a place of mind

ELEC 341: Systems and Control

Lecture 11

Root locus: Introduction

Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- Models for systems
  - ✓ Electrical
  - ✓ Electromechanical
  - ✓ Mechanical
- ✓ Linearization, delay

Analysis

- ✓ Stability
  - ✓ Routh-Hurwitz
  - Nyquist
- ✓ Time response
  - ✓ Transient
  - ✓ Steady state
- Frequency response
  - Bode plot

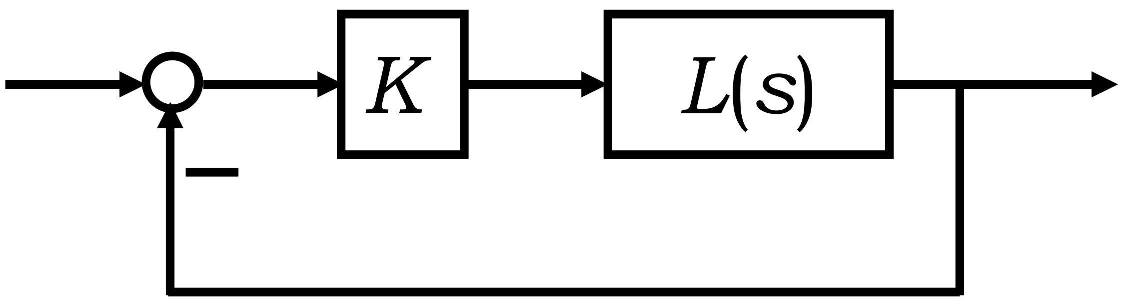
Design

- Design specs
- ➤ Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

Matlab simulations

What is Root Locus?

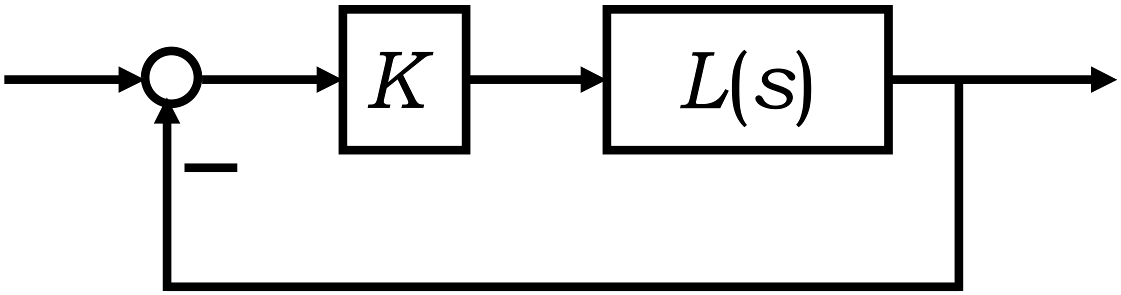
- *Pole locations* of the system characterizes **stability** and **transient properties**.
- Consider a feedback system that has one parameter (gain) $K > 0$ to be designed:



$K.L(s)$: open-loop TF

- *Root locus* (RL) graphically shows how poles of CL system vary as K varies from 0 to infinity.

Example 1



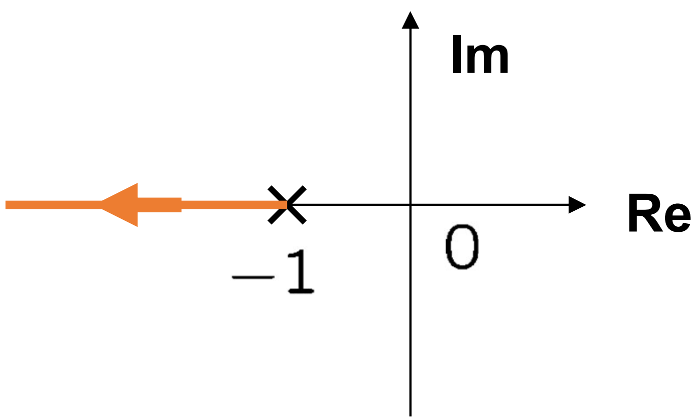
$$L(s) = \frac{1}{s+1}$$

- Characteristic eq. $1 + K\frac{1}{s+1} = 0$

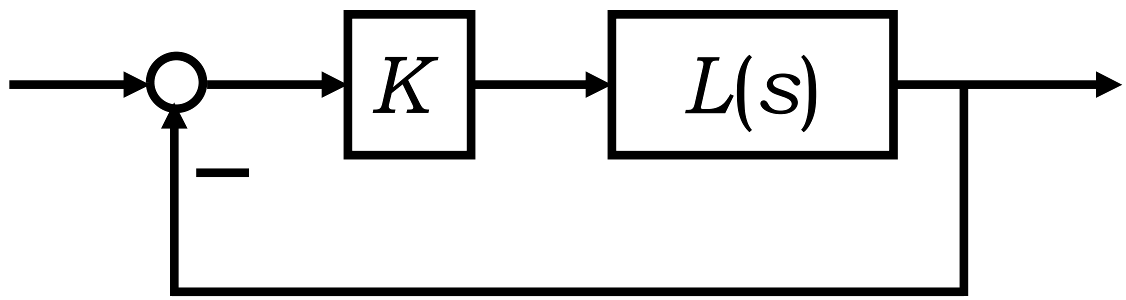
Closed-loop poles

$\Rightarrow$ $s + 1 + K = 0$ $\Rightarrow$ $s = -1 - K$

- K = 0: s = -1
- K = 1: s = -2
- K = 2: s = -3, etc.



