a place of mind
UBC

ELEC 341: Systems and Control

Lecture 13

Root locus: Controller design

Course roadmap

Modeling
- ✓ Laplace transform
- ✓ Transfer function
- Models for systems
  - ✓ Electrical
  - ✓ Electromechanical
  - ✓ Mechanical
- ✓ Linearization, delay

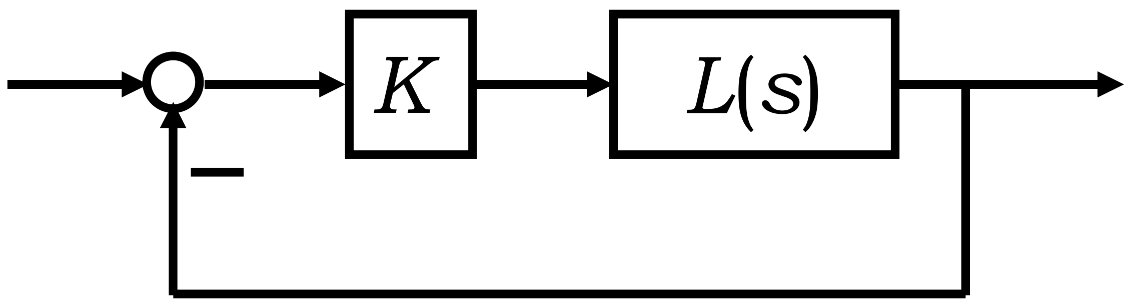
Analysis
- ✓ Stability
  - ✓ Routh-Hurwitz
  - Nyquist
- ✓ Time response
  - ✓ Transient
  - ✓ Steady state
- Frequency response
  - Bode plot

Design
- Design specs
- ➤ Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

Matlab simulations

What is Root Locus? (review)

- *Pole locations* of the system characterizes stability and transient properties.
- Consider a feedback system that has one parameter (gain) $K > 0$ to be designed.



$K.L(s)$: open-loop TF

- *Root locus* graphically shows how poles of CL system varies as K varies from 0 to infinity.

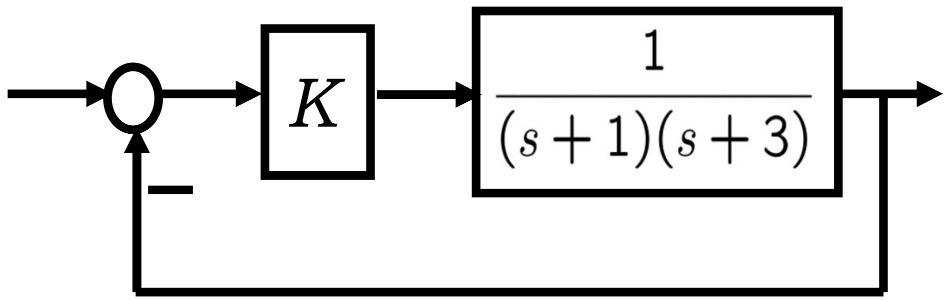
Today's topics

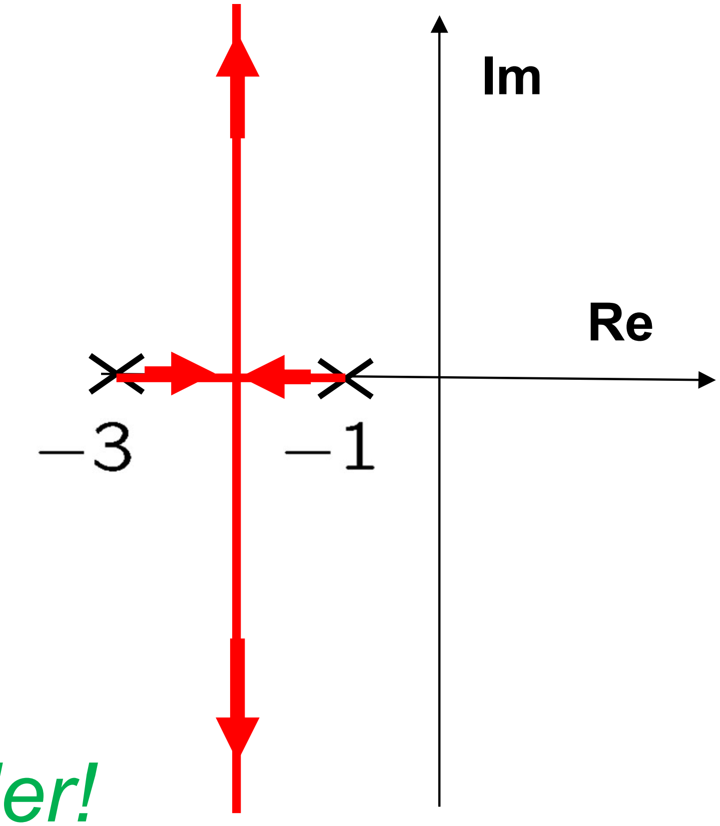
- How to use the root locus
  - **Example 1:** Gain design to meet design specifications
  - **Example 2:** Pole or zero location design (i.e., design based on pole or zero of OLTF)
  - **Example 3:** Multiple parameter design

  - In following lectures ….
    - Lead and lag compensator design
    - PID controller design

Example 1

- Design the gain K so that
  - Overshoot is at most 4.32%
  - 2% settling time is at most 2 sec
  - Error constant $K_p > 1$
- Find minimum SS error for unit step





$$\text{CLTF} = \frac{K}{(s+1)(s+3)+K}$$

2nd order!

Note: For your tests, there is no need to draw your graphs to scale. Please, just show the trend of variation.

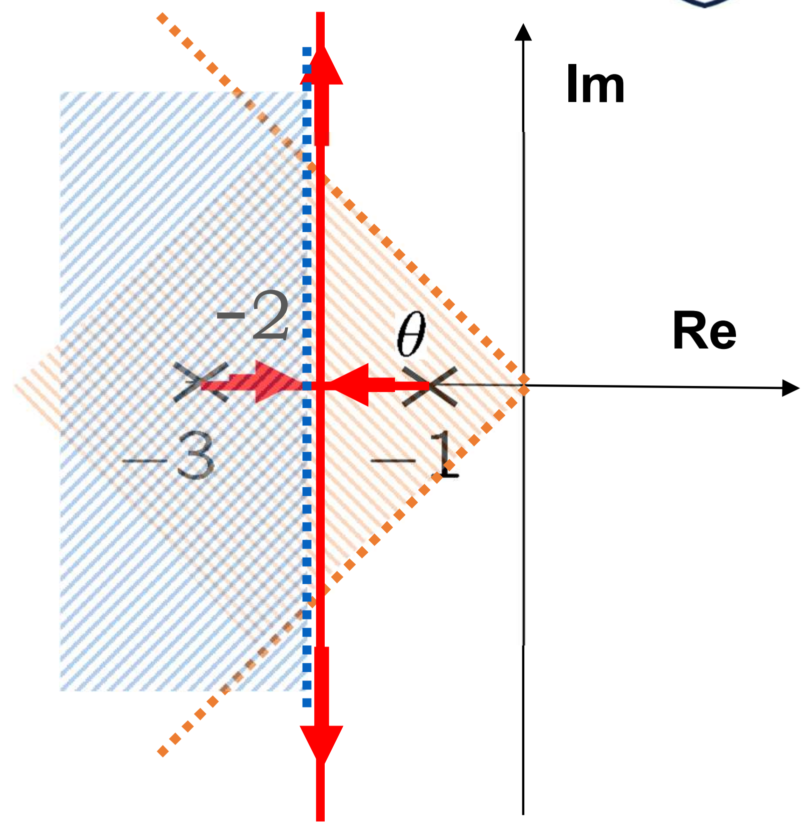
Example 1 (cont'd)

- Allowable region
  - Overshoot is at most 4.32%

$$\theta \leq 45°$$

  - 2% settling time is at most 2 sec

$$T_s = \frac{4}{|\mathrm{Re}|} \leq 2 \;\Leftrightarrow\; |\mathrm{Re}| \geq 2$$



- **Step 1:** Draw the allowable region using both settling time and overshoot constraints.
- **Step 2:** Draw the root locus diagram.
- **Step 3:** Find the *overlap region* of the root locus and the allowable region.

Example 1 (cont'd)

a) $PO \leq 4.32\%$

b) $2\%T_s \leq 2$ sec

c) $K_p > 1$

d) min e_{ss} for step input

---

a) $\ln \frac{PO}{100} = \frac{-\pi\xi}{\sqrt{1-\xi^2}} = -\pi/\tan\theta$

$$\ln \frac{4.32}{100} = -\pi/\tan\theta \implies \theta = 45^\circ \implies \boxed{\theta \leq 45^\circ}$$

b) $T_s = \frac{4}{|\text{Re}|} \leq 2 \implies \boxed{|\text{Re}| \geq 2}$

c) $L(s) = \frac{K}{(s+1)(s+3)}$

$K_p = \lim_{s\to 0} KL(s) = KL(0)$ ➡

$$K_p = KL(0); \quad K_p > 1 \implies KL(0) > 1 \implies K > \frac{1}{L(0)}$$

$$L(0) = \frac{1}{(0+1)(0+3)} = \frac{1}{3}$$

$$\implies K > \frac{1}{(1/3)} \implies \boxed{K > 3}$$

Example 1 (cont'd)

$$PO = 4.32\% \implies \boxed{\xi \approx 0.7}$$

$$2\% T_s = 2 \implies \frac{4}{\xi\omega_n} = 2 \implies \xi\omega_n = 2$$

$$\omega_n = \frac{2}{\xi} = \frac{2}{0.7} = 2.8571 \implies \boxed{\omega_n = 2.8571}$$

$$s = -\xi\omega_n \pm \omega_n\sqrt{1-\xi^2}j = -2 \pm 2.8571\sqrt{1-0.7^2}j = -2 \pm 2.8571\sqrt{0.51}j = -2 \pm 2j \implies$$

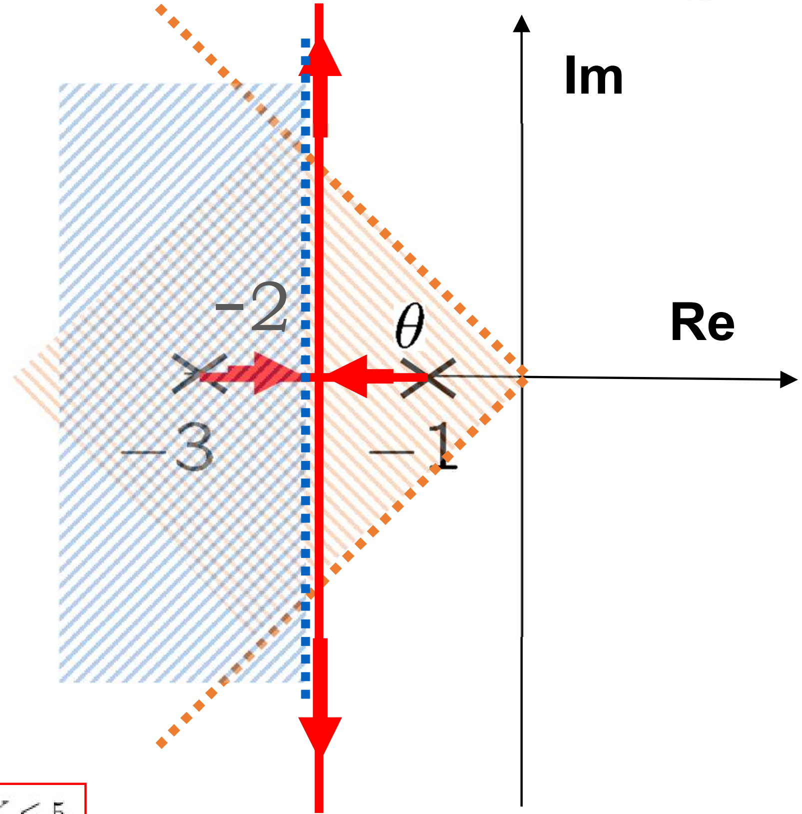
$$\boxed{s = -2 \pm 2j}$$

Example 1 (cont'd)

$$L(s=-2) = \frac{1}{(-2+1)(-2+3)} = \frac{1}{(-1)(1)} = -1$$

$$K = -\frac{1}{L(s)} \quad \text{or} \quad 1 + KL(s) = 0 \quad \text{(C.E.)}$$

$$s=-2: \quad K = -\frac{1}{(-1)} \implies K = 1$$



$$L(s=-2+2j) = \frac{1}{(-2+2j+1)(-2+2j+3)} = \frac{1}{(-1+2j)(1+2j)}$$

$$= \frac{1}{(-1)(1)+(-1)(2j)+(2j)(1)+(2j)(2j)}$$

$$= \frac{1}{-1-2j+2j-4} = \frac{1}{-5} = -\frac{1}{5}$$

$$K = -\frac{1}{L(s=-2+2j)} = -\frac{1}{(-1/5)} \implies K = 5 \implies$$

From the constraints, the acceptable gain range is $3 < K \leq 5$.

d) min e_{ss} for unit step input

$$r(t) = R \cdot u(t) = 1 \cdot u(t)$$

$$e_{ss} = \frac{R}{1+K_p} = \frac{1}{1+KL(0)} = \frac{1}{1+K(1/3)} = \frac{3}{3+K}$$

Example 1 (cont'd)

d) min e_{ss} for unit step input

$$r(t) = R \cdot u(t) = 1 \cdot u(t)$$
$$e_{ss} = \frac{R}{1 + K_p}$$

For minimum e_{ss}, we need maximum K_p (or K).

