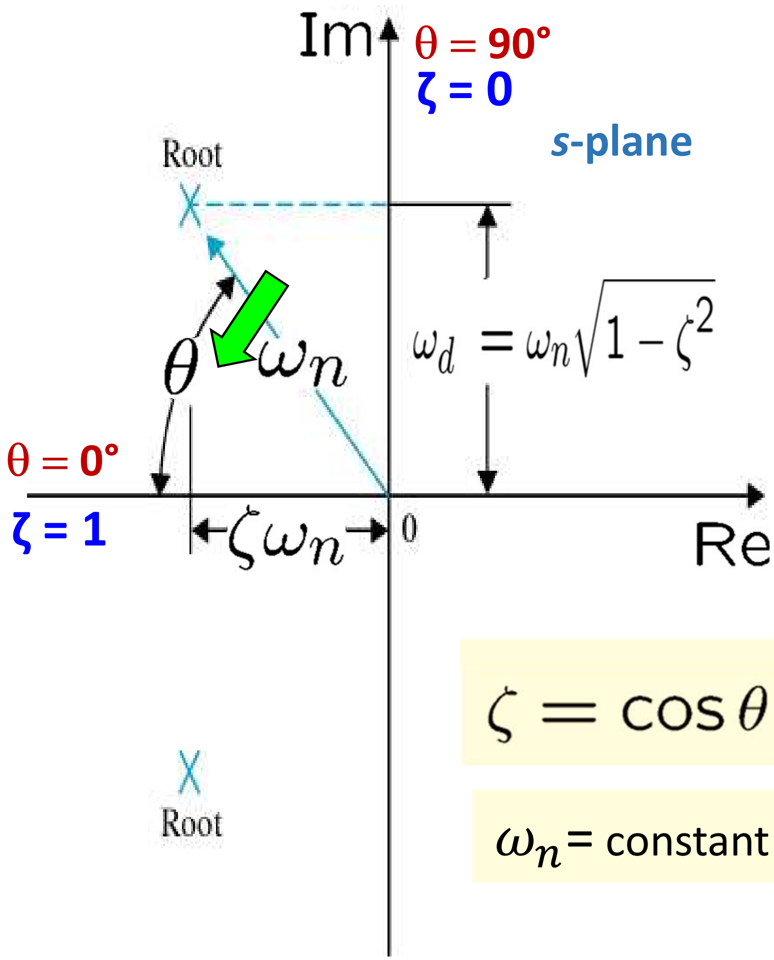

$$T_p = \frac{\pi}{\omega_d}$$

$T_s = \frac{3}{\zeta\omega_n}$ = constant



# Influence of angle of poles

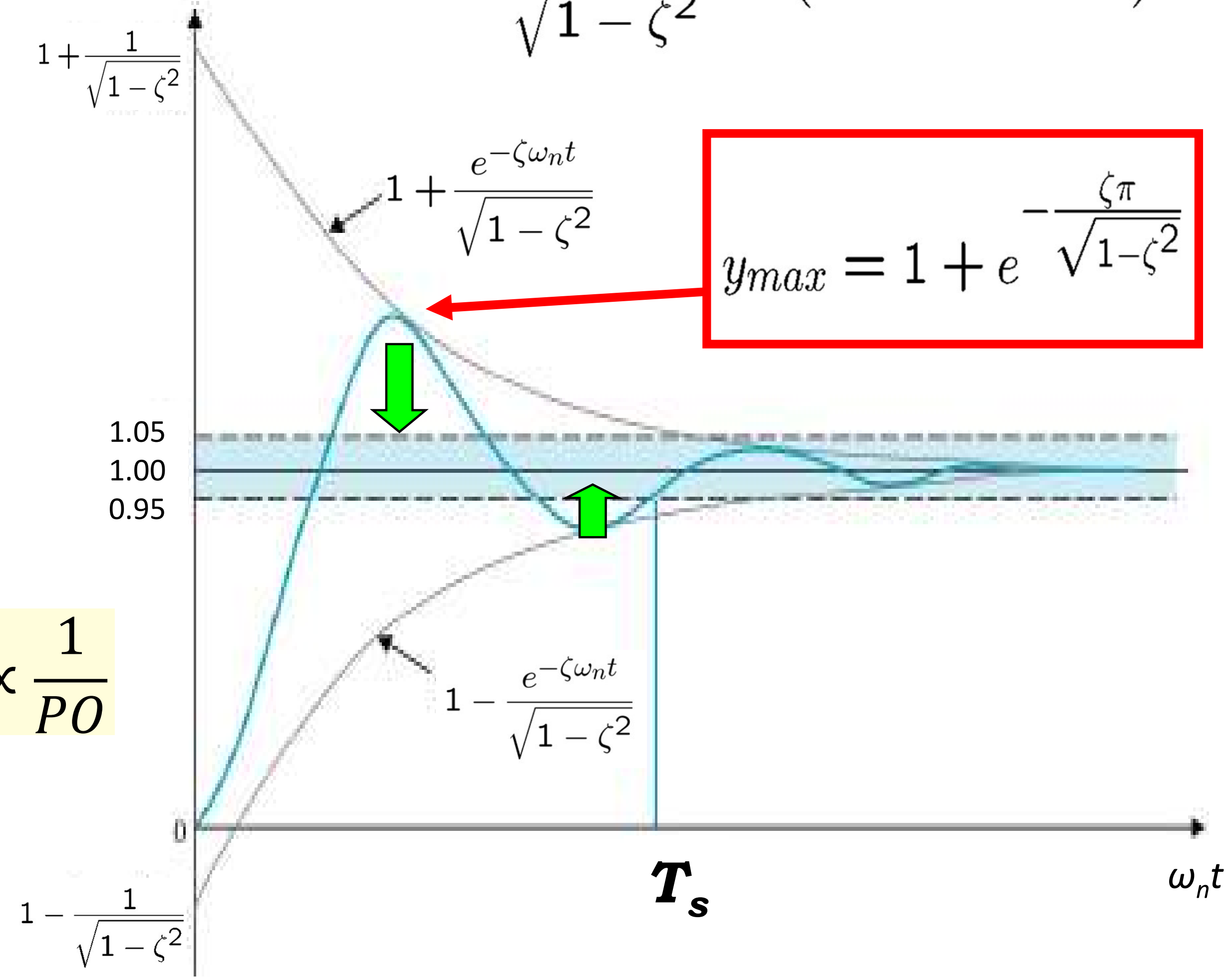
- As $\theta$ decreases from **90°** to **0°**, $\zeta$ increases from **0** to **1** and overshoot (or undershoot) decreases.