Example 2



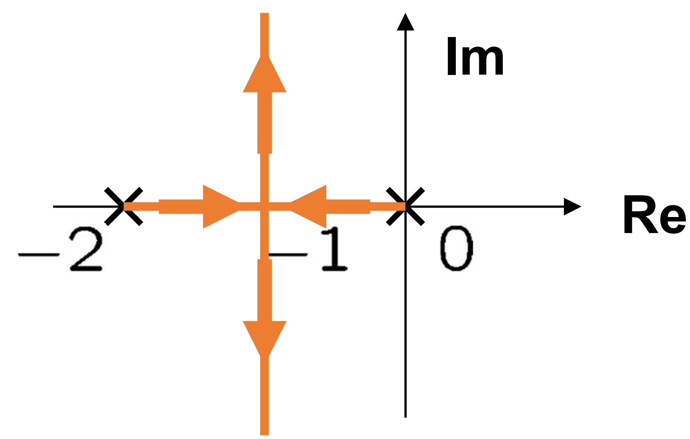
$$L(s) = \frac{1}{s(s+2)}$$

- Characteristic eq. $1 + K\frac{1}{s(s+2)} = 0$

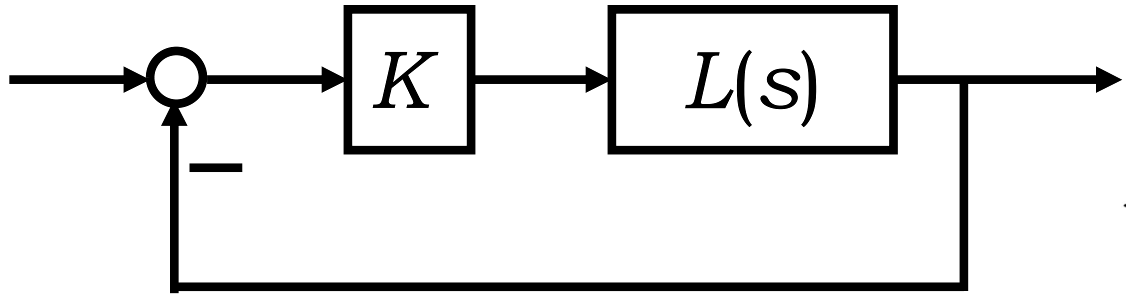
Closed-loop poles

$$\Rightarrow s^2 + 2s + K = 0 \Rightarrow s = -1 \pm \sqrt{1-K}$$

- $K = 0$: $s = 0, -2$
- $K = 1$: $s = -1, -1$
- $K > 1$: complex roots (numbers)



Example 3



$$L(s) = \frac{s+1}{s(s+2)(s+3)}$$

- Characteristic eq. $1 + K\dfrac{s+1}{s(s+2)(s+3)} = 0$

$\Rightarrow \quad s(s+2)(s+3) + K(s+1) = 0 \quad \Rightarrow \quad s = ???$

- It is hard to solve this analytically for each K.
- Is there some way to sketch roughly and quickly root locus by hand?

Example 3 (cont'd): Root locus sketching

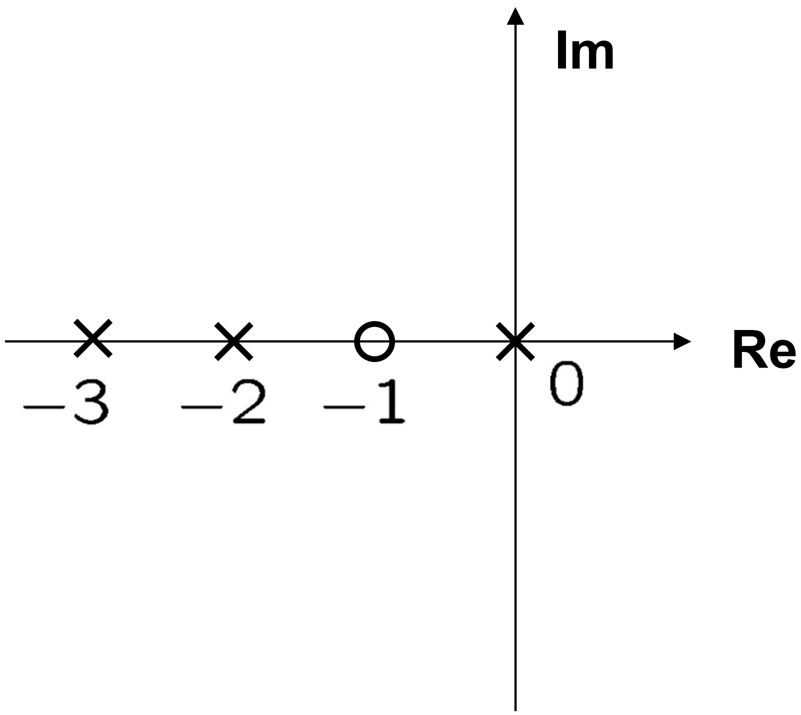
- **Step 0:** Mark open-loop poles and zeros
- **Step 1:** On the real axis (also called real-axis segments)
- **Step 2:** Asymptotes
- **Step 3:** Breakaway points
- **Step 4:** Angles of departures and arrivals

Note: For your tests, there is no need to draw your graphs to scale. Please, just show the trend of variation.

Example 3 (cont'd)
Root locus: Step 0

- *Mark poles of $L(s)$ with "x" and zeros of $L(s)$ with "o".*
- *Root locus is symmetric w.r.t. the real axis.*
- *The number of branches that go to infinity = # of poles - # of zeros*
- *The total number of branches = order of $L(s)$*

$$L(s) = \frac{s+1}{s(s+2)(s+3)}$$



Note: In most engineering designs, the total number of branches (or the order of $L(s)$) is equal to the number of poles.

Example 3 (cont'd): Root locus sketching

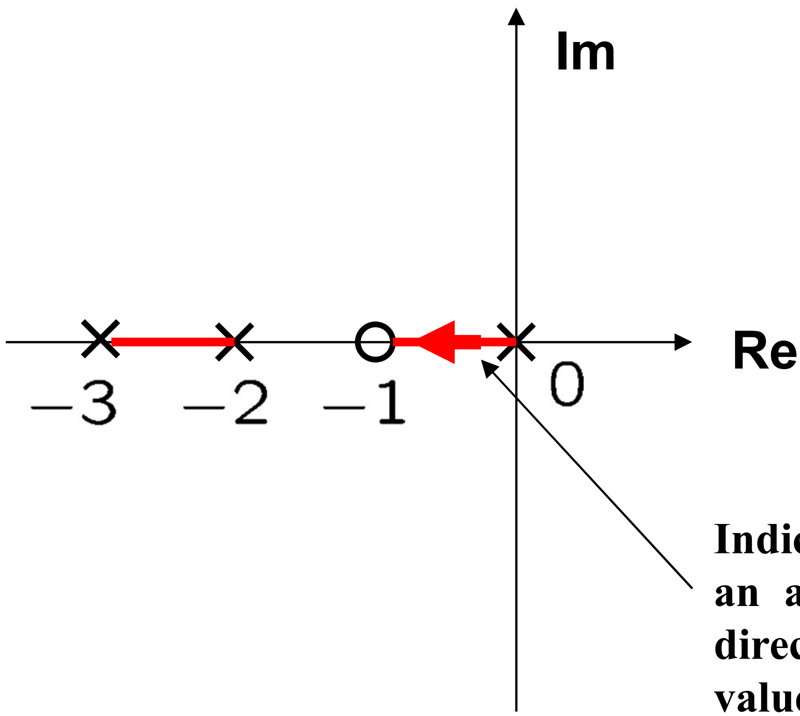
- **Step 0:** Mark open-loop poles and zeros
- **Step 1:** On the real axis
- **Step 2:** Asymptotes
- **Step 3:** Breakaway points
- **Step 4:** Angles of departures and arrivals

Example 3 (cont'd)
Root locus: Step 1 (On the real axis)

- *RL includes all points on real axis to the **left of an odd number** of roots (poles and zeros of $L(s)$).*
- *RL originates from the poles of $L(s)$ and terminates at the zeros of $L(s)$, including **infinity zeros**.*

How to count the number of infinity zeros of $L(s)$:
Let us say we have:
#Z = number of finite zeros of $L(s)$
#P = number of finite poles of $L(s)$

Case 1: If you DO see some finite zeros in $L(s)$, then the number of zeros at infinity is equal to #P – #Z.
Case 2: If you DO NOT see any finite zeros in $L(s)$, i.e., #Z = 0, then the number of infinity zeros is equal to #P.