$K_p = \lim_{s \to 0} KL(s)$; max K_p means max K (which we already found as 5);

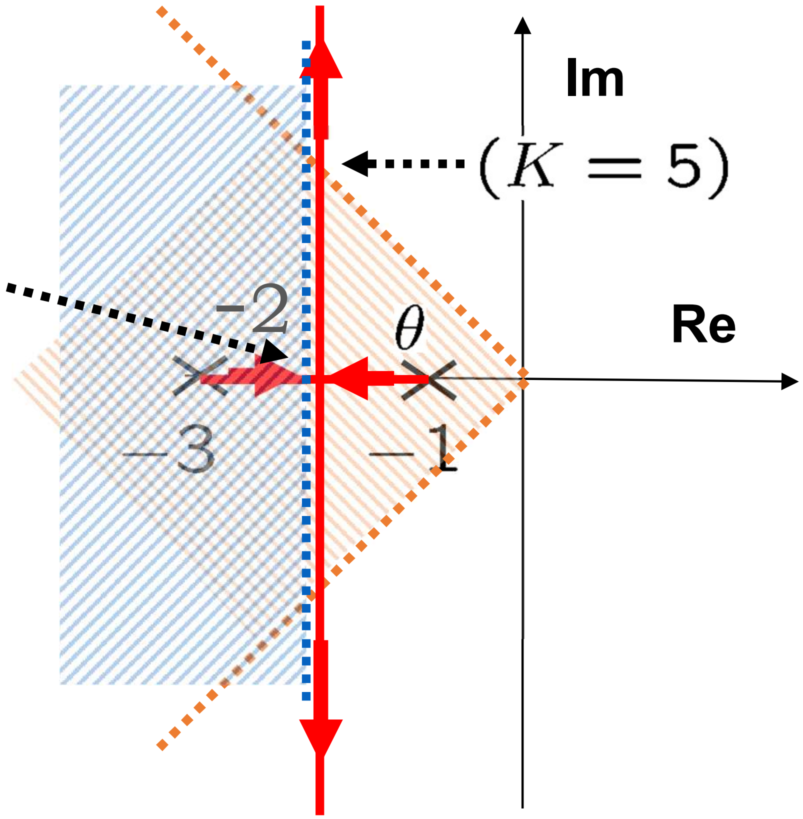
$$\text{maximum } K_p = \lim_{s \to 0} 5L(0) = 5 \times \frac{1}{3} = \frac{5}{3}$$
$$\text{minimum } e_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + \frac{5}{3}} = \frac{1}{\frac{8}{3}} = \frac{3}{8} = 0.375 \implies \boxed{\text{minimum } e_{ss} = 0.375}$$

Example 1 (cont'd)

Summary:

- Gain value computations

$$K = -\frac{1}{L(s)} \quad (\Leftrightarrow 1 + KL(s) = 0)$$



  - $s = -2$

$$K = -\frac{1}{L(-2)} = 1$$

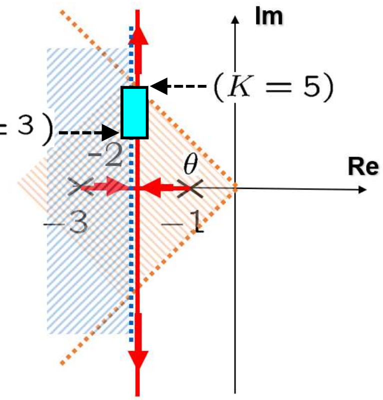
  - $s = -2 + 2j$

$$K = -\frac{1}{L(-2+2j)} = 5$$

Example 1 (cont'd)

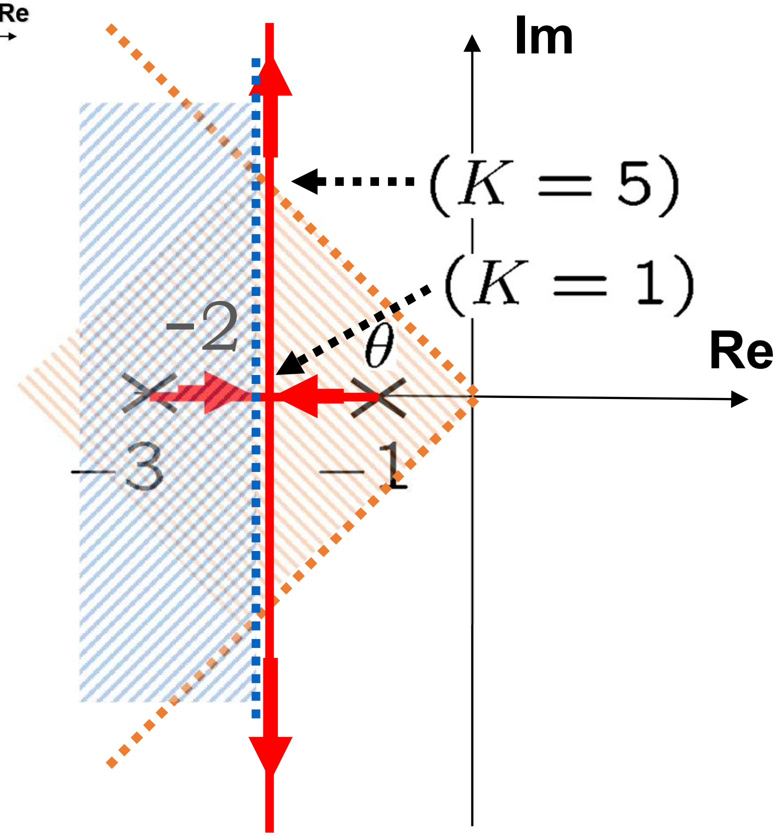
- Acceptable gain:

$3 < K \leq 5$ (shown by pale blue)



- Minimum SS error for unit step:

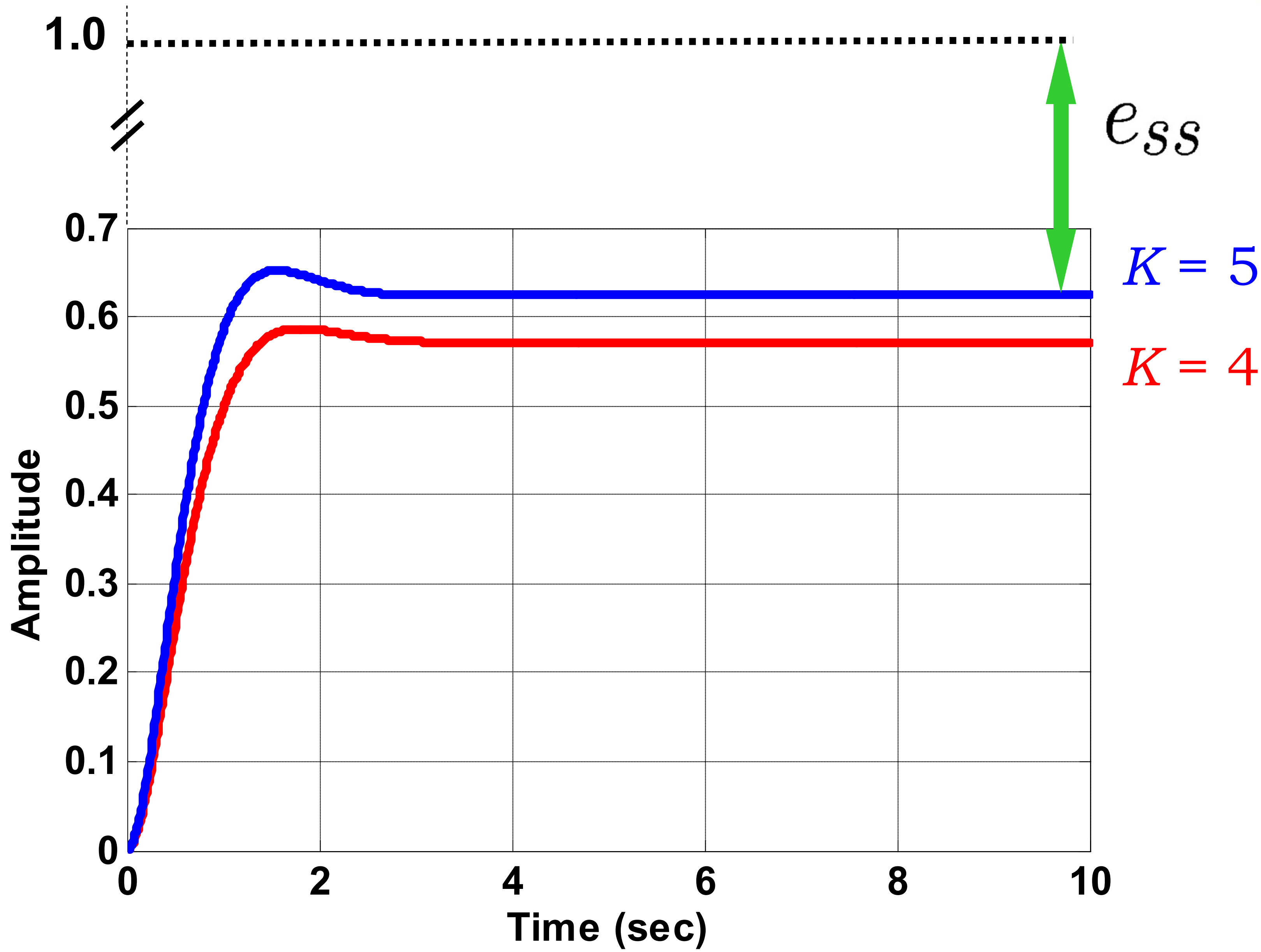
$$e_{ss} = \frac{1}{1 + 5/3} = \frac{3}{8} = 0.375 \Rightarrow e_{ss} = 0.375$$



- Limitations of gain controller:
  - T_s cannot be less than 2 sec
  - Overshoot and SS error cannot be improved simultaneously.

⇨ *Lead-lag or PID compensator design!*

Example 1 (cont'd): Step responses



Today's topics

- How to use the root locus
  - **Example 1:** Gain design to meet design specifications
  - **Example 2:** Pole or zero location design (i.e., design based on pole or zero of OLTF)
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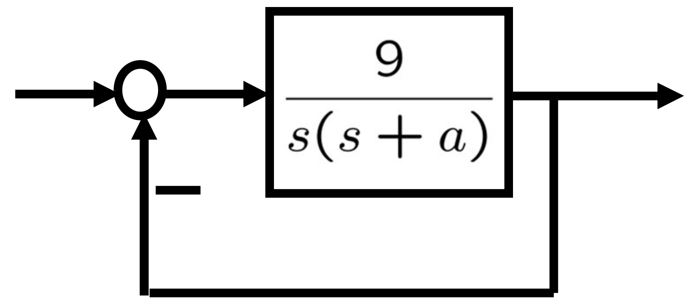
Example 2

Q: Draw root locus for $a > 0$



Characteristic eq.

$$1 + \frac{9}{s(s+a)} = 0$$

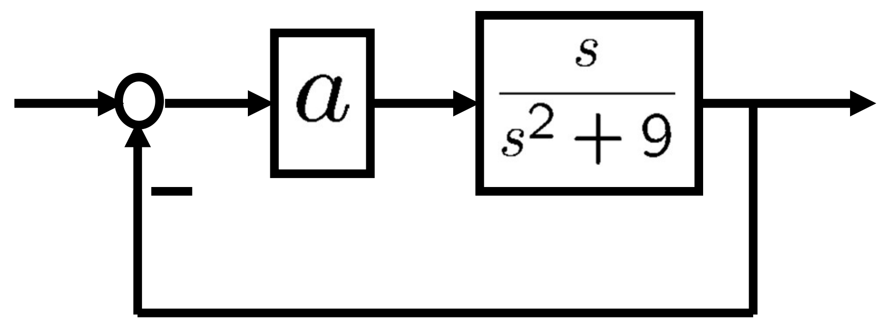
$$\Leftrightarrow \quad \underbrace{s^2 + 9}_{\text{Term without } a} + \underbrace{sa}_{\text{Term with } a} = 0$$

Term without a — Term with a

$$\Leftrightarrow \quad 1 + a\boxed{\frac{s}{s^2+9}} = 0$$

$L(s)$ **Fictitious OLTF**

Two CL systems have the same characteristic eq.