$$\zeta = \cos\theta \propto \frac{1}{PO}$$

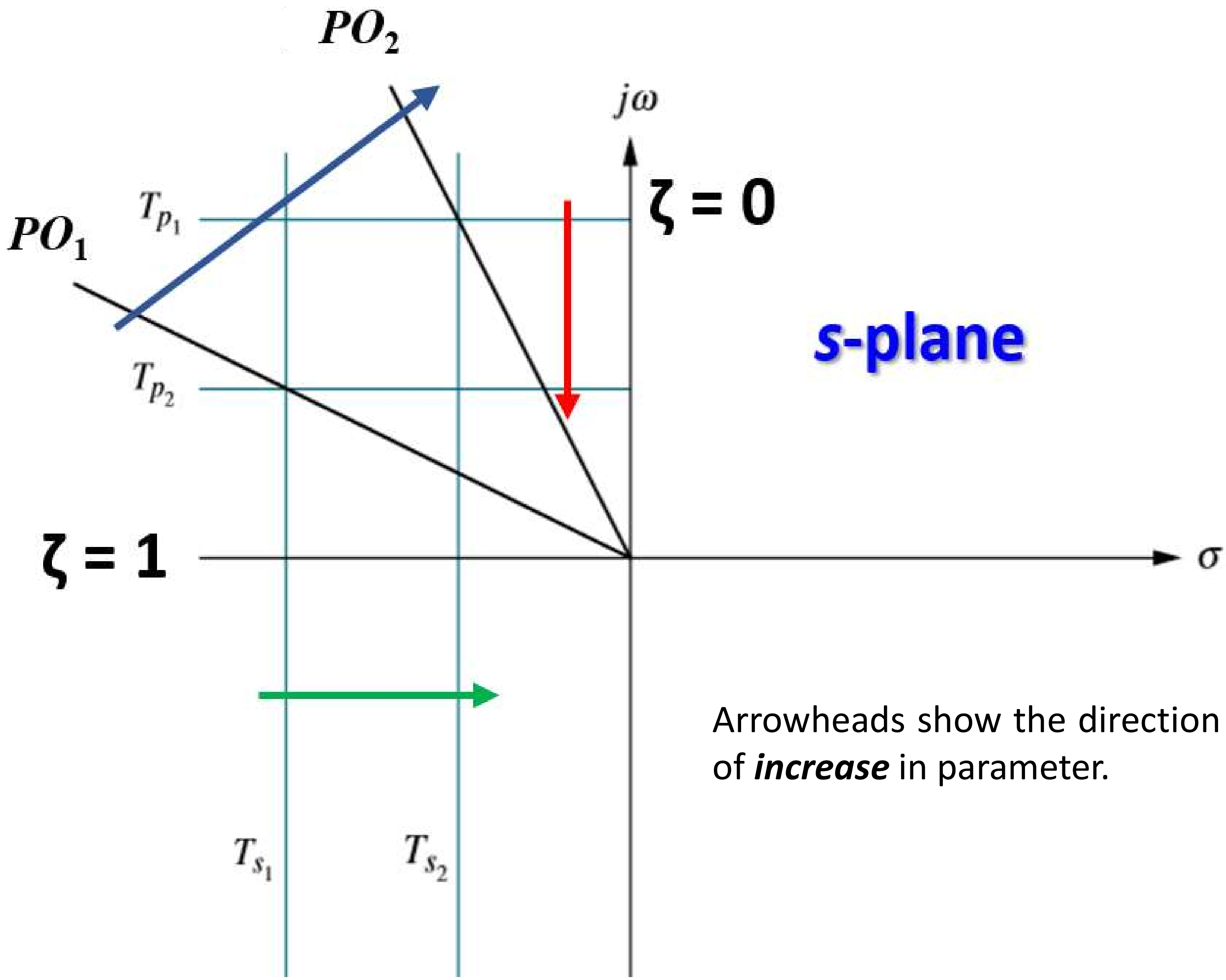
$\omega_n =$ constant

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)$$



# Variation trend of $T_p$, $T_s$, $PO$, and ζ in $s$-plane

$$\begin{cases} \zeta = \cos\theta \propto \dfrac{1}{PO} \\ T_p = \dfrac{\pi}{\omega_d} \\ T_s = \dfrac{3}{\zeta\omega_n} \end{cases}$$




Arrowheads show the direction of ***increase*** in parameter.

# Summary

- Step responses of
  - **1st-order system** is characterized by:
    - Time constant ($T$) and DC gain ($K$)
    - Pole location
  - **2nd-order system** is characterized by:
    - Damping ratio ($\zeta$) and undamped natural frequency ($\omega_n$)
    - Pole location
- Next
  - Time response examples