Indicate the direction with an arrowhead. This is the direction of increase in the value of *K*.

Example 3 (cont'd): Root locus sketching

- **Step 0:** Mark open-loop poles and zeros
- **Step 1:** On the real axis
- **Step 2:** Asymptotes (i.e., lines to which root locus converges)
- **Step 3:** Breakaway points
- **Step 4:** Angles of departures and arrivals

Example 3 (cont'd)
Root locus: Step 2 (Asymptotes)

- *Number of asymptotes = relative degree (r) of $L(s)$:*

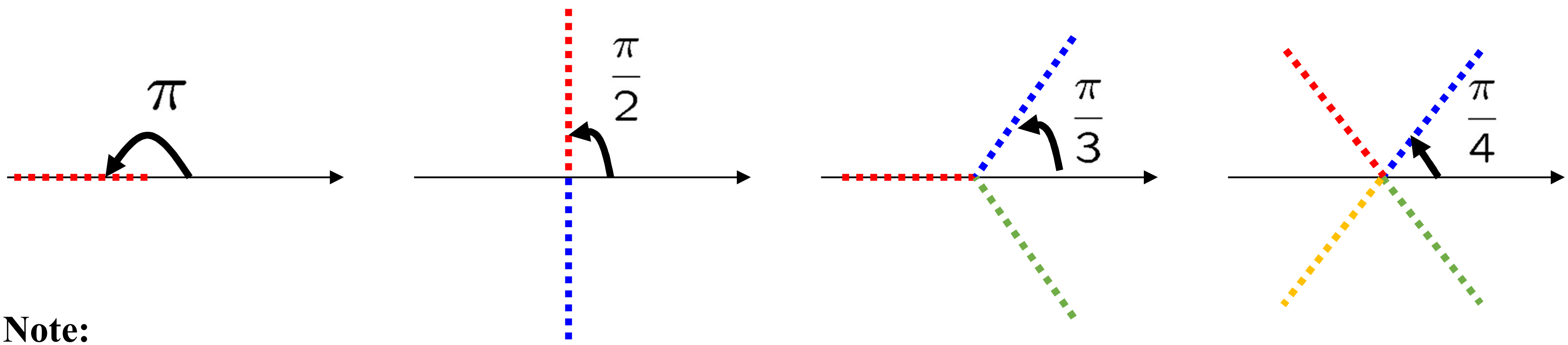
$$r = \underbrace{n}_{\text{deg (den)}} - \underbrace{m}_{\text{deg (num)}}$$

- *Angles of asymptotes are*

$$\frac{\pi}{r} \times (2k+1), \; k = 0, 1, \ldots, (r-1)$$

Odd number

$r = 1$ $\quad$ $r = 2$ $\quad$ $r = 3$ $\quad$ $r = 4$



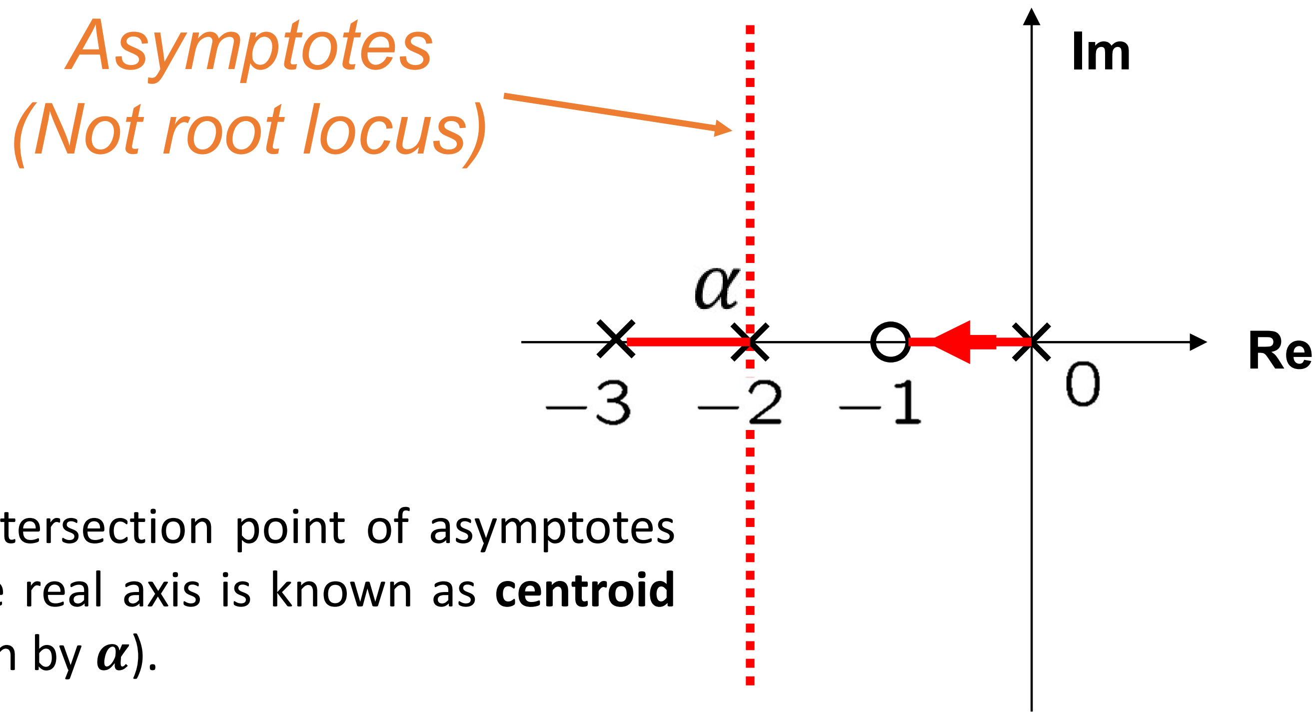
Note:
number of asymptotes = r = number of branches that will go to infinity = number of infinity zeros = #P – #Z

Example 3 (cont'd)
Root locus: Step 2 (Asymptotes)

- *Intersections of asymptotes* $\alpha = \dfrac{\sum \text{pole} - \sum \text{zero}}{r}$

$$L(s) = \frac{s+1}{s(s+2)(s+3)} \Rightarrow \frac{\sum \text{pole} - \sum \text{zero}}{r} = \frac{(0+(-2)+(-3))-(-1)}{2} = -2$$

$\alpha = -2$



The intersection point of asymptotes on the real axis is known as **centroid** (shown by $\boldsymbol{\alpha}$).

