ELEC 341: Systems and Control

a place of mind

Lecture 14

Root locus:
Lead-lag compensator design

Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- Models for systems
  - ✓ Electrical
  - ✓ Electromechanical
  - ✓ Mechanical
- ✓ Linearization, delay

Analysis

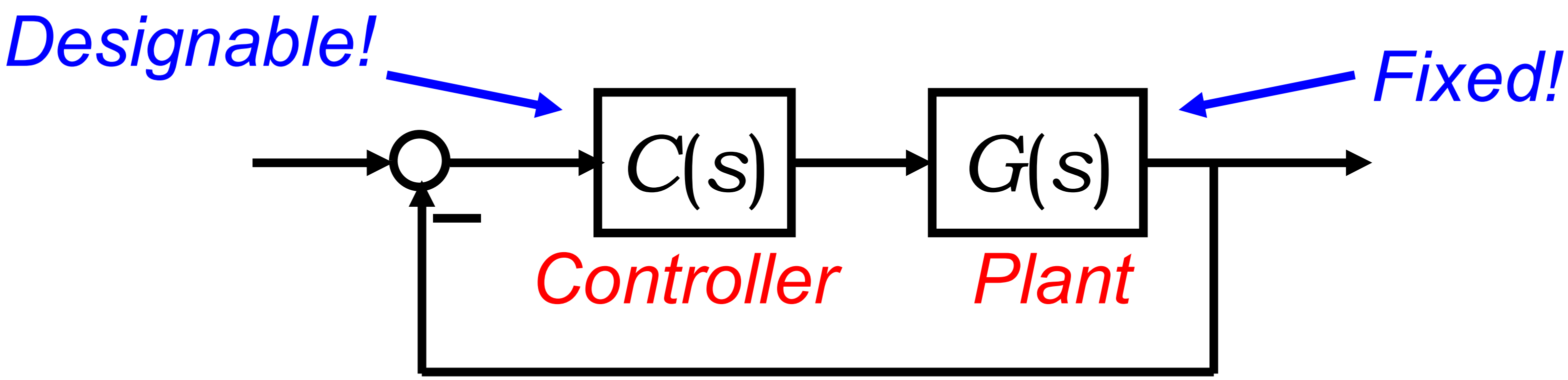
- ✓ Stability
  - ✓ Routh-Hurwitz
  - Nyquist
- ✓ Time response
  - ✓ Transient
  - ✓ Steady state
- Frequency response
  - Bode plot

Design

- Design specs
- ➤ Root locus
- Frequency domain
- ➤ PID & Lead-lag
- Design examples