Note: Despite the fact that both CL systems have the same characteristic equations, they are very different and we cannot use the fictitious OLTF for calculations of error constants, such as K_v.

Example 2 (cont'd)

- If a tuning parameter appears in the characteristic equation at a "non-standard location", to draw root locus with respect to the tuning parameter, we transform the equation into our "standard" form:

$$1 + (\text{parameter}).\,L(s) = 0$$

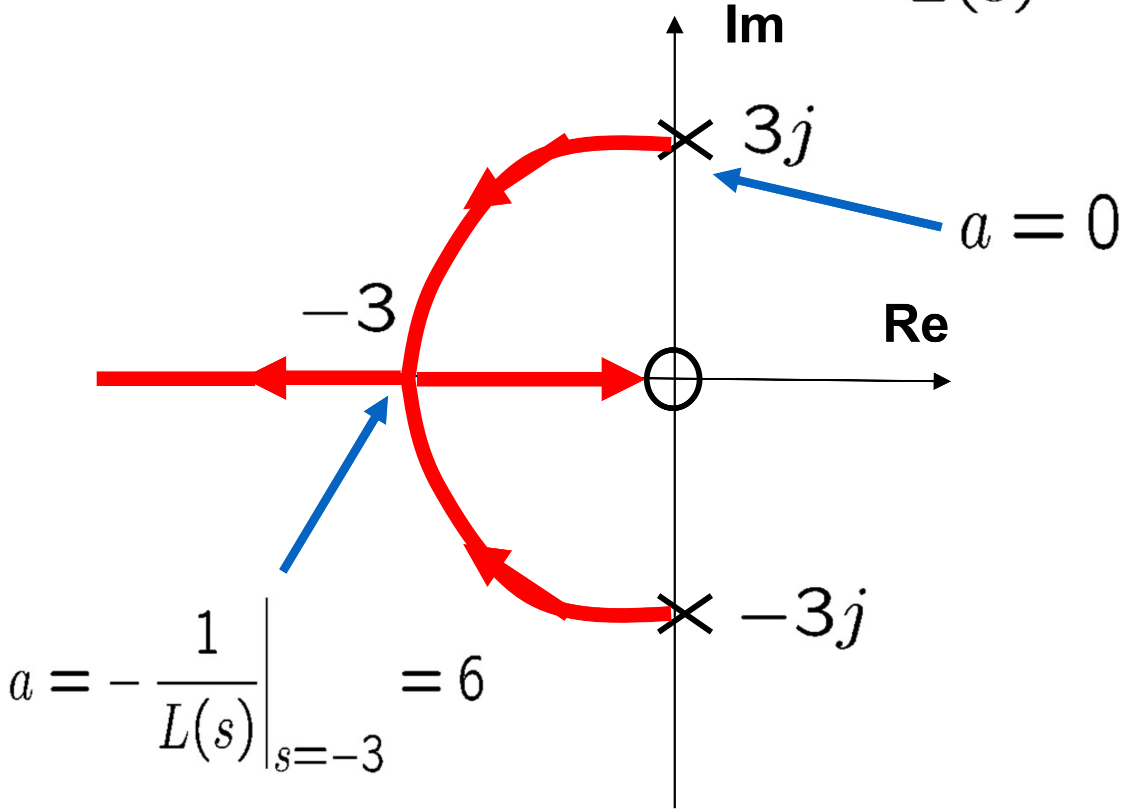
- Two examples:

$$1 + \frac{1}{s(s+p)} = 0 \;\Rightarrow\; s(s+p) + 1 = 0 \;\Rightarrow\; s^2 + 1 + ps = 0 \;\Rightarrow\; 1 + p\frac{s}{s^2+1} = 0$$

$$1 + \frac{s+z}{s^2} = 0 \;\Rightarrow\; s^2 + s + z = 0 \;\Rightarrow\; 1 + z\frac{1}{s(s+1)} = 0$$

Example 2 (cont'd)

Root locus for $1 + a\underbrace{\frac{s}{s^2+9}}_{L(s)} = 0$



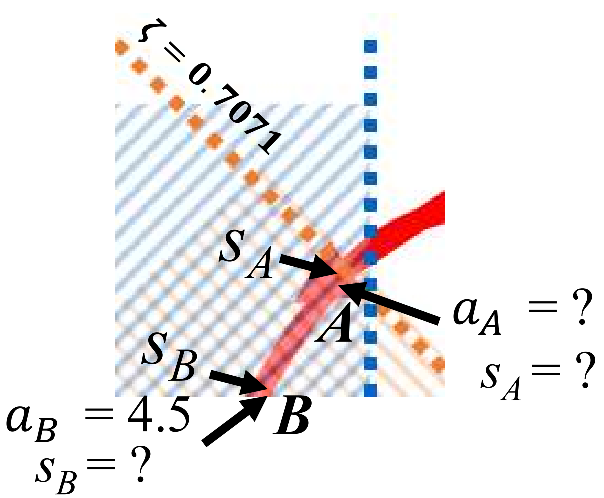
Example 2 (cont'd)

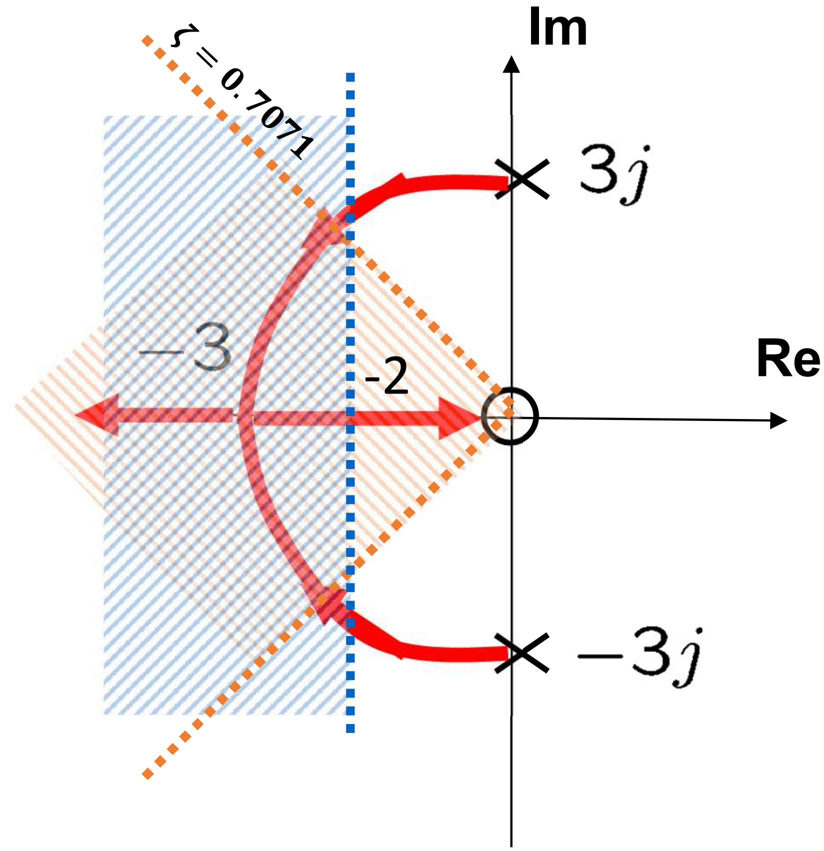
- Design the pole "a" satisfying the following :
  - Overshoot at most 4.32%
  - 2% settling time at most 2 sec
  - Error constant $K_v > 2$

$$K_v = \lim_{s \to 0} s \frac{9}{s(s+a)} = \frac{9}{a} > 2$$

$\Rightarrow$ $a < 4.5$ So, $a_B = 4.5$.

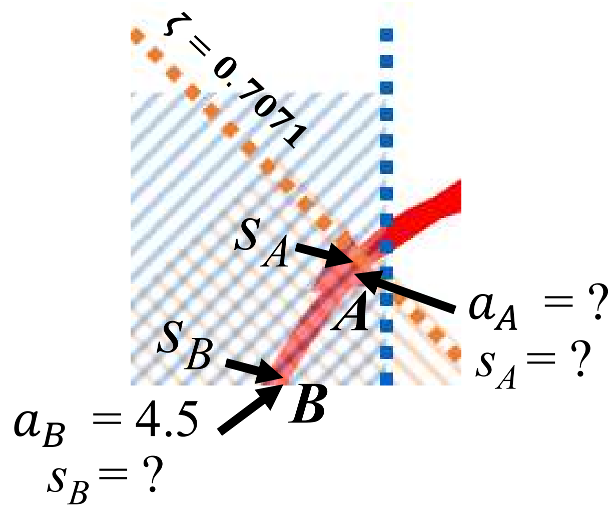
(This "a" corresponds to point ***B***)





Example 2 (cont'd)

Let us find the coordinates of point ***A*** and the numerical value of "*a*" at this point:



Point *A*: $s_A = b + cj$ (1)

$$\begin{cases} a_A = -\frac{1}{L(s_A)} = -\frac{1}{\frac{s_A}{{s_A}^2+9}} = -\frac{{s_A}^2+9}{s_A} & (2) \\ a_A = h & (3) \end{cases}$$

$$(2) = (3) \rightarrow -\frac{{s_A}^2+9}{s_A} = h \rightarrow (b+cj)^2 + 9 = -(b+cj)h \rightarrow (b^2 - c^2 + 9) + (2bc)j$$
$$= -(bh) - (ch)j \rightarrow$$

$$\begin{cases} b^2 - c^2 + 9 = -bh & (4) \\ 2bc = -ch \longrightarrow h = -2b & (5) \end{cases}$$

Substitute (5) in (4) $\rightarrow$ $b^2 + c^2 = 9$ (6)

$$s_A = -0.7071\omega_n + j\omega_n\sqrt{1 - 0.7071^2} \rightarrow s_A = -0.7071\omega_n + 0.7071\omega_n j \quad (7)$$

Example 2 (cont'd)

$$\begin{cases} s_A = b + cj & (1) \\ s_A = -0.7071\omega_n + 0.7071\omega_n j & (7) \end{cases}$$

We know that (1) = (7) $\rightarrow \begin{cases} b = -0.7071\omega_n \\ c = +0.7071\omega_n \end{cases}$ Substitute these in (6) $\rightarrow b^2 + c^2 = 9$

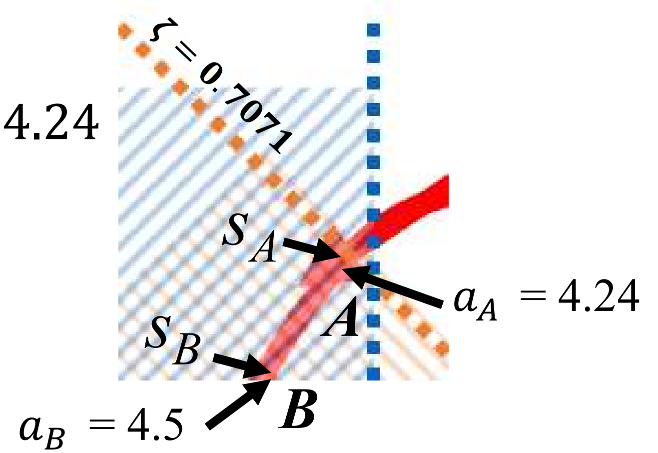
$$(-0.7071)^2{\omega_n}^2 + (+0.7071)^2{\omega_n}^2 = 9 \rightarrow \omega_n = 3 \rightarrow s_A = (-0.7071)(3) + (0.7071)(3)j$$

$$\rightarrow \boxed{s_A = -2.1213 + 2.1213j}$$

Let us find "a" at s_A. Substitute s_A in "a", i.e., in (2):

$$a_A = -\frac{{s_A}^2 + 9}{s_A} = -\frac{(-2.1213 + 2.1213j)^2 + 9}{(-2.1213 + 2.1213j)} = 4.24 \rightarrow a_A = 4.24$$

$$\boxed{4.24 \le a < 4.50}$$



We can also show that $s_B = -2.25 \pm 1.9843j$.

