a place of mind

ELEC 341: Systems and Control

Lecture 12

Root locus: Examples

Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- Models for systems
  - ✓ Electrical
  - ✓ Electromechanical
  - ✓ Mechanical
- ✓ Linearization, delay

Analysis

- ✓ Stability
  - ✓ Routh-Hurwitz
  - Nyquist
- ✓ Time response
  - ✓ Transient
  - ✓ Steady state
- Frequency response
  - Bode plot

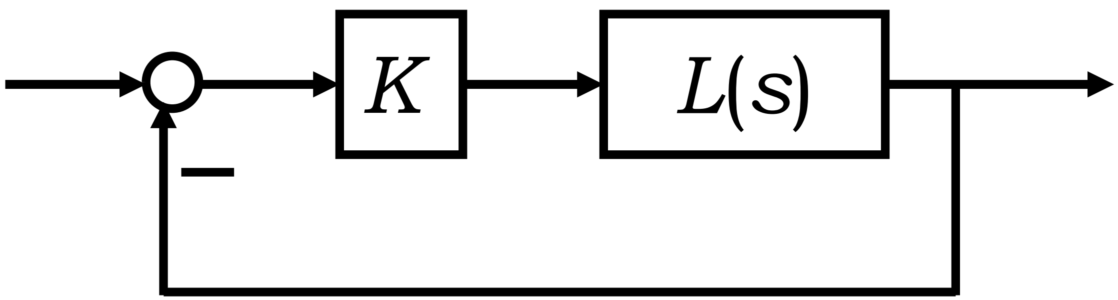
Design

- Design specs
- ➤ Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

Matlab simulations

What is Root Locus? (review)

- *Pole locations* of the system characterizes stability and transient properties.
- Consider a feedback system that has one parameter (gain) $K > 0$ to be designed:



$K.L(s)$: open-loop TF

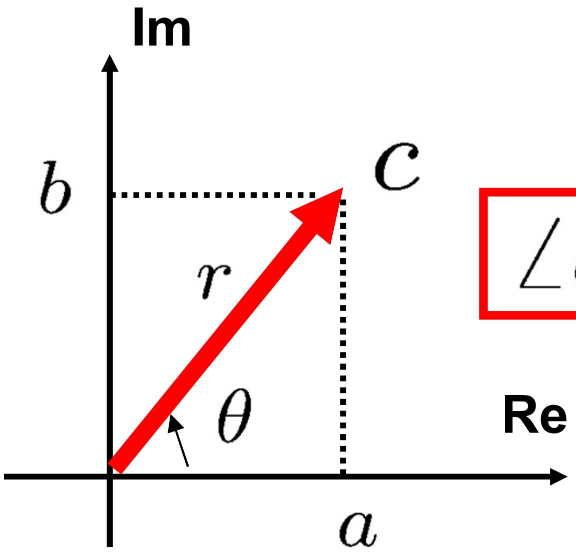
- *Root locus* graphically shows how poles of CL system vary as K varies from 0 to infinity.

RL sketching (review)

- **Step 0:** Mark open-loop poles and zeros
- **Step 1:** On the real axis
- **Step 2:** Asymptotes
- **Step 3:** Breakaway points
- **Step 4:** Angles of departures and arrivals (to be explained next)

Complex numbers

- Representation:
  - Cartesian form
    $$c = a + bj$$
  - Polar form
    $$c = re^{j\theta}$$



$$\angle c = \theta$$

- Multiplication & division in the polar form:

$$\left.\begin{aligned} c_1 &= r_1 e^{j\theta_1} \\ c_2 &= r_2 e^{j\theta_2} \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 c_2 &= r_1 r_2 e^{j(\theta_1+\theta_2)} \\ \frac{c_1}{c_2} &= \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)} \end{aligned}$$

$$\angle L(s) = \angle \frac{s - z_1}{(s-p_1)(s-p_2)} = \angle(s - z_1) - \angle(s - p_1) - \angle(s - p_2) = \phi$$

Angle condition

- For an open-loop transfer function $L(s)$,

$$s = s_0 \text{ is on the root locus} \;\Leftrightarrow\; \angle L(s_0) = 180^\circ$$

(or an odd multiple of 180°)

- Why?

Note: "s.t." is short for "so that".

$s = s_0$ is on RL

$\Rightarrow$ There exists a $K > 0$ s.t. $1 + KL(s_0) = 0$.

$\Rightarrow L(s_0) = -\frac{1}{K} < 0$

$\Rightarrow \angle L(s_0) = 180^\circ$

$\angle L(s_0) = 180^\circ$

$\Rightarrow$ There exists a $K > 0$ s.t. $L(s_0) = -\frac{1}{K}$.