Matlab simulations

Controller design by root locus

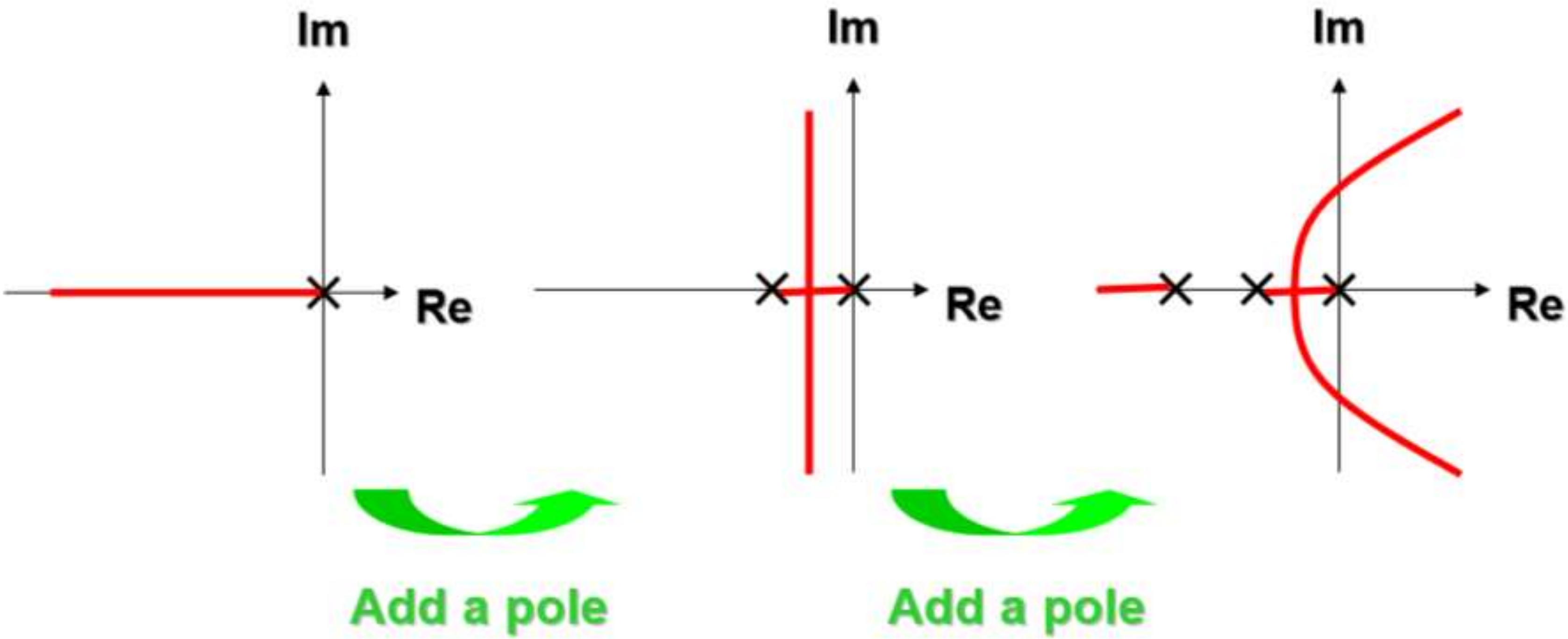


- Place closed-loop poles at desired locations…
  - by tuning the gain $C(s) = K$. (for time domain specs)
- If root locus does not pass through the desired location, then **reshape** the root locus…
  - by adding poles/zeros to $C(s)$. How?