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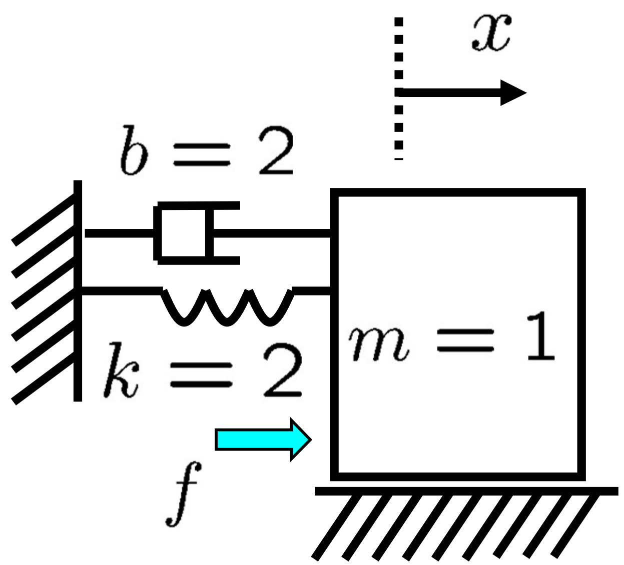
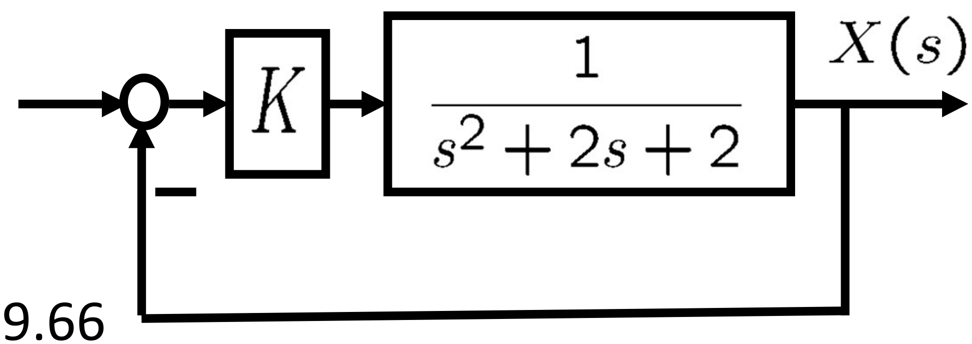
  - In following lectures ….
    - Lead and lag compensator design
    - PID controller design

Example 3

- Mass-spring-damper system
  - Shock absorber (car, train)
  - Seismic isolator (building)
  - Accelerometer (Lec 3)
- Design a controller so that
  - 2% settling at most 1 sec
  - No overshoot
  - e_{ss} for unit step < 0.048



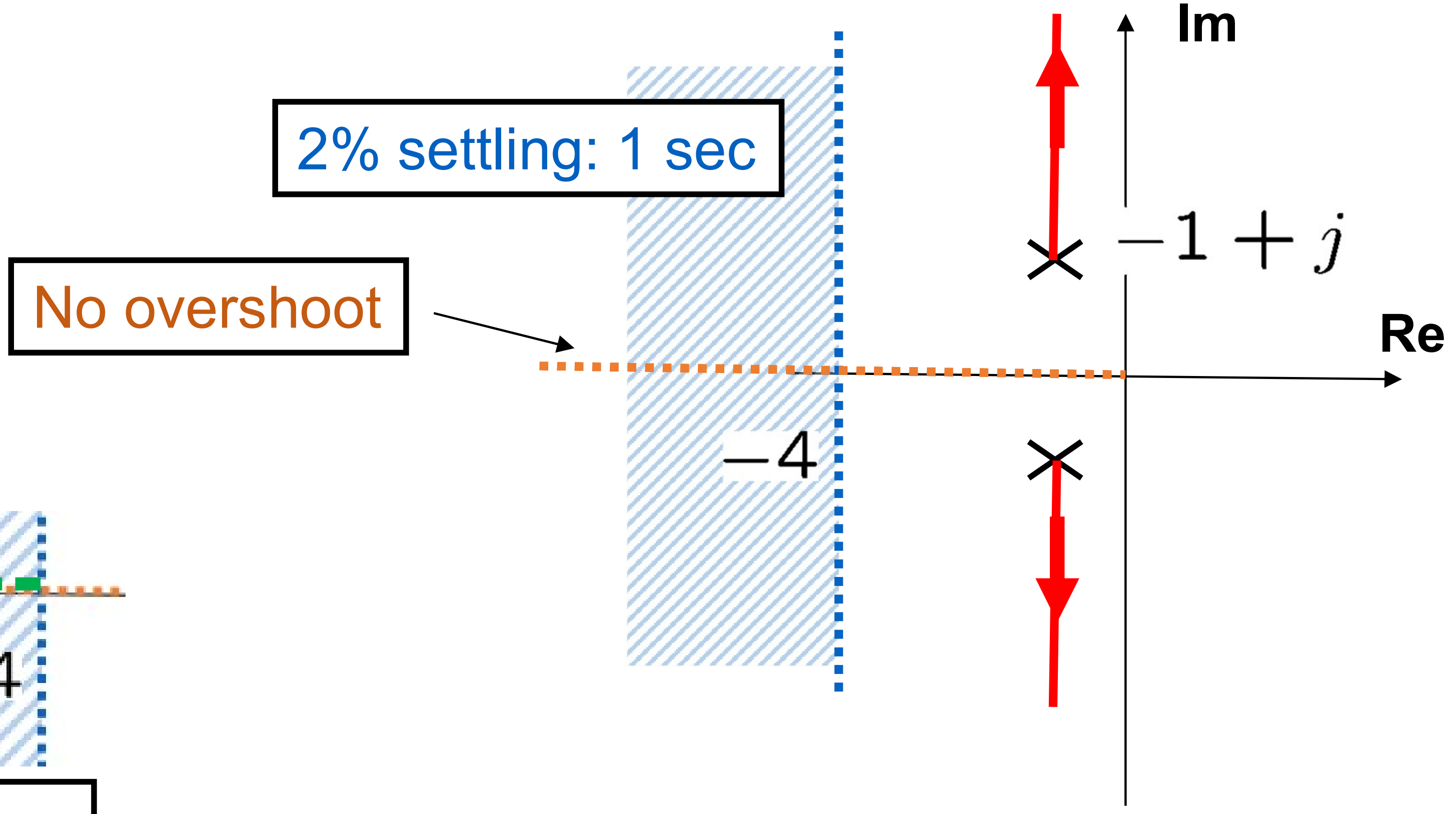


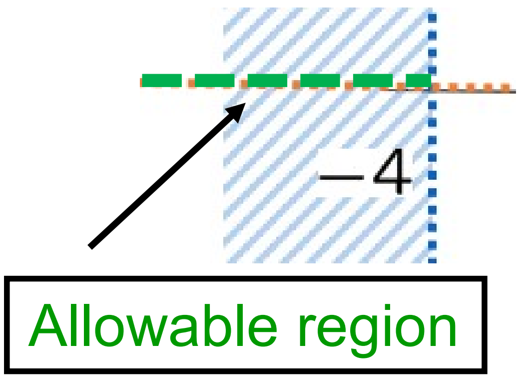
$\Rightarrow$ $e_{ss} = \frac{1}{1 + K_p} < 0.048$

$\Rightarrow$ $K_p > 19.83$ $\Rightarrow$ $K > 39.66$

Example 3 (cont'd)

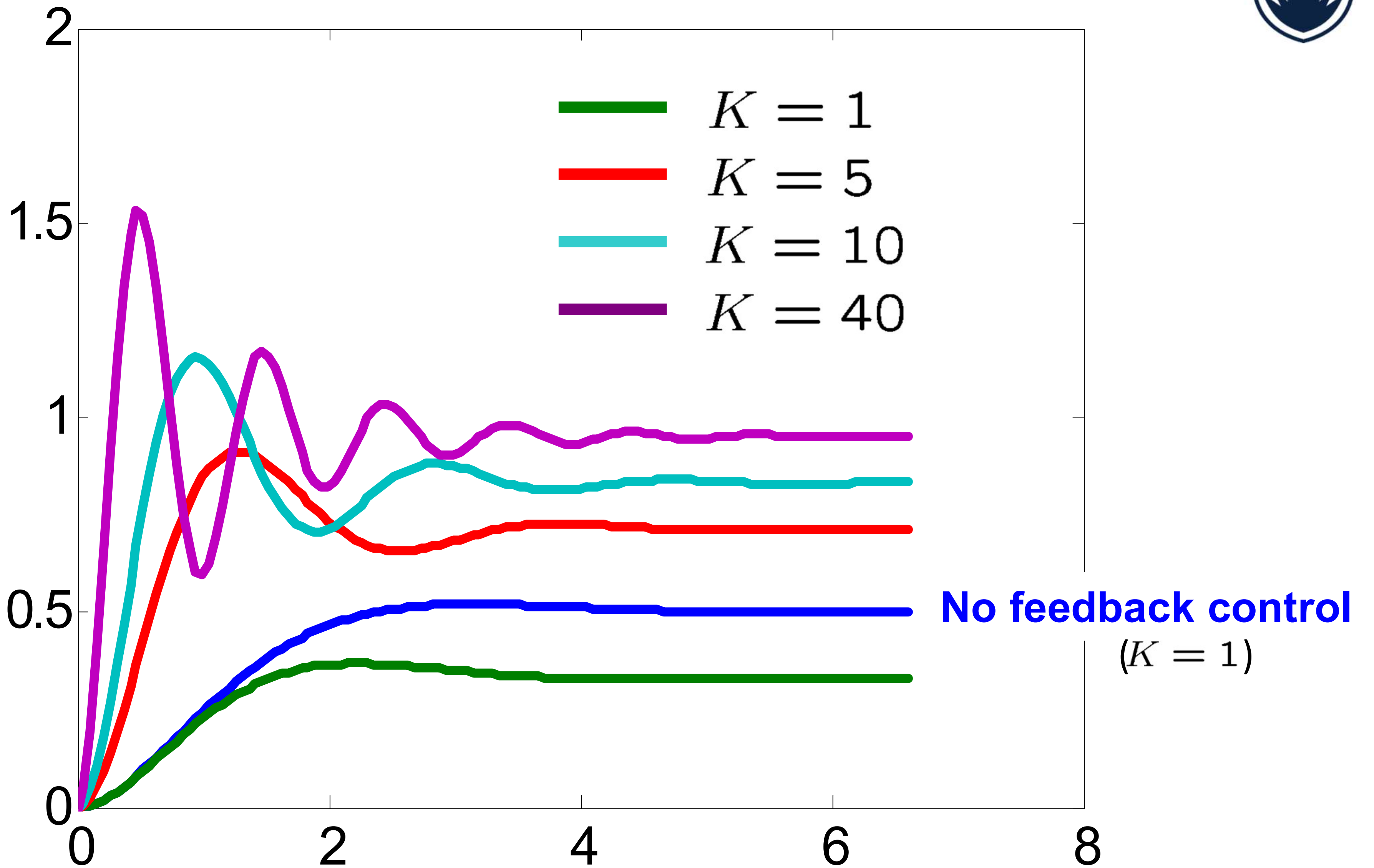
- Root locus
  - Poles: $s = -1 \pm j$





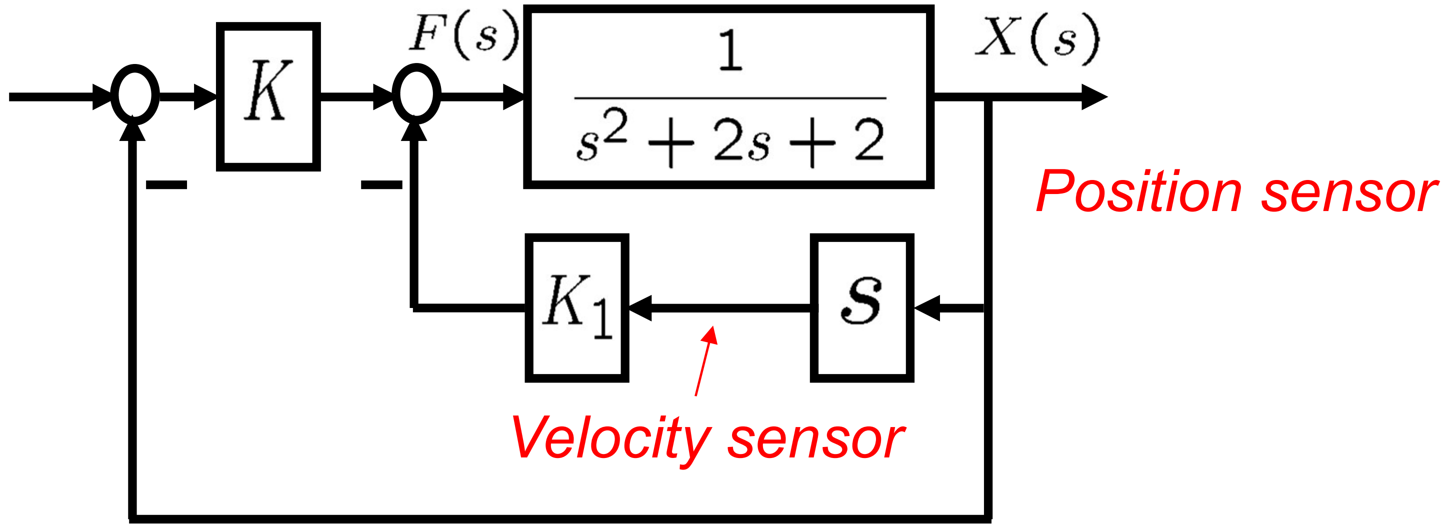
We cannot achieve the design specs with the position feedback gain controller!