$\Rightarrow 1 + KL(s_0) = 0$

$\Rightarrow s = s_0$ is on RL

Example 1: With complex poles

$$L(s) = \frac{s}{s^2 + s + 1}$$

zero	0
pole	$-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

$$\frac{dL(s)}{ds} = 0$$

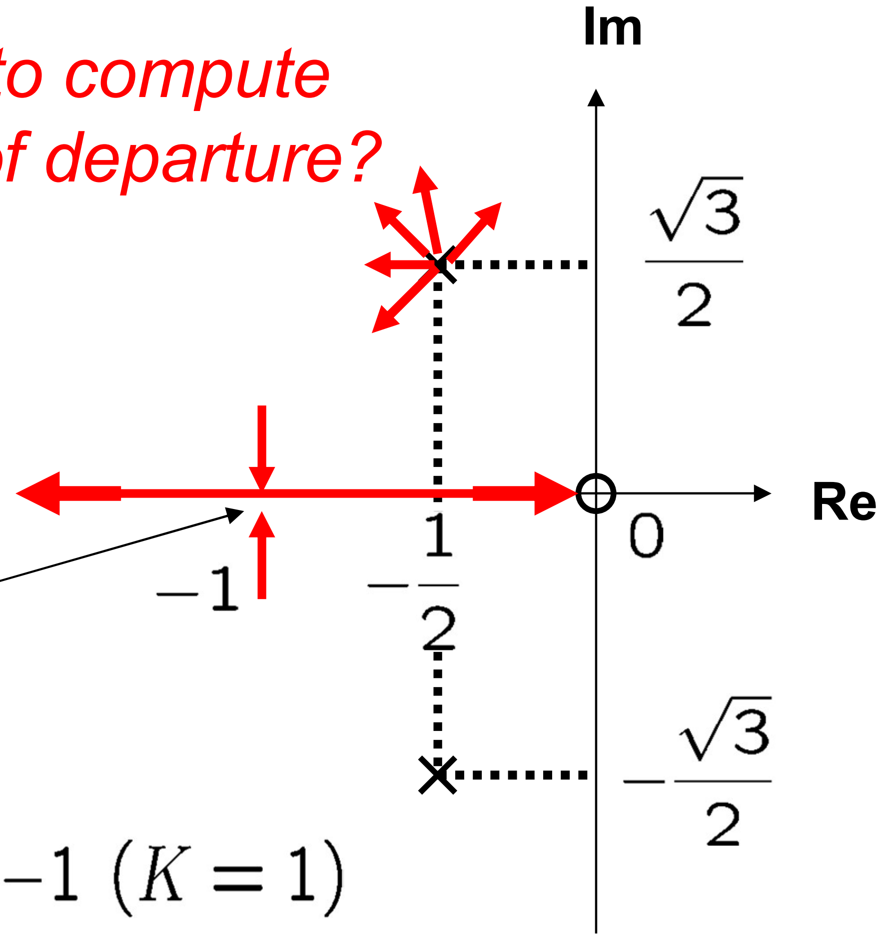
Breakaway point

$$s^2 + s + 1 - s(2s+1) = 0$$

$$\Rightarrow s = \pm 1 \quad \Rightarrow s = -1 \ (K = 1)$$

After Steps 0,1,2,3, we obtain

How to compute angle of departure?



Note that in this case, breakaway point is actually a break-in point. Hence, you can also use the term *break-in point*.

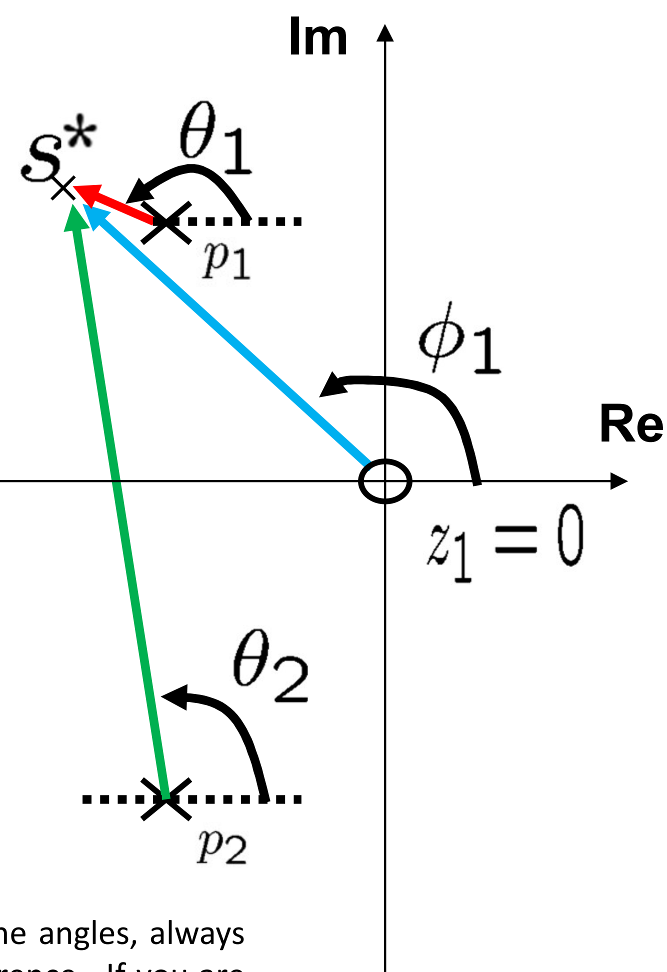
Example 1 (cont'd): Step 4: Angle of departure

- Select a point "s^*" near p_1. Use angle condition.

$$s^* \approx -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$p_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$p_2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$



$$\angle L(s^*) \approx 180^\circ$$

$$\angle L(s^*) = \angle \frac{s^* - z_1}{(s^* - p_1)(s^* - p_2)}$$

$$= \angle(s^* - z_1) - \angle(s^* - p_1) - \angle(s^* - p_2)$$

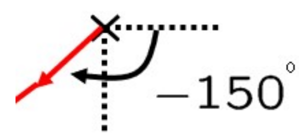
$$= \phi_1 - \theta_1 - \theta_2 \approx 180^\circ$$

Method 1 (Graphical Method):

Since s^* is close to p_1

$$\phi_1 \approx 120, \ \ \theta_2 \approx 90$$

$$\theta_1 = -150$$



Note: When substituting for the angles, always use positive Re axis as the reference. If you are moving counterclockwise, it will be positive.