Example 3 (cont'd): Root locus sketching

- **Step 0:** Mark open-loop poles and zeros
- **Step 1:** On the real axis
- **Step 2:** Asymptotes
- **Step 3:** Breakaway points
- **Step 4:** Angles of departures and arrivals

Breakaway points **(BAP)** **are the points where two or more branches meet on the real axis and then break away.**

Example 3 (cont'd)
Root locus: Step 3

- *Breakaway points are among roots of* $\frac{dL(s)}{ds} = 0$

$$L(s) = \frac{s+1}{s(s+2)(s+3)} \quad \Rightarrow \quad \frac{dL(s)}{ds} = -2\frac{s^3+4s^2+5s+3}{(\ s(s+2)(s+3)\ )^2} = 0$$

BAP (break-away point)

$$\Rightarrow \quad s = -2.4656, \ -0.7672 \pm 0.7926i$$

For each candidate s*, check the positivity of* $K = -\frac{1}{L(s)}$

Positivity Test

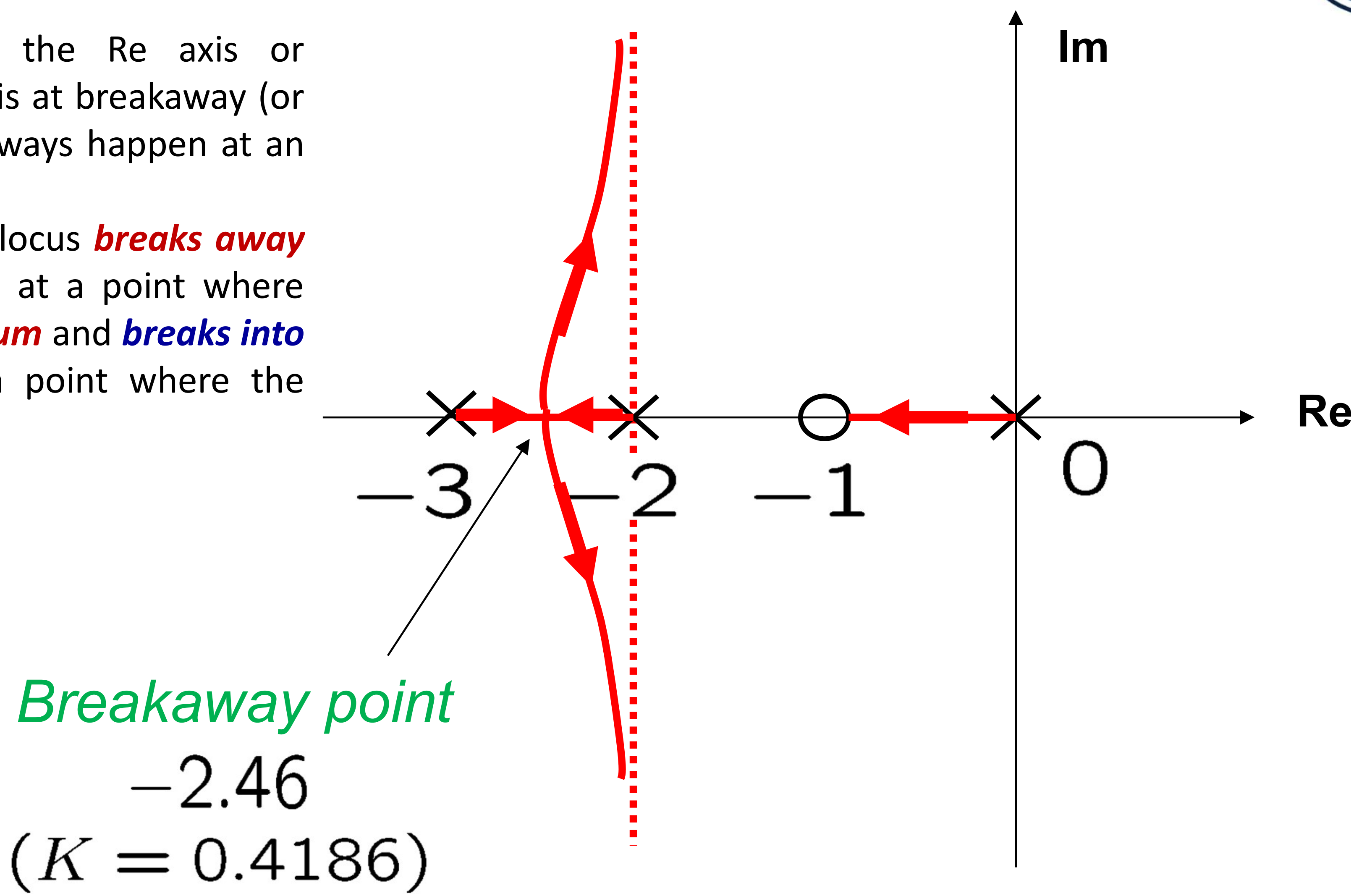
$$\Rightarrow \quad K = +0.4186$$

K **value corresponding to the BAP**

Example 3 (cont'd)
Root locus: Step 3

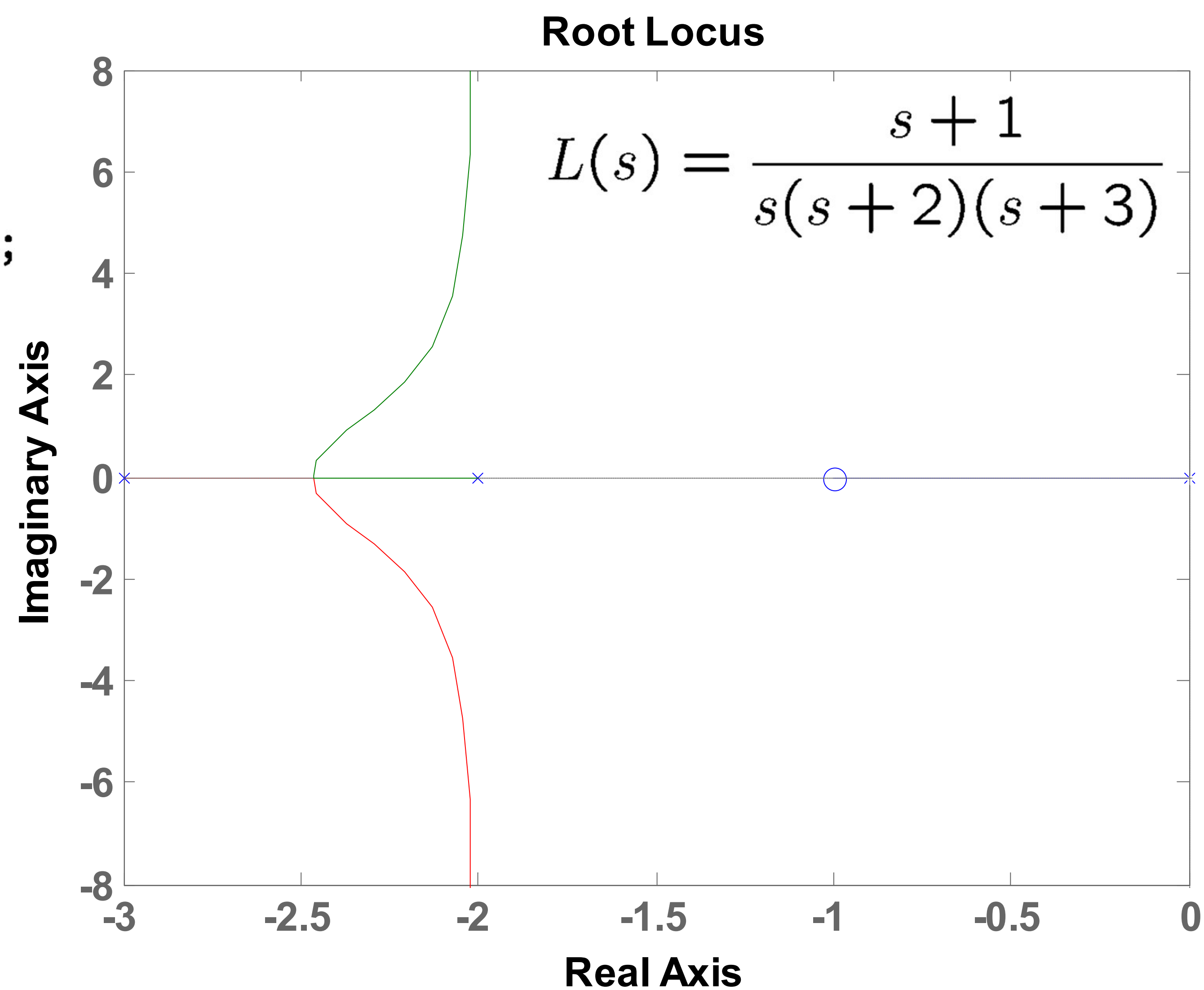
Note 1: Leaving the Re axis or entering the Re axis at breakaway (or break-in) points always happen at an **angle of 90°.**

Note 2: The root locus ***breaks away*** from the real axis at a point where the ***gain is maximum*** and ***breaks into*** the real axis at a point where the ***gain is minimum***.