Example 3 (cont'd): Step responses



Example 3 (cont'd)

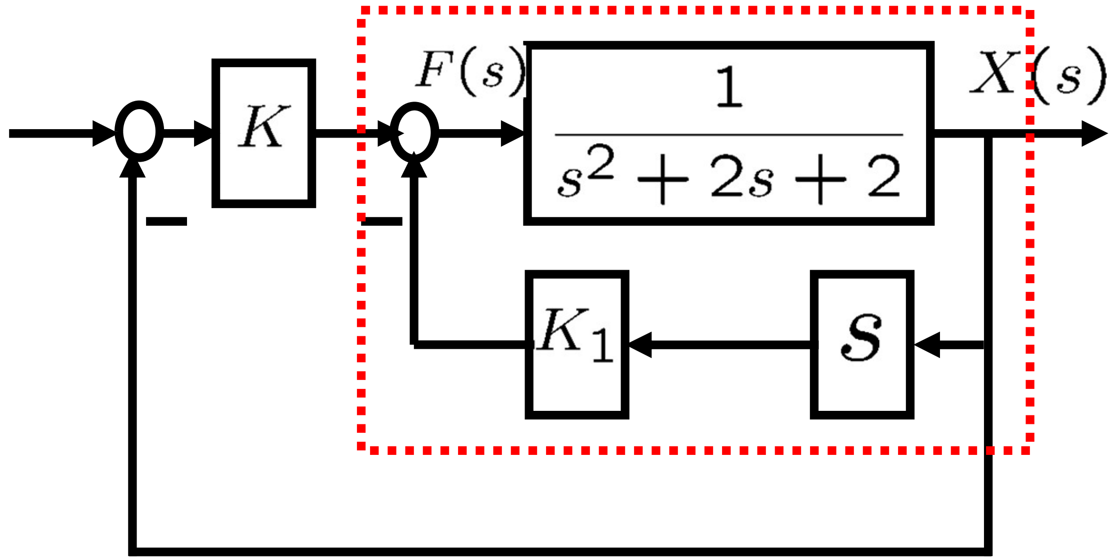
- Position & velocity (rate) feedback



- We have two design parameters to be tuned to satisfy the design specs, i.e., K & K_1.
- How to use root locus technique?

Note: On your tests and for these sorts of design problems, the above block diagram (or similar block diagrams) will be given to you.

Example 3 (cont'd)



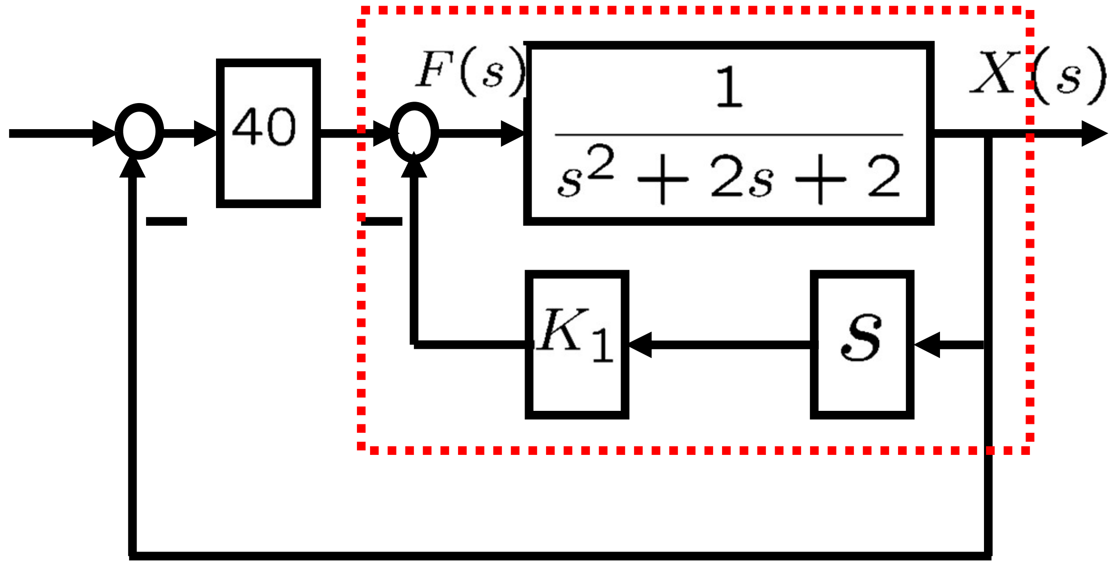
Transfer function for $= \dfrac{\frac{1}{s^2+2s+2}}{1+\frac{K_1 s}{s^2+2s+2}} = \dfrac{1}{s^2+(K_1+2)s+2}$

- SS error spec:

$e_{ss} = \dfrac{1}{1+K_p} < 0.048$ ➡ $K_p > 19.83$ ➡ $K > 39.66$ ➡ Set $K = 40$

Note: On your tests, just round up the value of K to the nearest next integer.

Example 3 (cont'd): Set $K = 40$



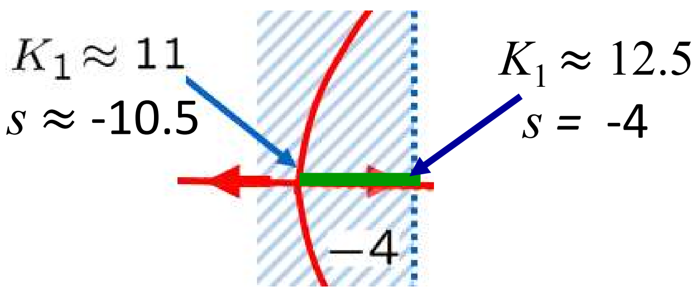
Characteristic eq. $$1 + 40 \times \frac{1}{s^2 + (K_1 + 2)s + 2} = 0$$

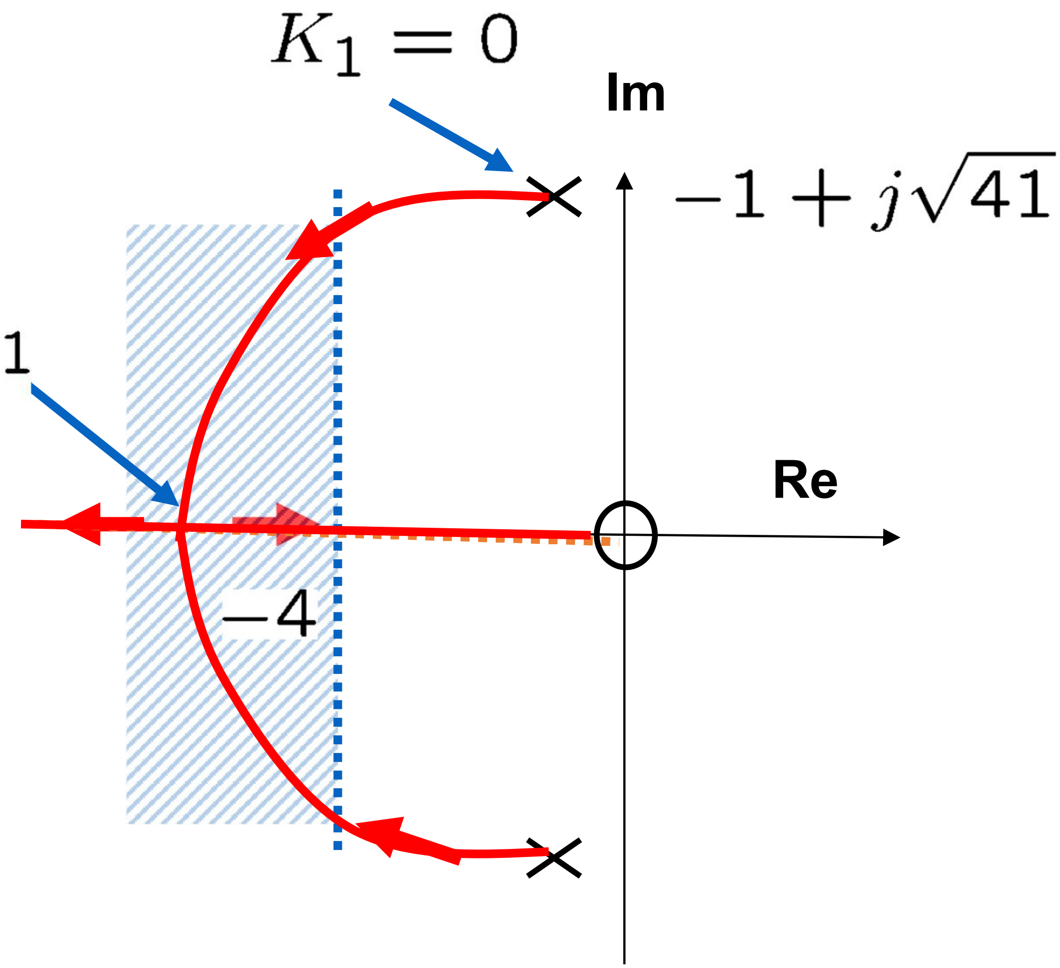
$\Rightarrow$ $s^2 + 2s + 42 + K_1 s = 0$ $\Rightarrow$ $1 + K_1 \underbrace{\frac{s}{s^2 + 2s + 42}}_{L(s)} = 0$