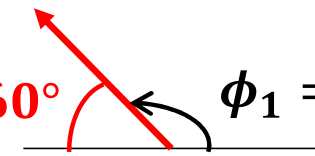
Example 1 (cont'd): Step 4: Angle of departure

Method 2 (Analytical Method): To be used in your tests.

$$s^* \approx -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

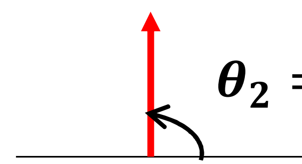
$$\phi_1 = \angle(s^* - z_1) = \angle\left\{\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) - (0 + j0)\right\} = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right) = -60° \rightarrow \phi_1 = -60°$$

$$\rightarrow \boldsymbol{\phi_1 = +120°}$$



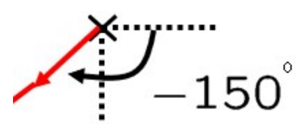
$$\theta_2 = \angle(s^* - p_2) = \angle\left\{\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\right\} = \angle\{(0 + j\sqrt{3})\} = \tan^{-1}\left(\frac{\sqrt{3}}{0}\right)$$

$$= +90° \rightarrow \boldsymbol{\theta_2 = +90°}$$



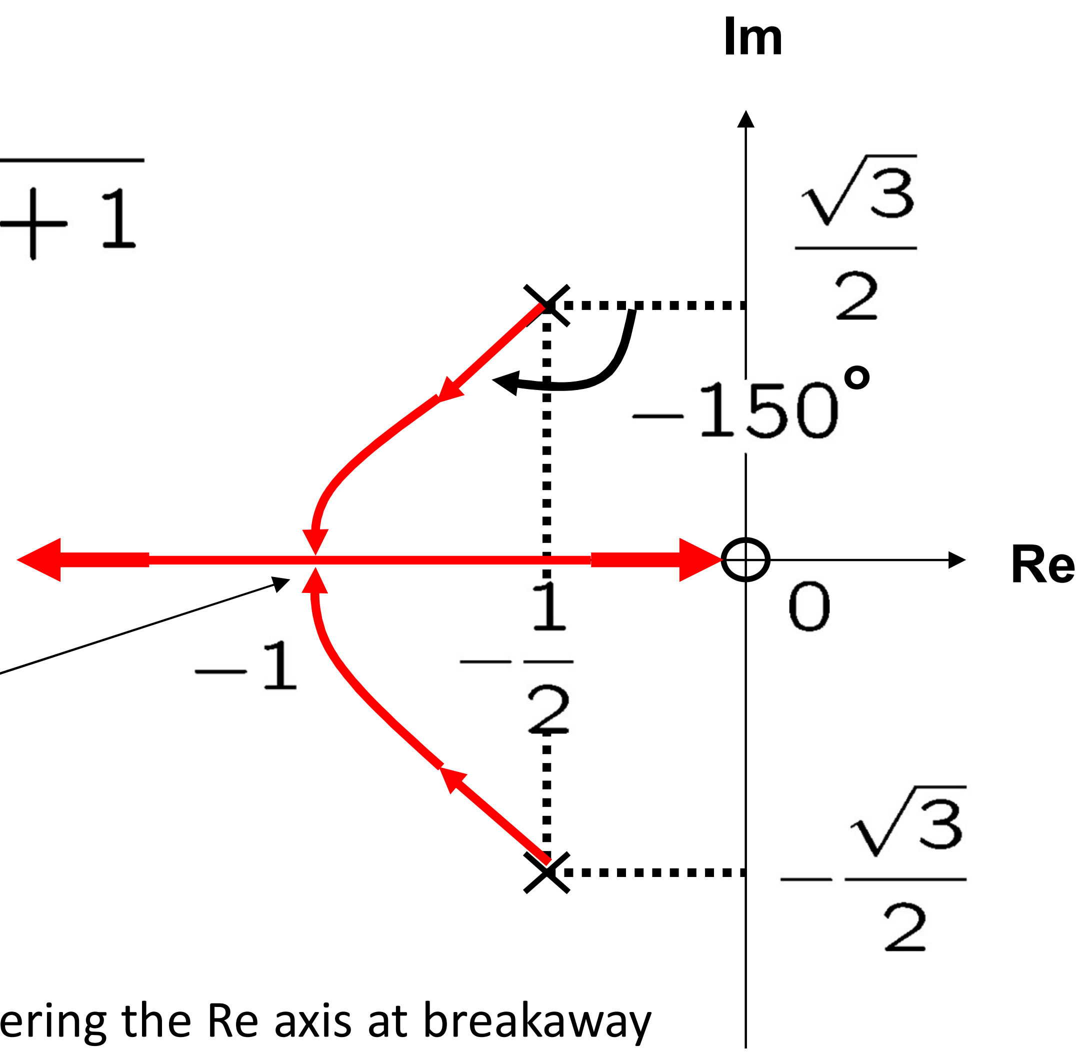
Angle condition: $\phi_1 - \theta_1 - \theta_2 \approx 180°$

$$120° - \theta_1 - 90° = 180° \rightarrow \boxed{\boldsymbol{\theta_1 = -150°}}$$