Example 3 (cont'd)
Matlab command "rlocus"

```
num=[1 1];
den=[1 5 6 0];
sys=tf(num,den);
rlocus(sys)
```



Example 3 (cont'd)
Root locus: Step 3

Find the number of root locus branches (as a prediction):

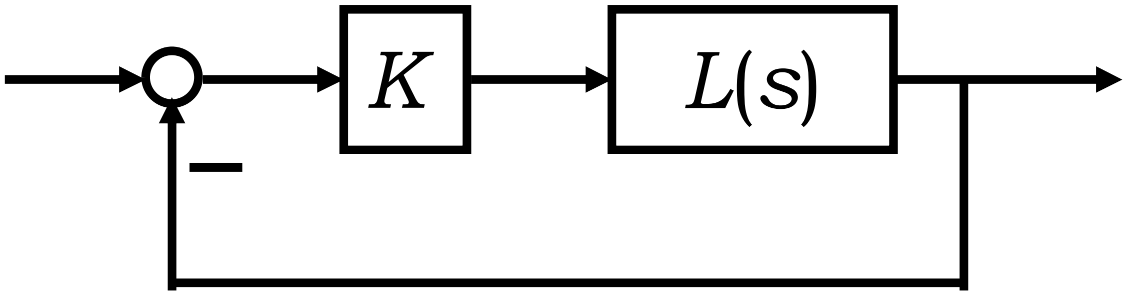
- We know that the root locus branches start at the open loop poles and end at open loop zeros. So, the number of root locus branches $\boldsymbol{N_b}$ is equal to the number of finite open loop poles $\#\boldsymbol{P}$ or the number of finite open loop zeros $\#\boldsymbol{Z}$, whichever is greater.
- Mathematically, we can write the number of root locus branches $\boldsymbol{N_b}$ as:

$$\text{If} \quad \#\boldsymbol{P} \geq \#\boldsymbol{Z} \quad \rightarrow \quad \boldsymbol{N_b} = \#\boldsymbol{P}$$

$$\text{If} \quad \#\boldsymbol{P} < \#\boldsymbol{Z} \quad \rightarrow \quad \boldsymbol{N_b} = \#\boldsymbol{Z}$$

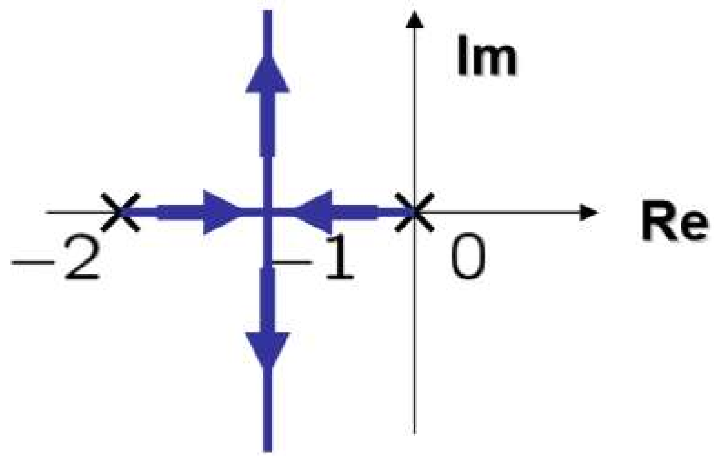
- In this example, $\#\boldsymbol{P} = 3$, $\#\boldsymbol{Z} = 1$.
- So, $\#\boldsymbol{P} \geq \#\boldsymbol{Z} \rightarrow \boldsymbol{N_b} = \#\boldsymbol{P} = 3 =$ Number of branches.