Example 3 (cont'd): $K = 40$

$$L(s) = \frac{s}{s^2 + 2s + 42}$$

$$K_1 = -\left.\frac{1}{L(s)}\right|_{s=-\sqrt{42}} \approx 11$$

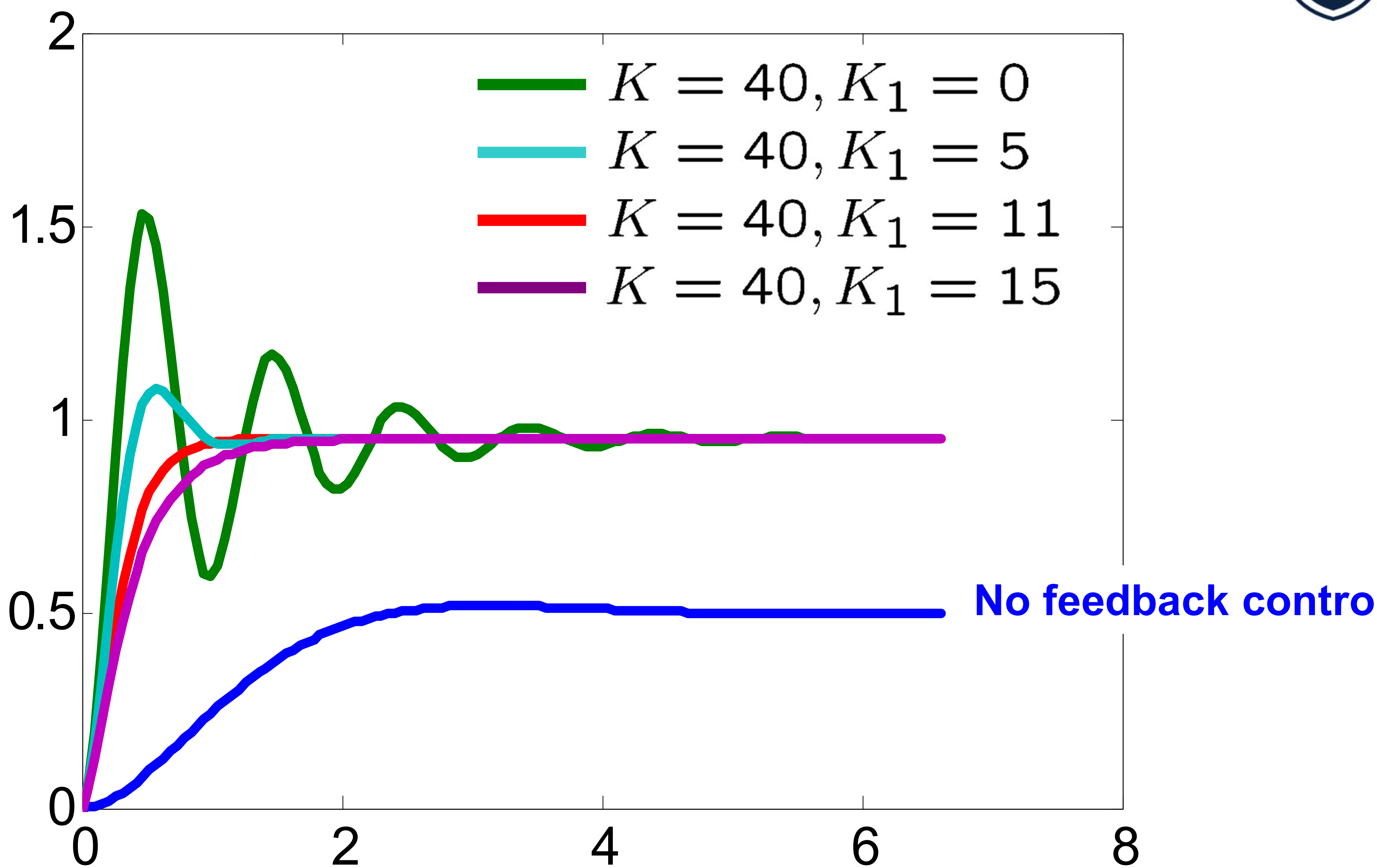




$$11 \le K_1 \le 12.5$$

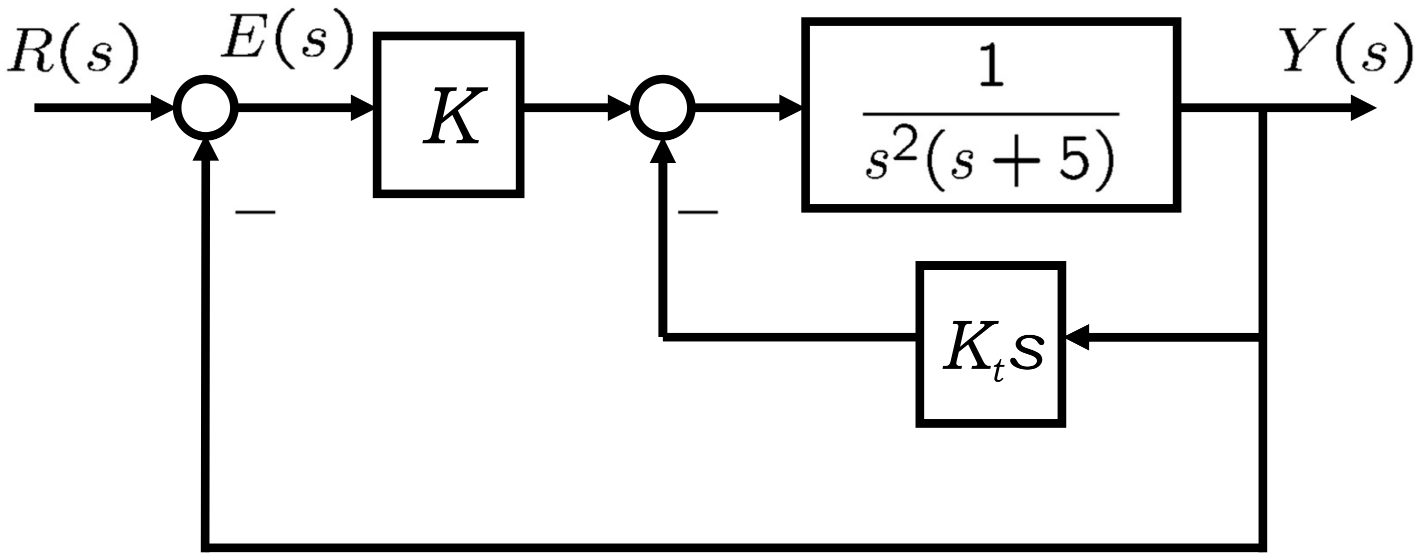
By increasing K_1, we can get a satisfactory closed-loop system.

Example 3 (cont'd): Step responses



Note: At around $K_1 = 12.5$ we will pass the threshold of "-4" on the Re axis. So, $K_1 = 15$ is a slightly above the acceptable K_1 value.

Example 4



a) Set $K_t = 0$. Draw root locus for $K > 0$.

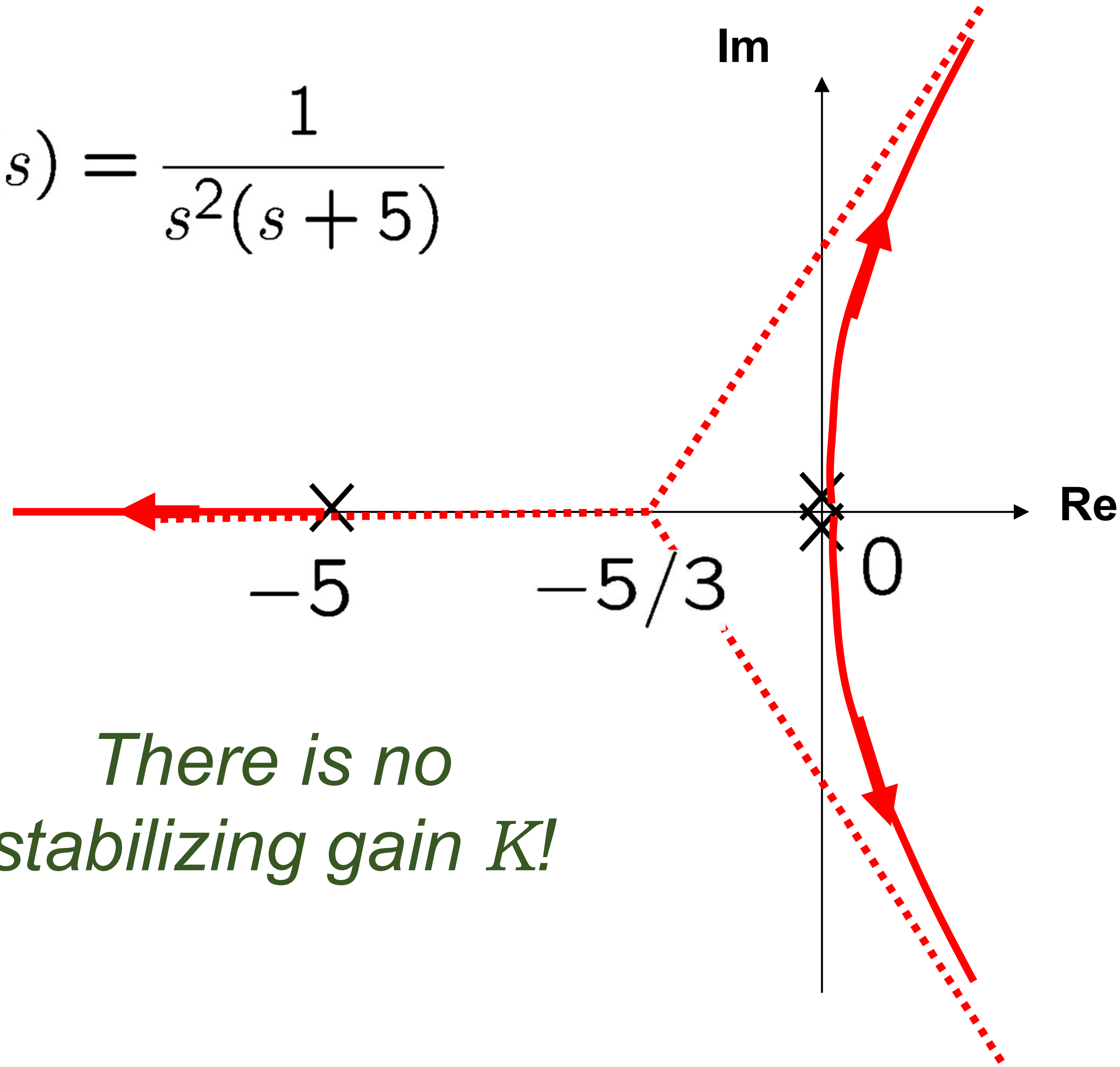
b) Set $K = 10$. Draw root locus for $K_t > 0$.

c) Set $K = 5$. Draw root locus for $K_t > 0$.

Example 4 (cont'd) (a): $K_t = 0$

a)

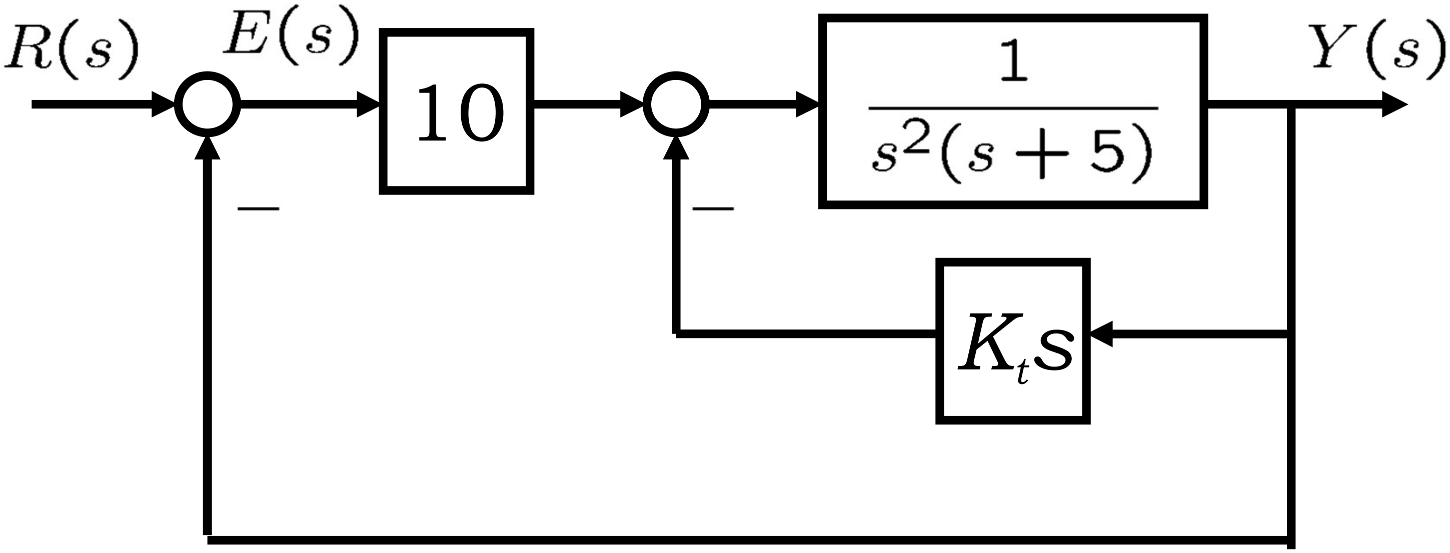
$$L(s) = \frac{1}{s^2(s+5)}$$



There is no stabilizing gain K!