Example 1 (cont'd): Root locus

$$L(s) = \frac{s}{s^2 + s + 1}$$

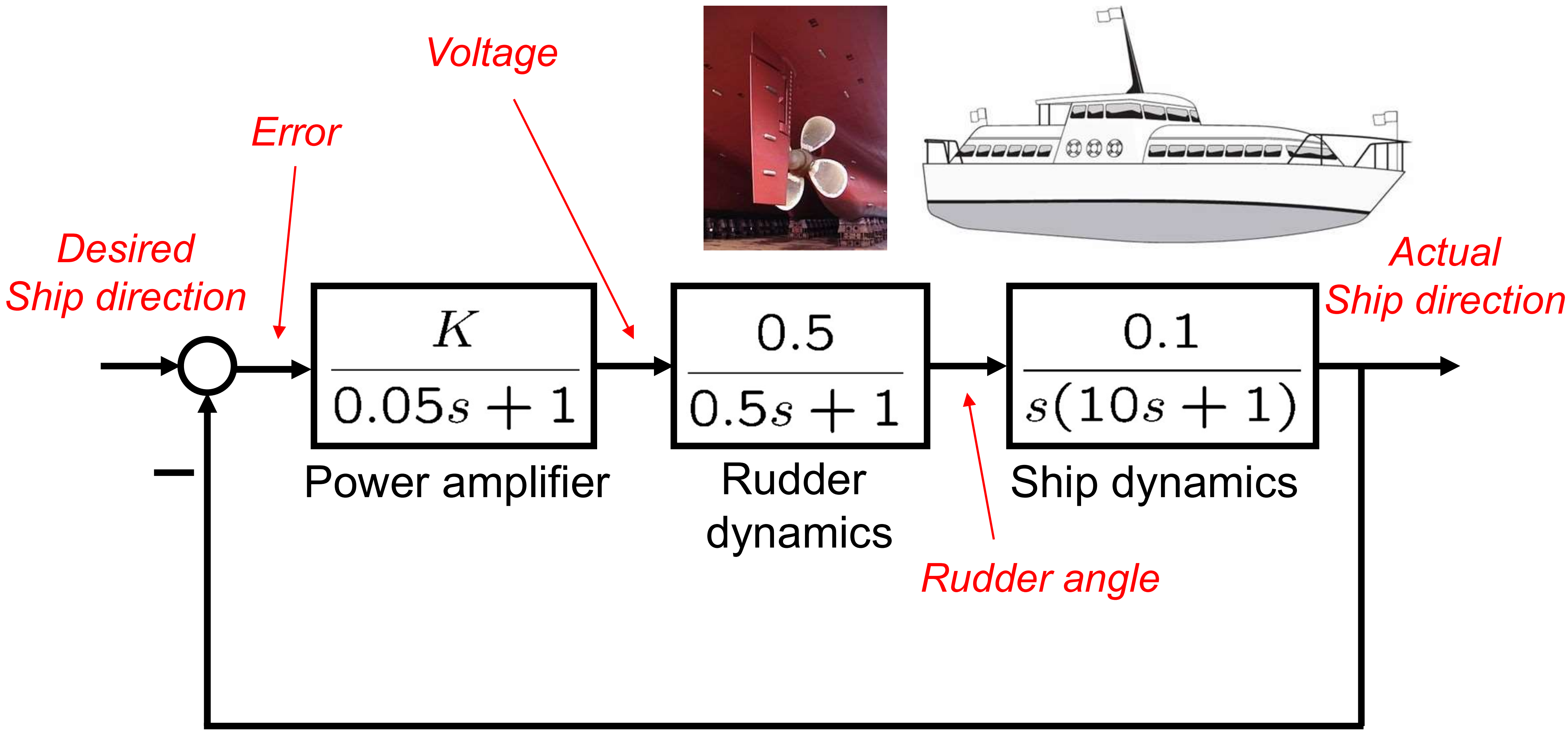


Break-in point

$$s = -1 \; (K = 1)$$

Note: Leaving the Re axis or entering the Re axis at breakaway (or break-in) points always happen at an angle of 90°.

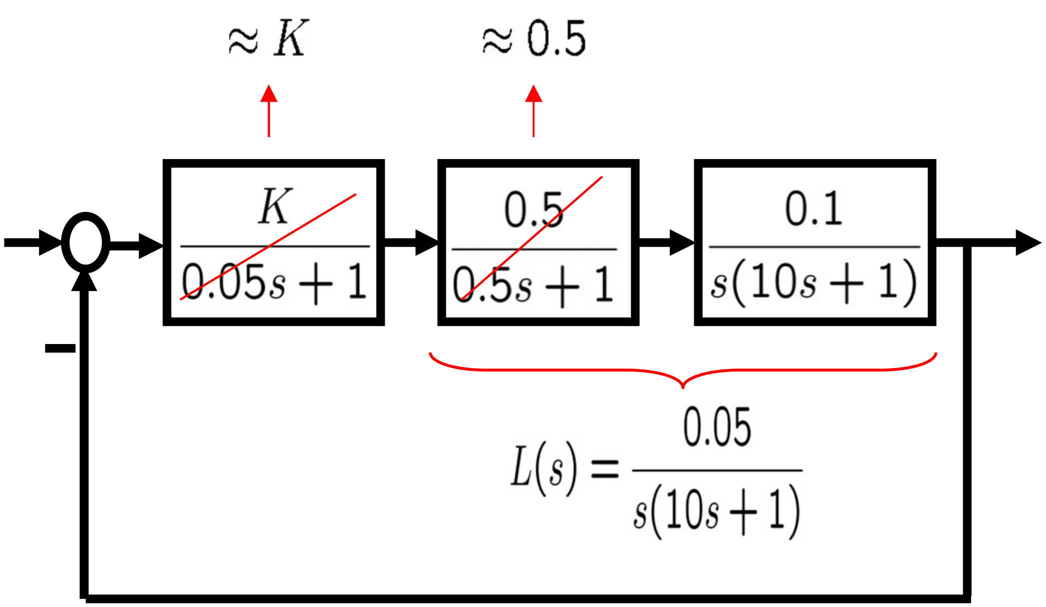
Example 2: Ship-steering system



Example 2 (cont'd):

- Time constants:
  - Power amplifier dynamics: 0.05s (fastest)
  - Rudder dynamics: 0.5s
  - Ship dynamics: 10s (slowest)
- We draw root locus for three cases.
  - We take into account mainly ship dynamics.
    - ➢ Assume transfer function of power amplifier is roughly equal to K and transfer function of the rudder is roughly equal to 0.5.
  - We take into account mainly rudder & ship dynamics.
    - ➢ Assume transfer function of power amplifier is roughly equal to K.
  - We take into account all three dynamics without any simplifications.