Example 4



$$L(s) = \frac{1}{s(s+2)}$$

- Asymptotes:
  - Relative degree: $r = 2$
  - Intersection: $\frac{0+(-2)}{2} = -1$



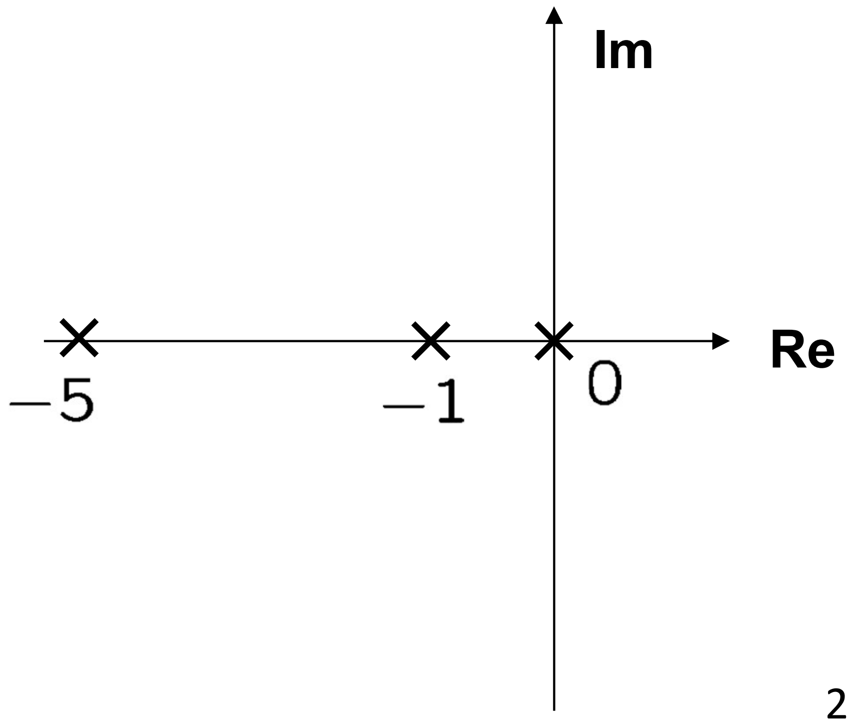
- Breakaway point:

$$L'(s) = \frac{-(2s+2)}{s^2(s+2)^2} = 0 \Rightarrow s = -1$$

Example 5
Root locus: Step 0

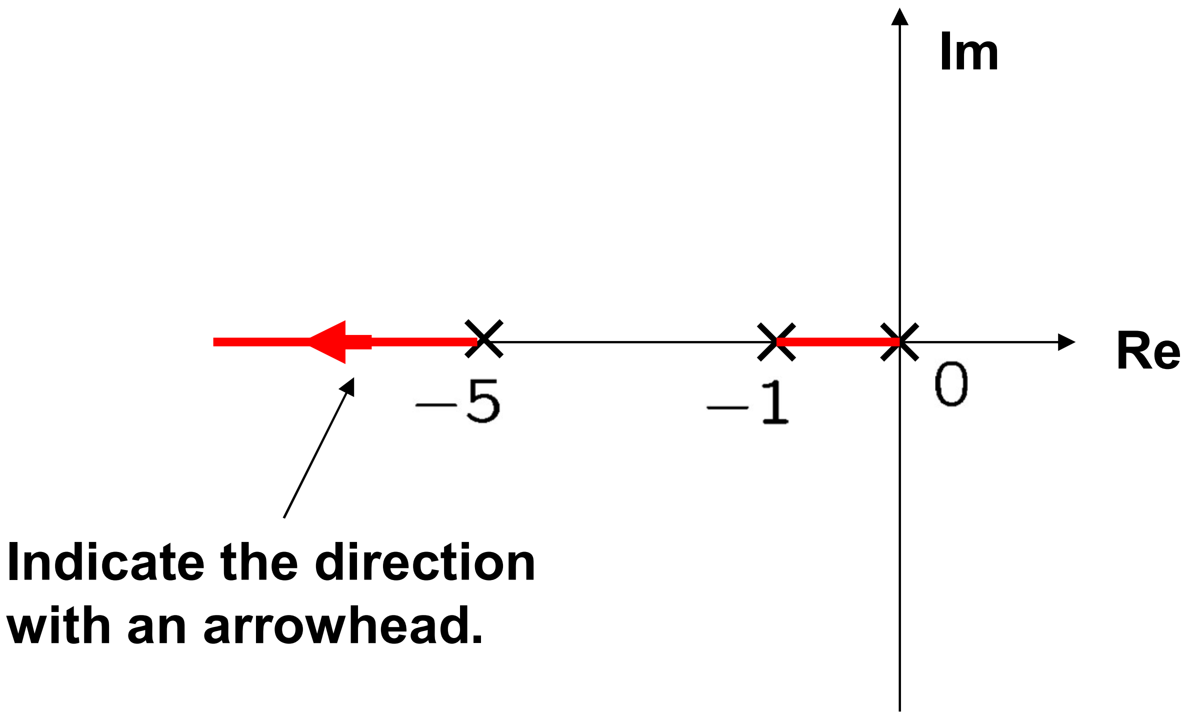
$$L(s) = \frac{1}{s(s+1)(s+5)}$$

- *Mark poles of L with "x" and zeros of L with "o".*
- *Root locus is symmetric w.r.t. the real axis.*
- *The total number of branches = order of $L(s)$*



Example 5 (cont'd)
Root locus: Step 1 (Real axis)

- *RL includes all points on real axis to the left of an odd number of roots (poles and zeros).*
- *RL originates from the poles of L and terminates at the zeros of L, including infinity zeros.*



Example 5 (cont'd)
Root locus: Step 2 (Asymptotes)

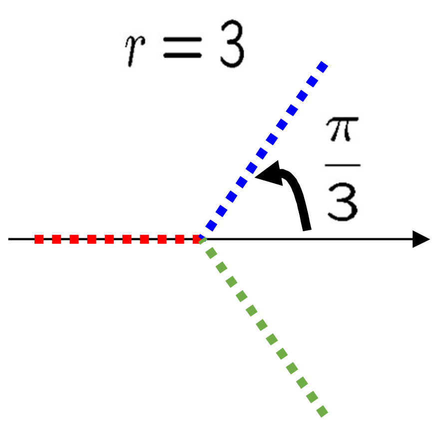
- *Number of asymptotes = relative degree (r) of L:*