Example 4 (cont'd) (b): $K = 10$

b)

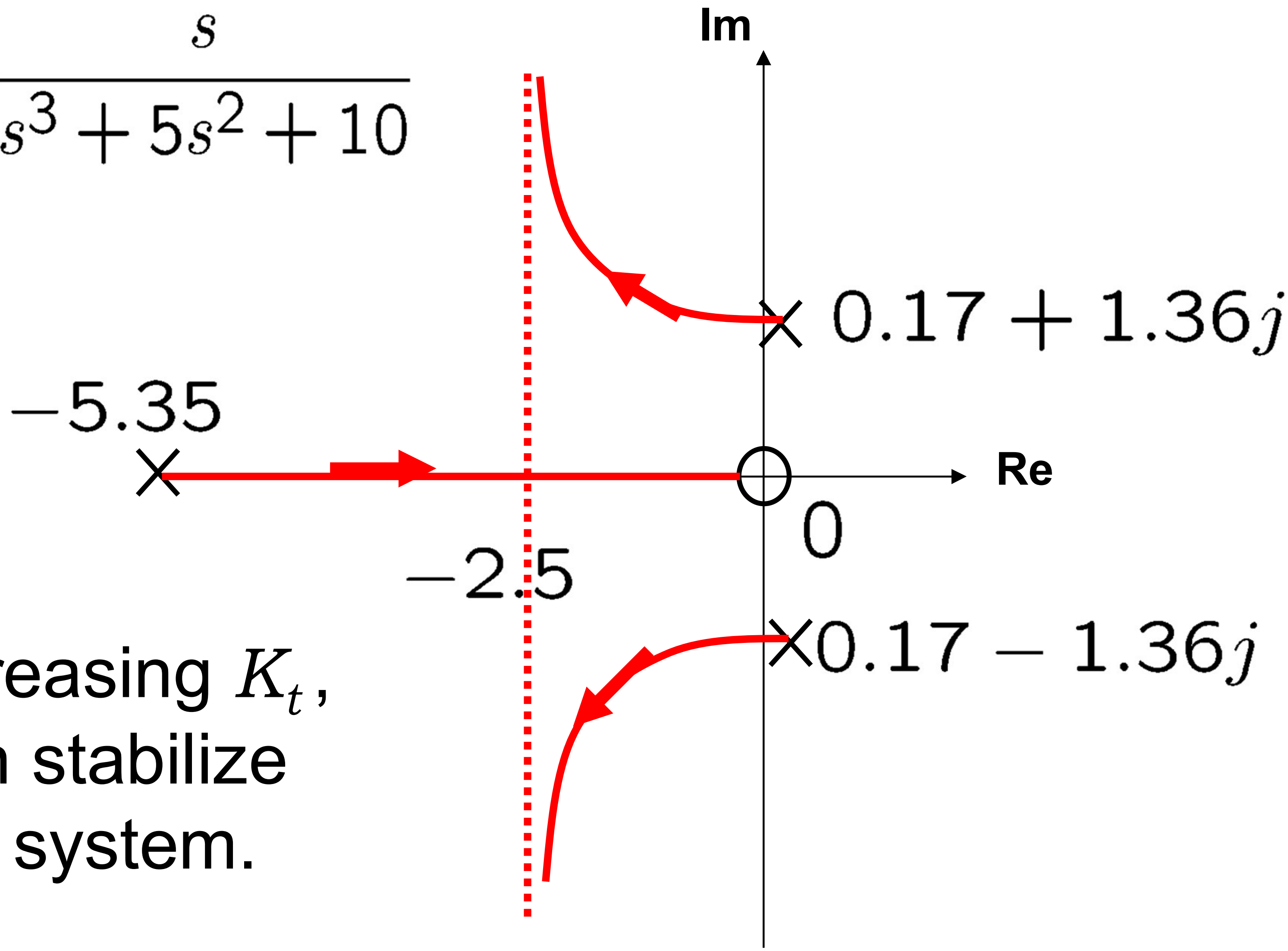


Characteristic eq. $$1 + 10\left(\frac{\frac{1}{s^2(s+5)}}{1+\frac{K_t s}{s^2(s+5)}}\right) = 0$$

$\Rightarrow$ $s^2(s+5) + K_t s + 10 = 0$ $\Rightarrow$ $1 + K_t \underbrace{\boxed{\frac{s}{s^3+5s^2+10}}}_{L(s)} = 0$

Example 4 (cont'd) (b): $K = 10$

$$L(s) = \frac{s}{s^3 + 5s^2 + 10}$$



By increasing K_t, we can stabilize the CL system.

Example 4 (cont'd): Find the range of K_t

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- Characteristic equation:

$$1+\frac{K_t s}{s^3+5s^2+10}=0 \Leftrightarrow s^3+5s^2+K_t s+10=0$$

- Routh array:

s^3	1	K_t
s^2	5	10
s^1	$\frac{5K_t-10}{5}$	
s^0	10	

Stability condition

$K_t > 2$

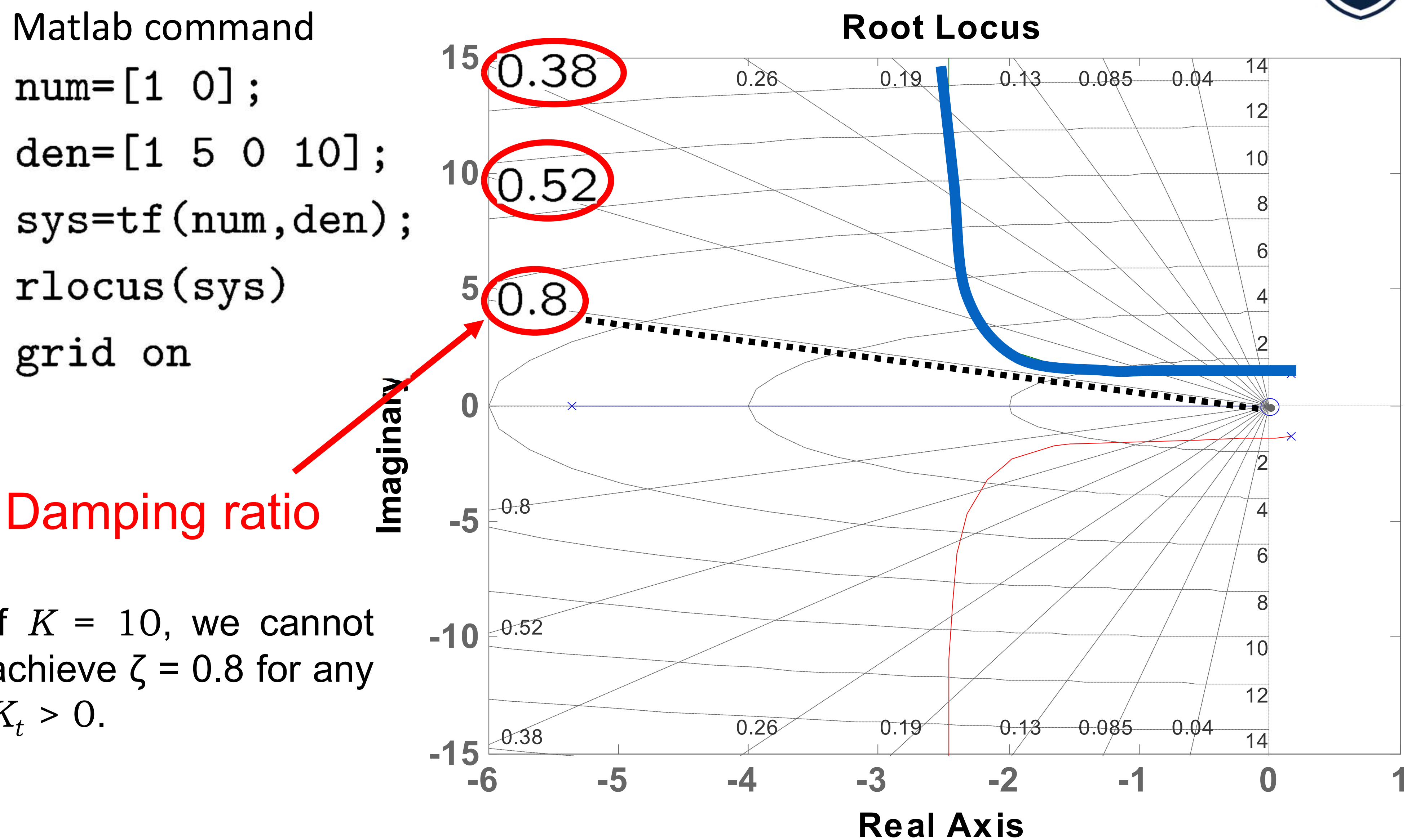
- When $K_t = 2$:

$$5s^2+10=0 \Rightarrow s=\pm\sqrt{2}j$$

Example 4 (cont'd)

Matlab command

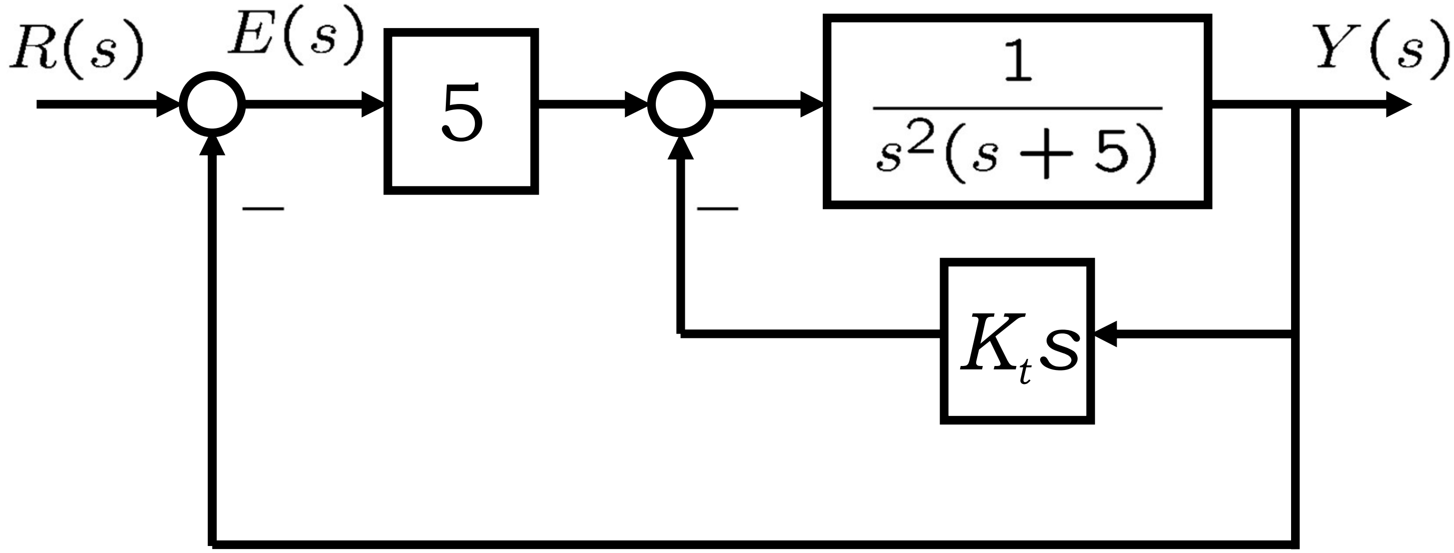
```
num=[1 0];
den=[1 5 0 10];
sys=tf(num,den);
rlocus(sys)
grid on
```



If $K = 10$, we cannot achieve $\zeta = 0.8$ for any $K_t > 0$.

Example 4 (cont'd) (c): $K = 5$

c)

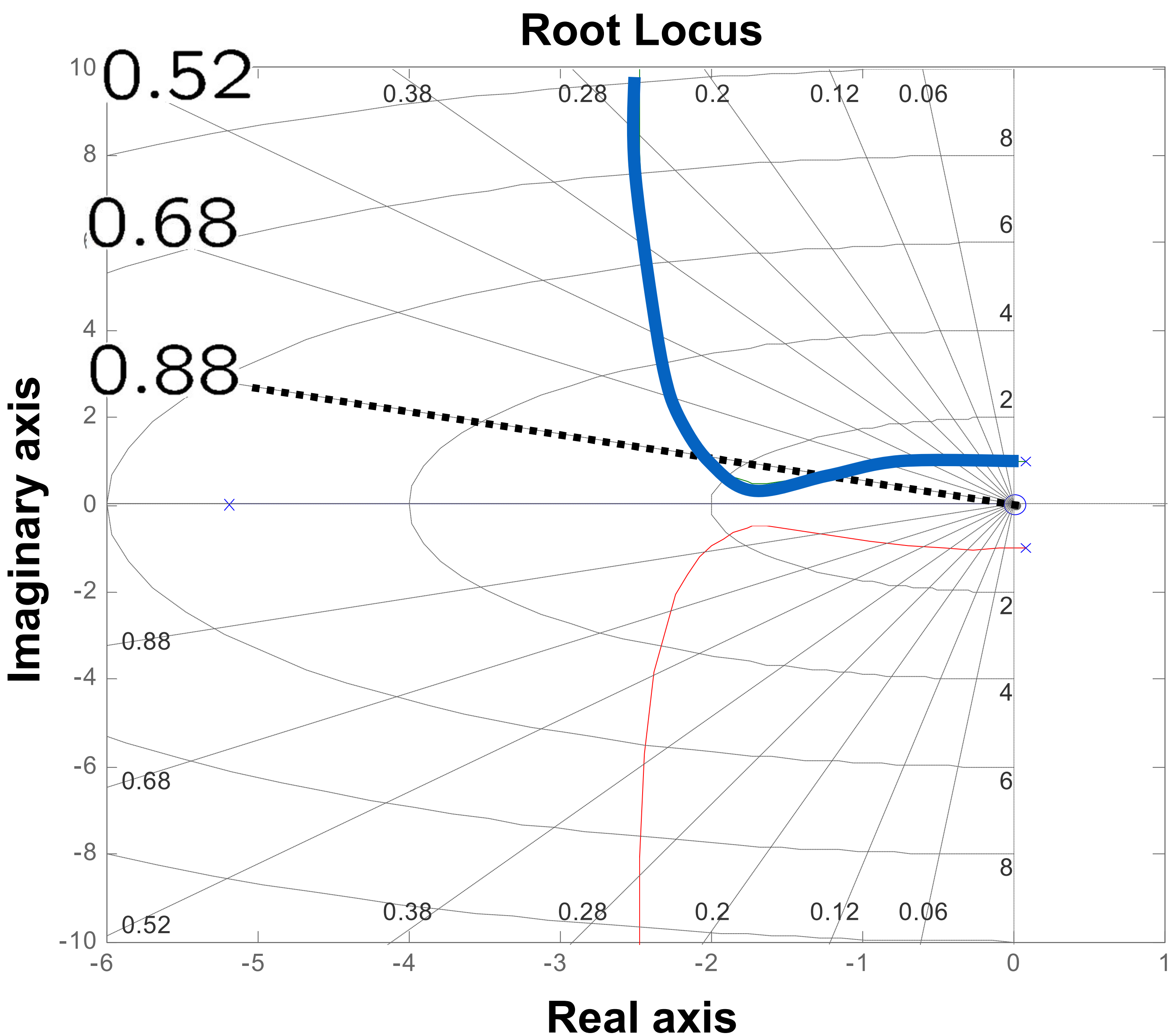


Characteristic eq.