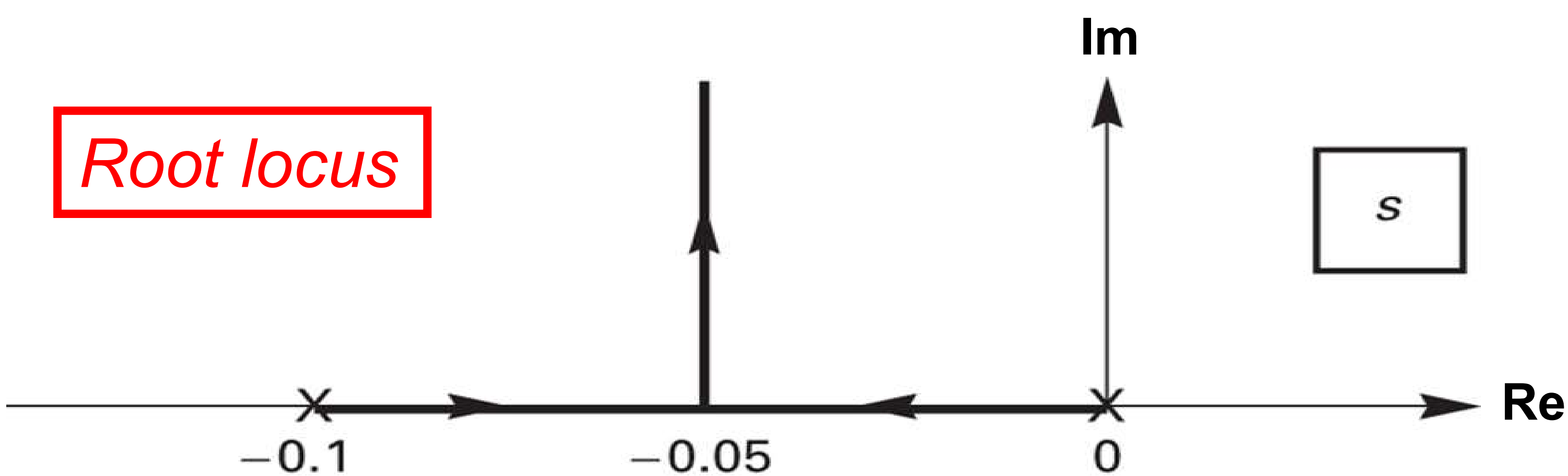
Example 2 (cont'd):
1. Mainly ship dynamics



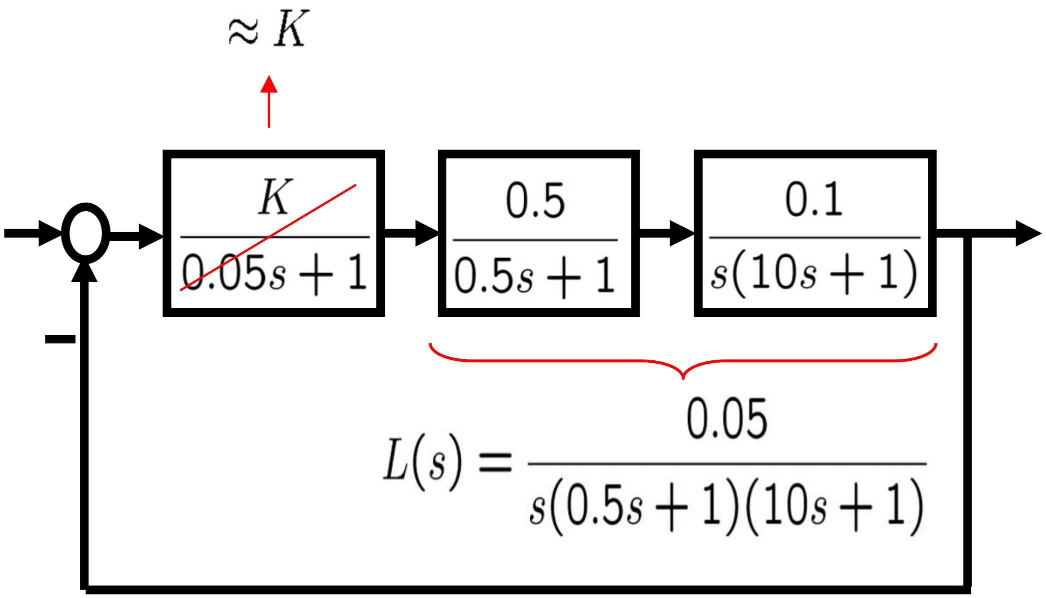
Q: Plot the root locus for the given $L(s)$ and find the 2% settling time.

(Ans: 80 seconds)