$$r = \deg(\text{den}) - \deg(\text{num})$$

- *Angles of asymptotes are*

$$\frac{\pi}{r} \times (2k+1), \; k = 0, 1, \ldots, (r-1)$$

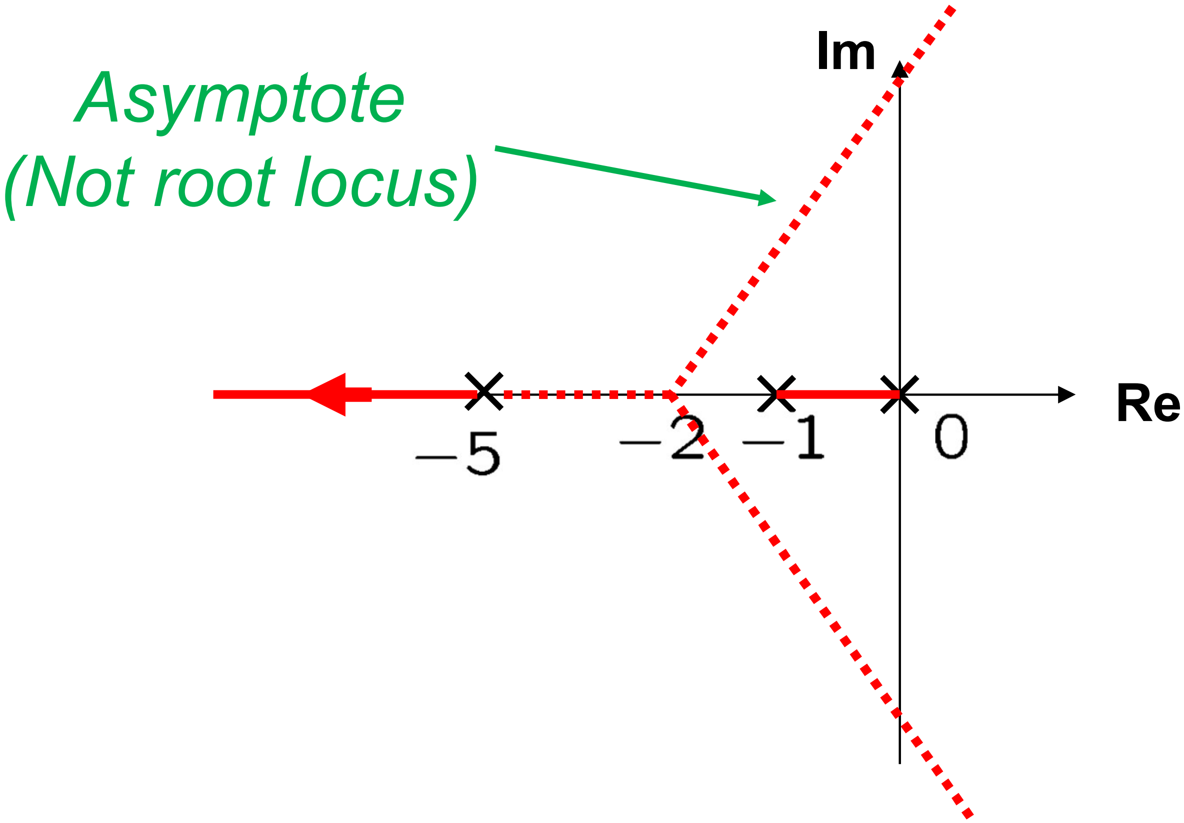
$$L(s) = \frac{1}{s(s+1)(s+5)} \Rightarrow r = 3 - 0 = 3 \Rightarrow$$



Example 5 (cont'd)
Root locus: Step 2 (Asymptotes)

- *Intersections of asymptotes* $\alpha = \dfrac{\sum \text{pole} - \sum \text{zero}}{r}$

$$L(s) = \frac{1}{s(s+1)(s+5)} \Rightarrow \alpha = \frac{\sum \text{pole} - \sum \text{zero}}{r} = \frac{0 + (-1) + (-5)}{3} = -2$$



Example 5 (cont'd)
Root locus: Step 3 (Breakaway)

- *Breakaway points are among roots of* $\frac{dL(s)}{ds} = 0$

$$L(s) = \frac{1}{s(s+1)(s+5)} \Rightarrow \frac{dL(s)}{ds} = -\frac{3s^2+12s+5}{\left(s(s+1)(s+5)\right)^2} = 0$$

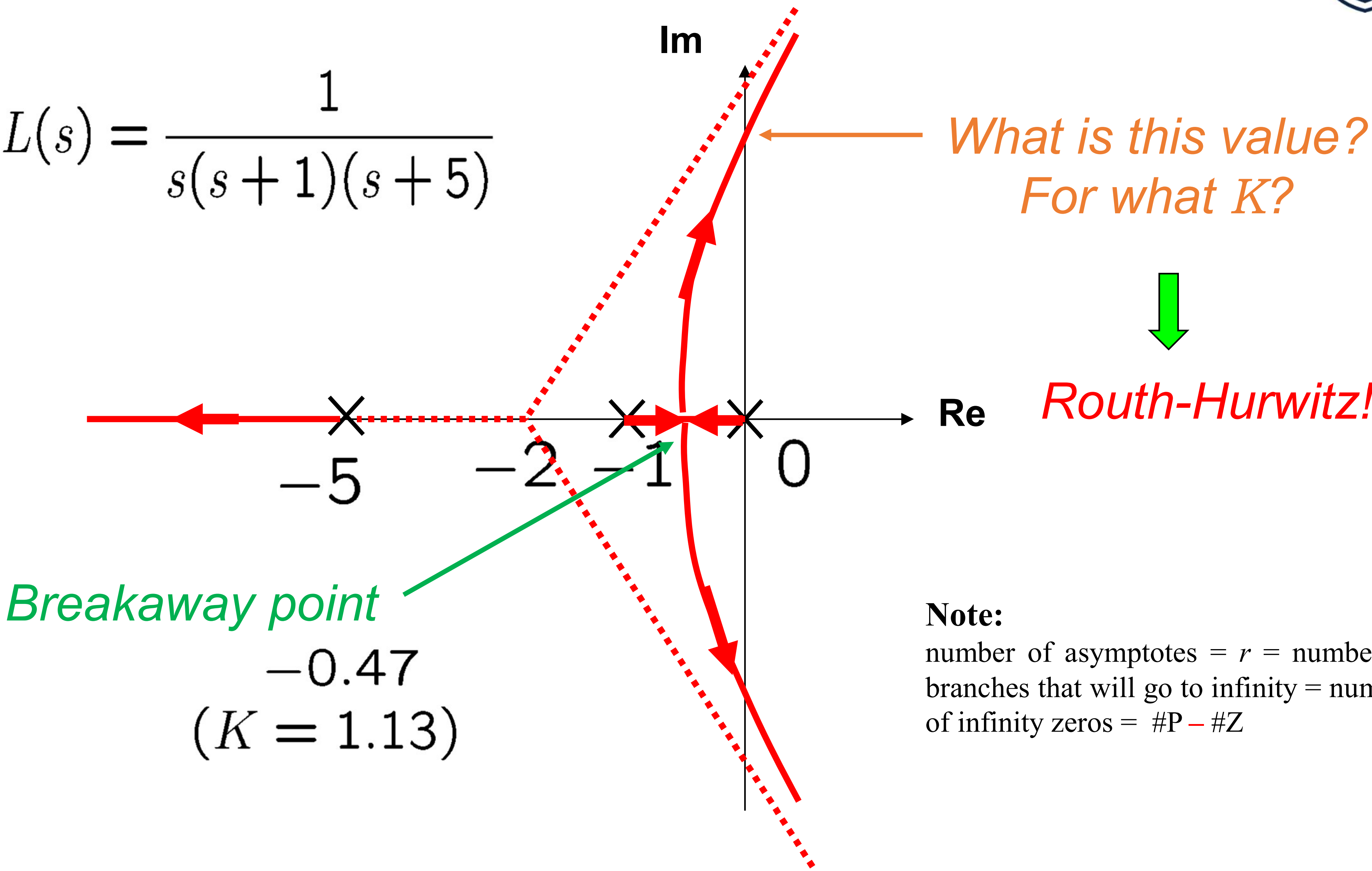
$$\Rightarrow s = -2 \pm \frac{\sqrt{21}}{3}$$