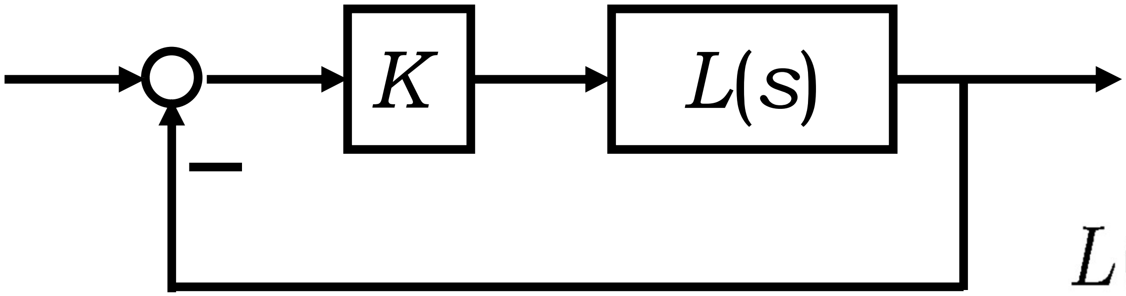
$$1+5\left(\frac{\frac{1}{s^2(s+5)}}{1+\frac{K_t s}{s^2(s+5)}}\right)=0$$

$$\Rightarrow \quad s^2(s+5)+K_t s+5=0 \quad \Rightarrow \quad 1+K_t \underbrace{\boxed{\frac{s}{s^3+5s^2+5}}}_{L(s)}=0$$

Example 4 (cont'd) (c): Root locus plot



Example 5



$$L(s) = \frac{1+Ts}{s(s+1)(s+2)}$$

a) Set $T = 0$. Draw root locus for $K > 0$.

$$L(s) = \frac{1}{s(s+1)(s+2)}$$

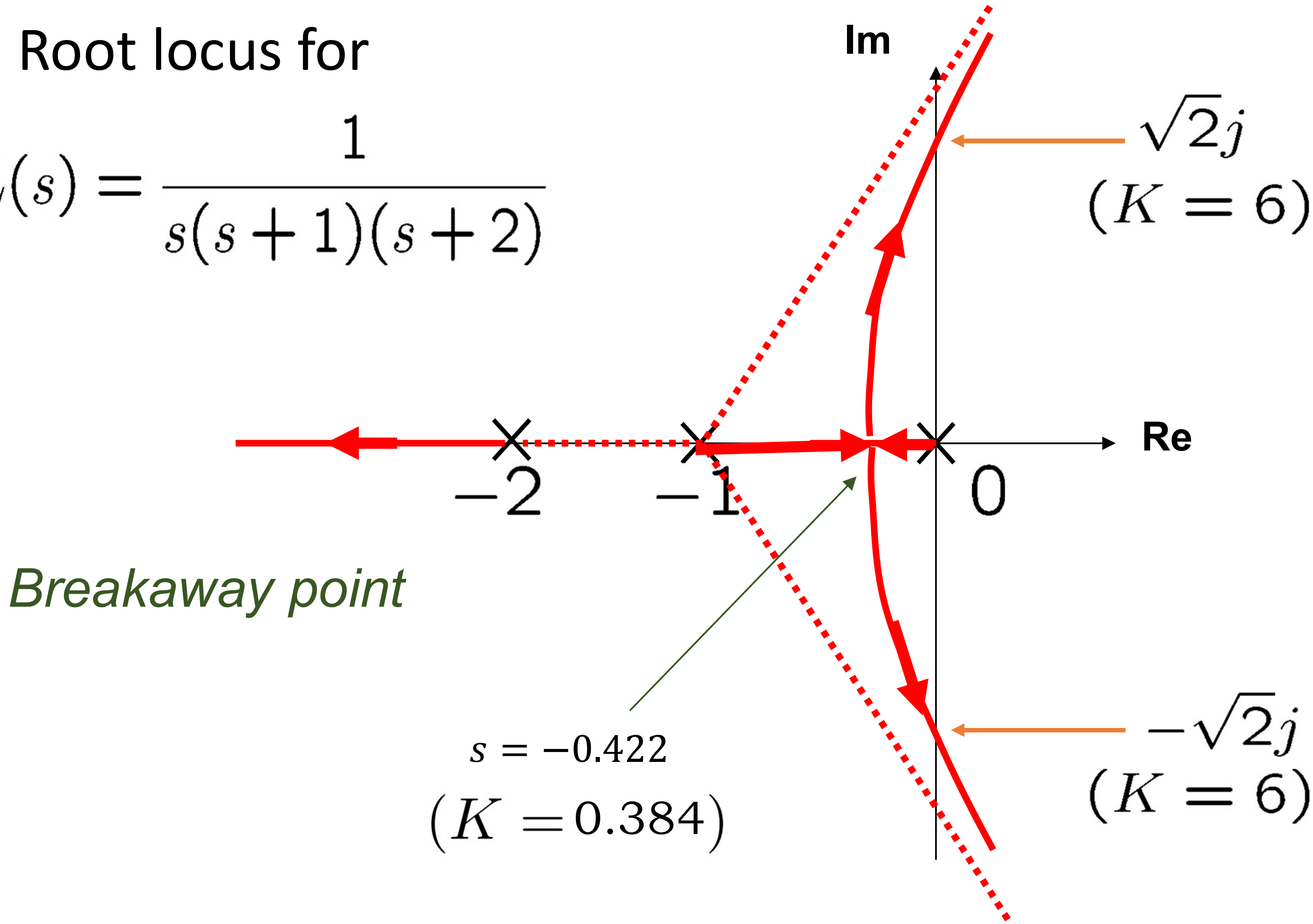
b) Vary T to see the effect of a zero on root locus. Let $K = 1.3$.

Example 5 (a) (cont'd)

a)

- Root locus for

$$L(s) = \frac{1}{s(s+1)(s+2)}$$



Example 5 (b) (cont'd)

b)

- When K is fixed and T is a positive parameter, the characteristic equation can be written as:

$$1+K\frac{1+Ts}{s(s+1)(s+2)}=0$$

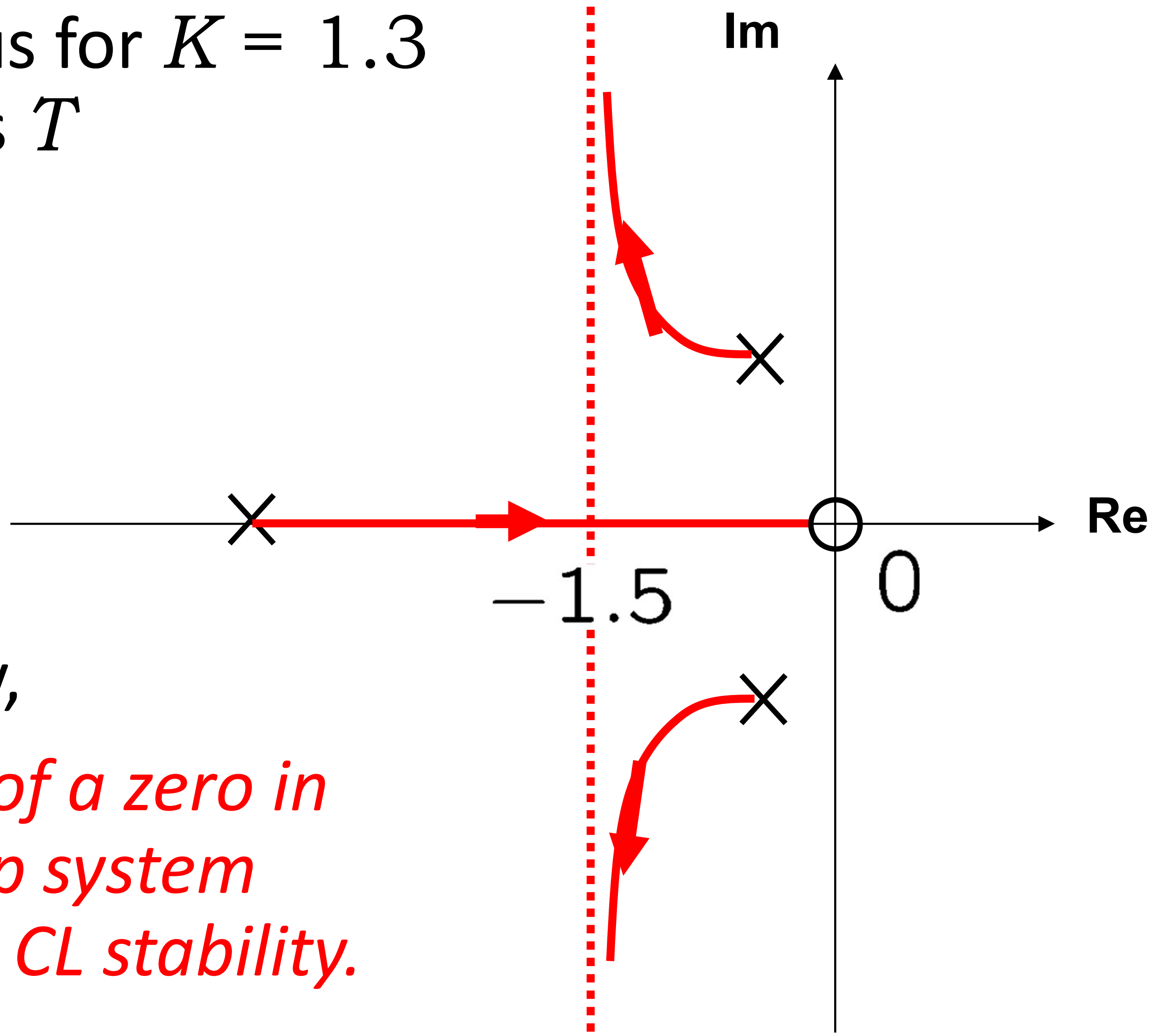
$$\Rightarrow \quad \underbrace{s(s+1)(s+2)+K}_{\text{Term without } T}+\underbrace{TKs}_{\text{Term with } T}=0$$

$$\Rightarrow \quad 1+T\frac{Ks}{s(s+1)(s+2)+K}=0 \qquad K=1.3 \Rightarrow$$

$$\Rightarrow \quad 1+T\frac{1.3s}{s(s+1)(s+2)+1.3}=0$$

Example 5 (b) (cont'd)

- Root locus for $K = 1.3$ & various T



- Generally,

addition of a zero in open-loop system improves CL stability.

Summary

- How to use the root locus
  - **Example 1:** Gain design to meet design specifications
  - **Example 2:** Pole or zero location design (i.e., design based on pole or zero of OLTF)
  - **Example 3:** Multiple parameter design
    - Velocity (rate) feedback can be used to improve the transient property.
- Next
  - We will study how to use the root locus to design lead-lag compensators.