Root locus



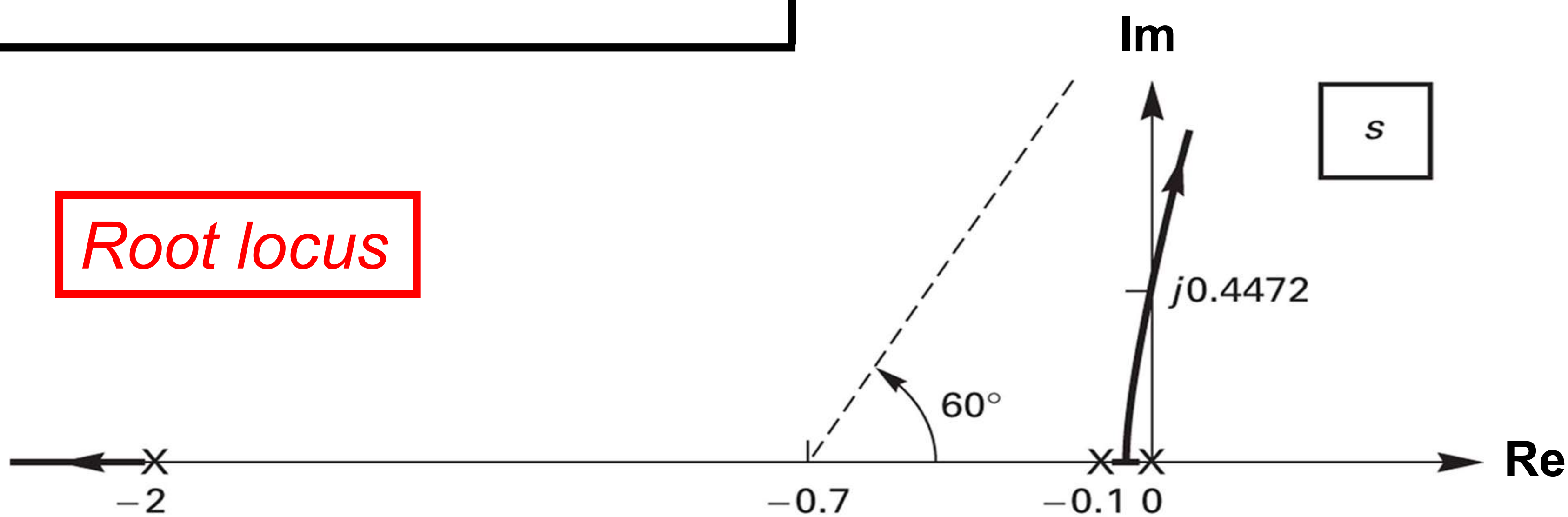
Example 2 (cont'd):
2. Mainly rudder & ship dynamics



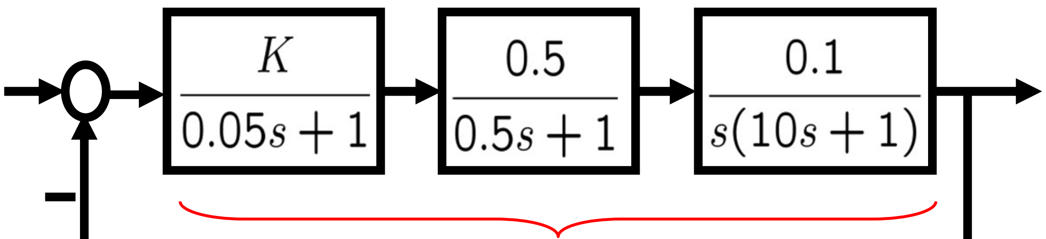
$$L(s) = \frac{0.05}{s(0.5s+1)(10s+1)}$$

Q: Plot the root locus for the given $L(s)$. What is the allowable K for closed-loop stability?
(Ans: $0 < K < 42$)

Root locus



Example 2 (cont'd): 3. All dynamics included

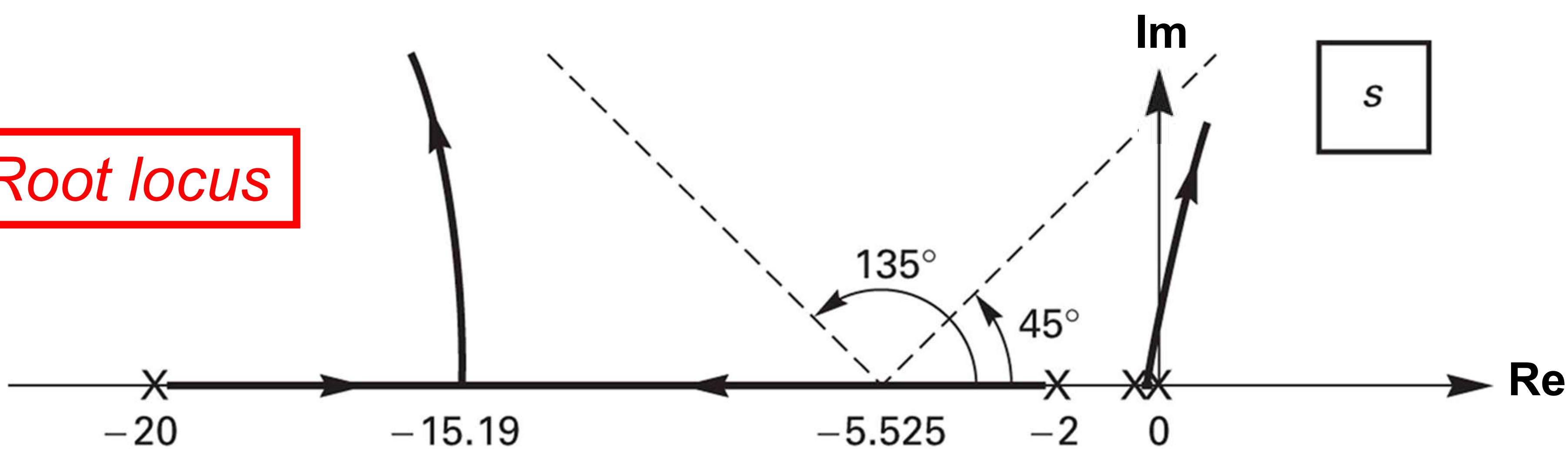


$$L(s) = \frac{0.05}{s(0.05s+1)(0.5s+1)(10s+1)}$$

Q: Plot the root locus for the given $L(s)$. What is the allowable K for closed-loop stability?