For each candidate s, *check the positivity of* $K = -\frac{1}{L(s)}$

$$\Rightarrow \begin{cases} s = -2 + \frac{\sqrt{21}}{3} \approx -0.47 \Rightarrow K \approx 1.13 \\ s = -2 - \frac{\sqrt{21}}{3} \approx -3.52 \Rightarrow K \approx -13.1 \end{cases}$$

Example 5 (cont'd)
Root locus: Step 3 (Breakaway)

$$L(s) = \frac{1}{s(s+1)(s+5)}$$



Note:
number of asymptotes = r = number of branches that will go to infinity = number of infinity zeros = #P – #Z

Example 5 (cont'd)
Finding stability condition for the range of K

- Characteristic equation:

$$1+\frac{K}{s(s+1)(s+5)}=0 \Leftrightarrow s^3+6s^2+5s+K=0$$

- Routh array:

s^3	1	5
s^2	6	K
s^1	$\frac{30-K}{6}$	
s^0	K	

Stability condition

$$0 < K < 30$$

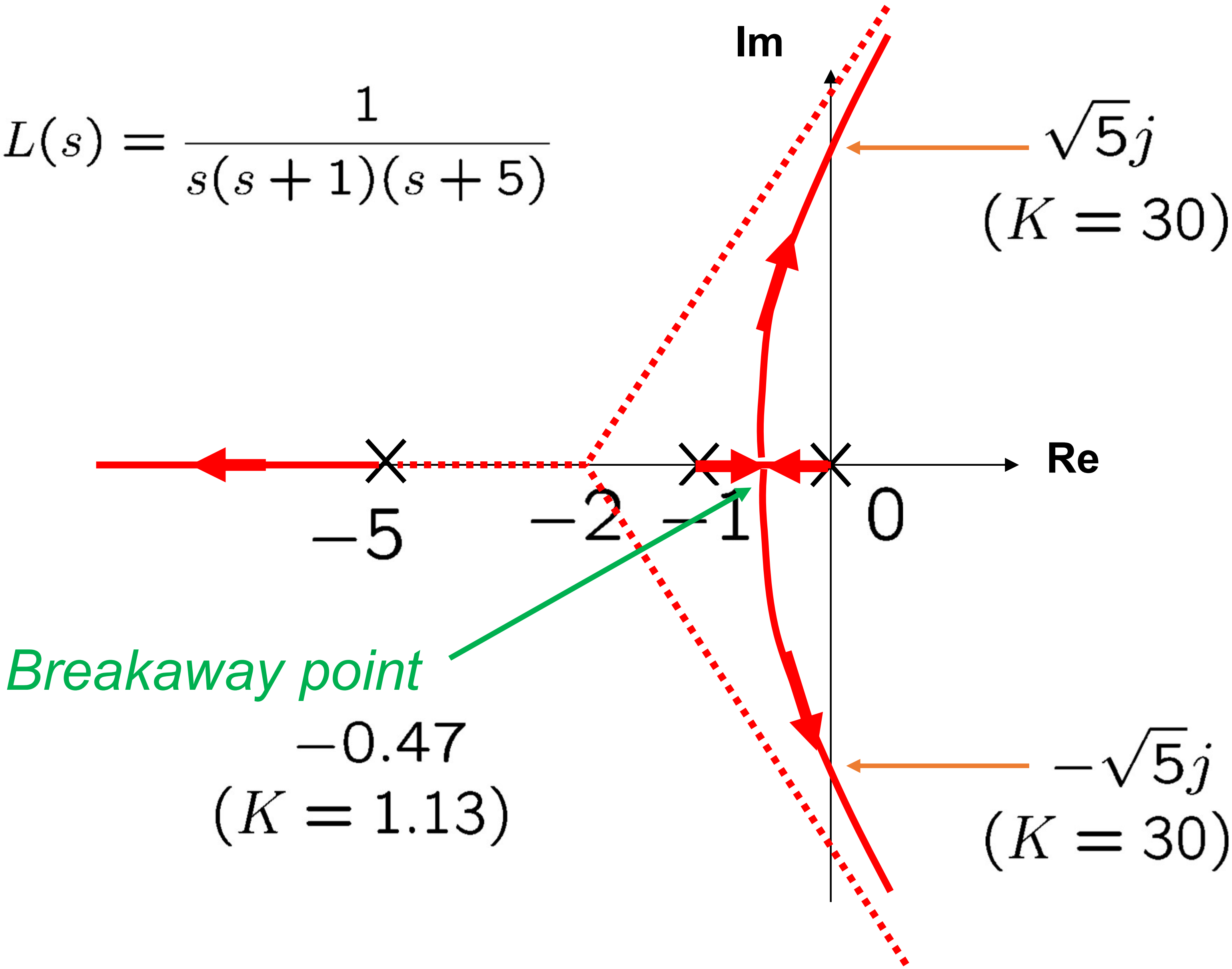
- When $K = 30$

$$6s^2+30=0 \Rightarrow s=\pm\sqrt{5}j$$

These are called ***jω*-axis crossing**.

Example 5 (cont'd)
Root locus

$$L(s) = \frac{1}{s(s+1)(s+5)}$$



Summary

- Root locus
  - What is root locus
  - How to roughly and quickly sketch root locus
- Sketching root locus relies heavily on experience. Please practice!
- To accurately draw root locus, use Matlab.
- Next
  - Root locus examples
  - Step 4 (Angles of departures and arrivals)