Compensation

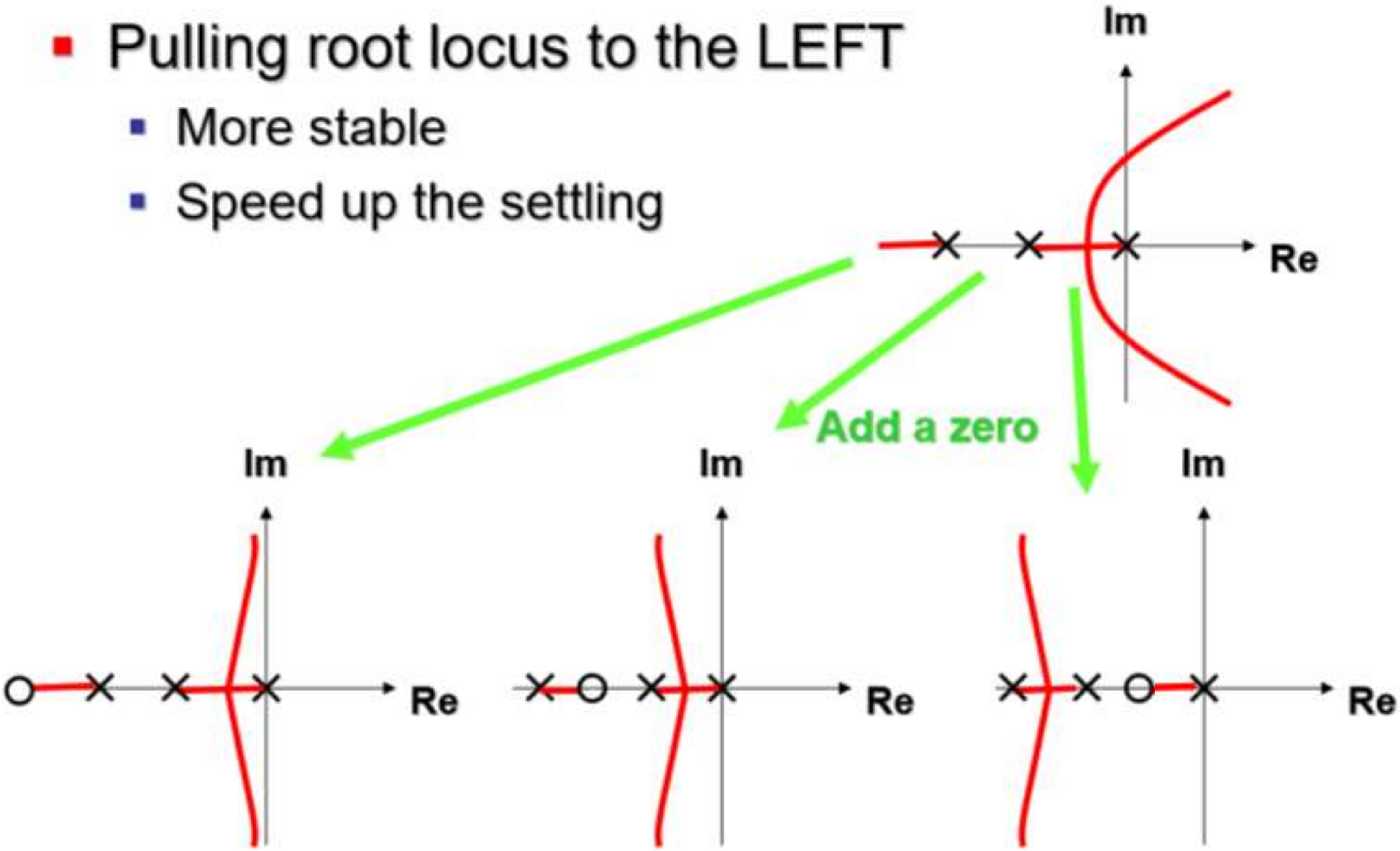
General effect of addition of poles

- Pulling root locus to the RIGHT
  - Less stable
  - Slow down the settling



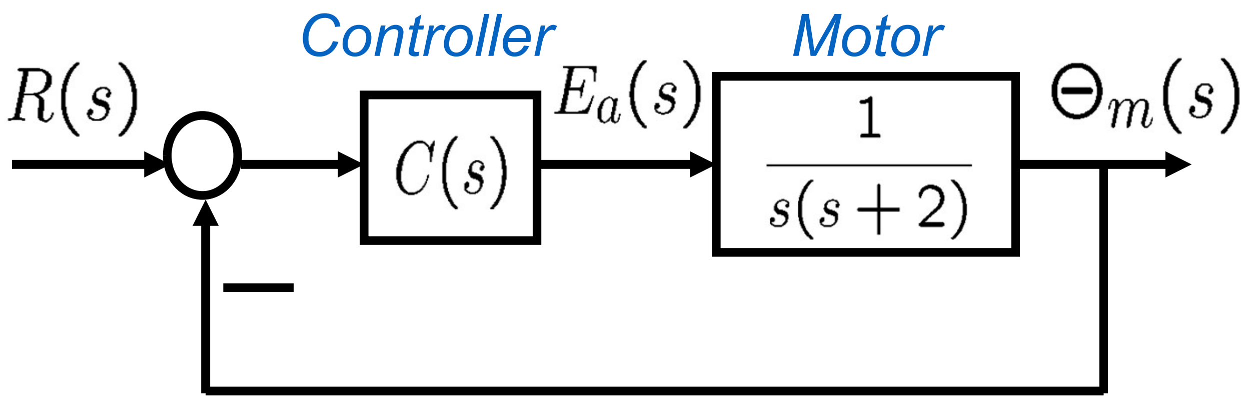
General effect of addition of zeros

- Pulling root locus to the LEFT
  - More stable
  - Speed up the settling