(Ans: $0 < K < 38.03$)

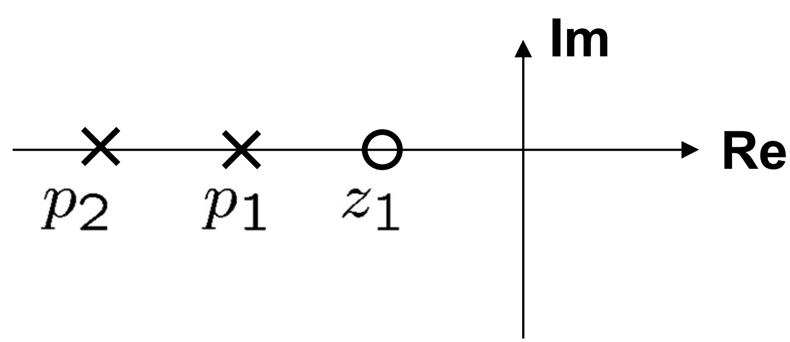
Root locus



Root locus: Step 0

- *Root locus is symmetric w.r.t. the real axis.*
  - Characteristic equation is an equation with real coefficients. Hence, if a complex number is a root, its complex conjugate is also a root.
- *The total number of branches = order of $L(s)$ = number of $L(s)$ poles*
  - If $L(s) = n(s)/d(s)$, then Ch. Eq. is $d(s)+K.n(s)=0$, which has roots as many as the order of $d(s)$.
- *Mark poles of $L(s)$ with "x" and zeros of $L(s)$ with "o".*

$$L(s) = \frac{s - z_1}{(s-p_1)(s-p_2)}$$

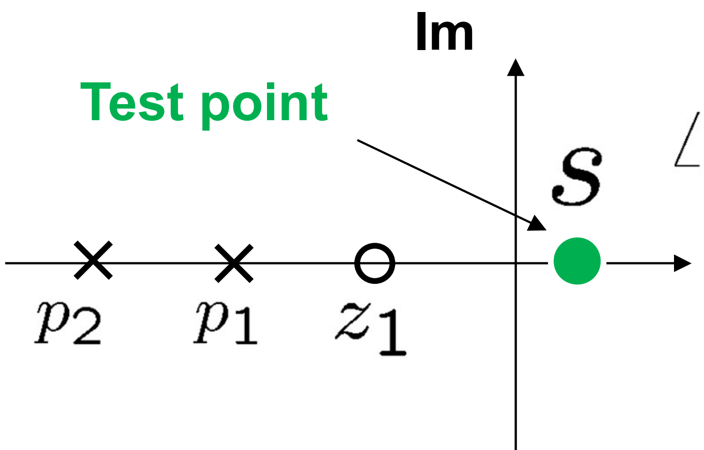


Root locus: Step 1

- **Step 1-1:** *RL includes all points on real axis to the left of an odd number of roots (poles and zeros).*
- **Step 1-2:** *RL originates from the poles of $L(s)$, and terminates at the zeros of $L(s)$, including infinity zeros.*

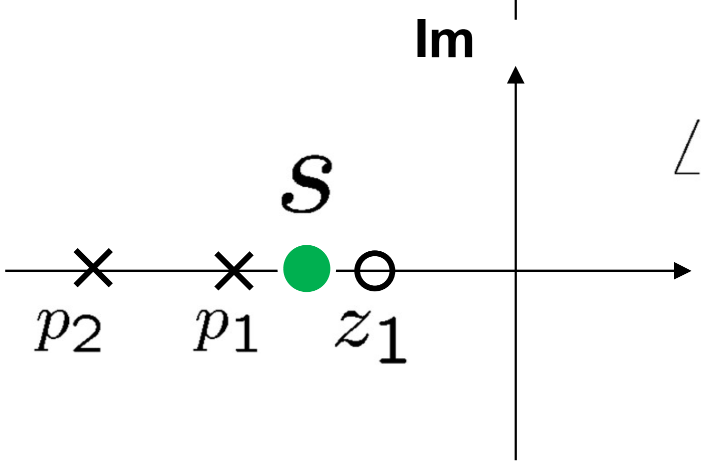
Root locus: Step 1-1

- **Step 1-1:** *RL includes all points on real axis to the left of an odd number of roots (poles and zeros).*



$$\angle L(s) = \underbrace{\angle(s - z_1)}_{0} - \underbrace{\angle(s - p_1)}_{0} - \underbrace{\angle(s - p_2)}_{0}$$

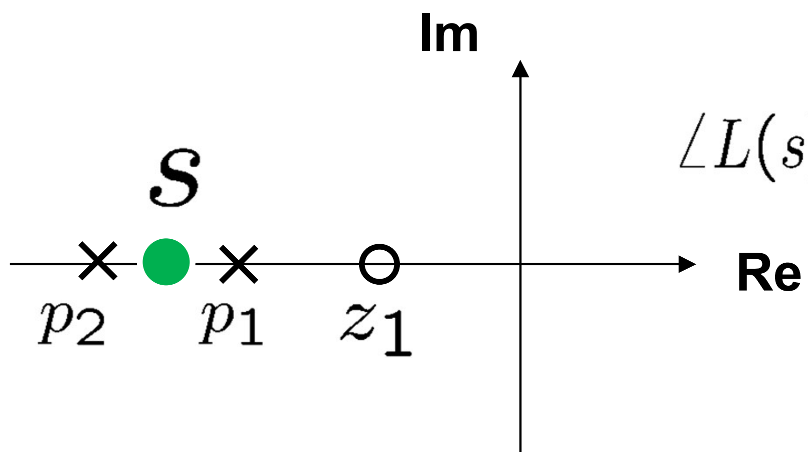
It does not satisfy angle condition!



$$\angle L(s) = \underbrace{\angle(s - z_1)}_{180} - \underbrace{\angle(s - p_1)}_{0} - \underbrace{\angle(s - p_2)}_{0}$$

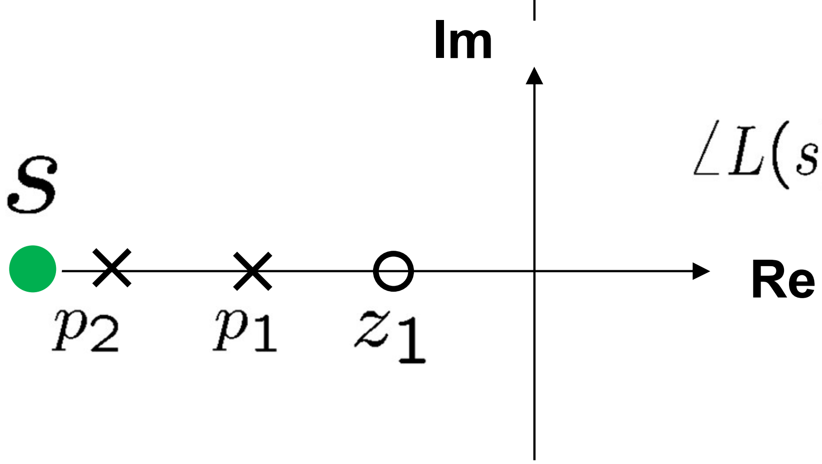
It satisfies angle condition!

Root locus: Step 1-1 (cont'd)



$$\angle L(s) = \underbrace{\angle(s - z_1)}_{180} - \underbrace{\angle(s - p_1)}_{180} - \underbrace{\angle(s - p_2)}_{0}$$

It does not satisfy angle condition!



$$\angle L(s) = \underbrace{\angle(s - z_1)}_{180} - \underbrace{\angle(s - p_1)}_{180} - \underbrace{\angle(s - p_2)}_{180}$$

It satisfies angle condition!

Note: In general, for Angle Condition, negative values of k are also acceptable. That is, $k = 0, \pm 1, \pm 2, \ldots$

Root locus: Step 1-2

- **Step 1-2:** *RL originates from the poles of L(s), and terminates at the zeros of L(s), including infinity zeros.*

$$1+K\underbrace{\frac{n(s)}{d(s)}}_{L(s)}=0 \Leftrightarrow d(s)+Kn(s)=0 \Leftrightarrow \frac{1}{K}+\frac{n(s)}{d(s)}=0$$

$K=0$ → $d(s)=0$

s: Poles of $L(s)$

$K=\infty$ → $\frac{n(s)}{d(s)}=0$ → $n(s)=0$

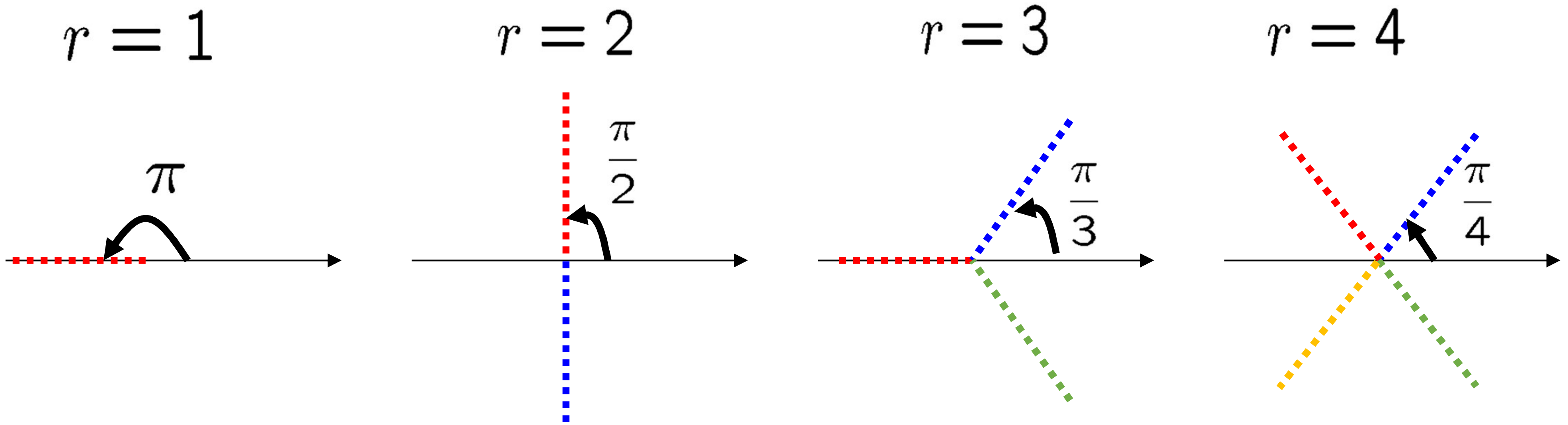
s: Zeros of $L(s)$

Root locus: Step 2

- *Angles of asymptotes are:*

$$\frac{\pi}{r} \times (2k+1), \;\; k = 0, 1, \ldots, (r-1)$$

Odd number



Number of asymptotes = relative degree (r) of $L(s)$:

$$r = \underbrace{n}_{\deg(\text{den})} - \underbrace{m}_{\deg(\text{num})}$$

Root locus: Step 2 (cont'd)

- For a very large s,

$$L(s) = \frac{n_0 s^m + \cdots}{s^n + \cdots} = \frac{n_0 s^{n-r} + \cdots}{s^n + \cdots} \approx \frac{n_0}{s^r}$$

- Ch. Eq. is approximately:

$$r = \underbrace{n}_{\text{deg (den)}} - \underbrace{m}_{\text{deg (num)}}$$

$$1 + KL(s) = 0 \Rightarrow 1 + K\frac{n_0}{s^r} = 0 \Rightarrow s^r + Kn_0 = 0$$

$$\Rightarrow s^r = -Kn_0 < 0 \ \text{(we assume } n_0 > 0)$$

$$\Rightarrow \angle s^r = \pi \times (2k+1),\ k = 0, 1, 2, \ldots$$

$$\Rightarrow \boxed{\angle s = \frac{\pi}{r} \times (2k+1),\ k = 0, 1, 2, \ldots}$$

Summary

- Angle condition
- Examples of sketching root locus
- Next
  - Controller design based on root locus