Example 1

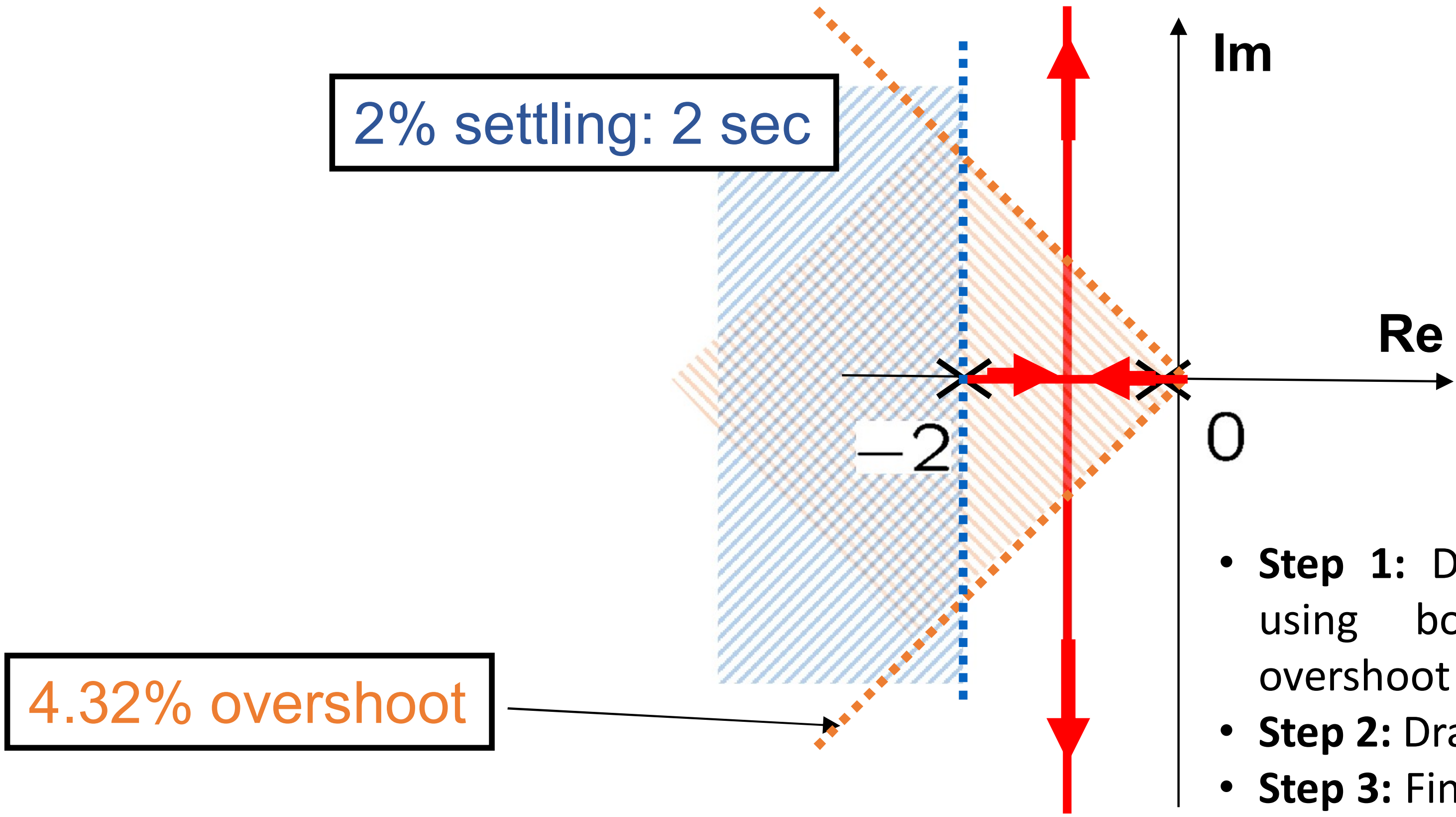
- A feedback system:



- Design specifications:
  - 2% settling time at most 2 seconds
  - Overshoot at most 4.32%
  - Steady state error:
    - Zero for unit step $r(t) = u(t)$
    - At most 0.05 for unit ramp $r(t) = tu(t)$

Example 1 (cont'd)

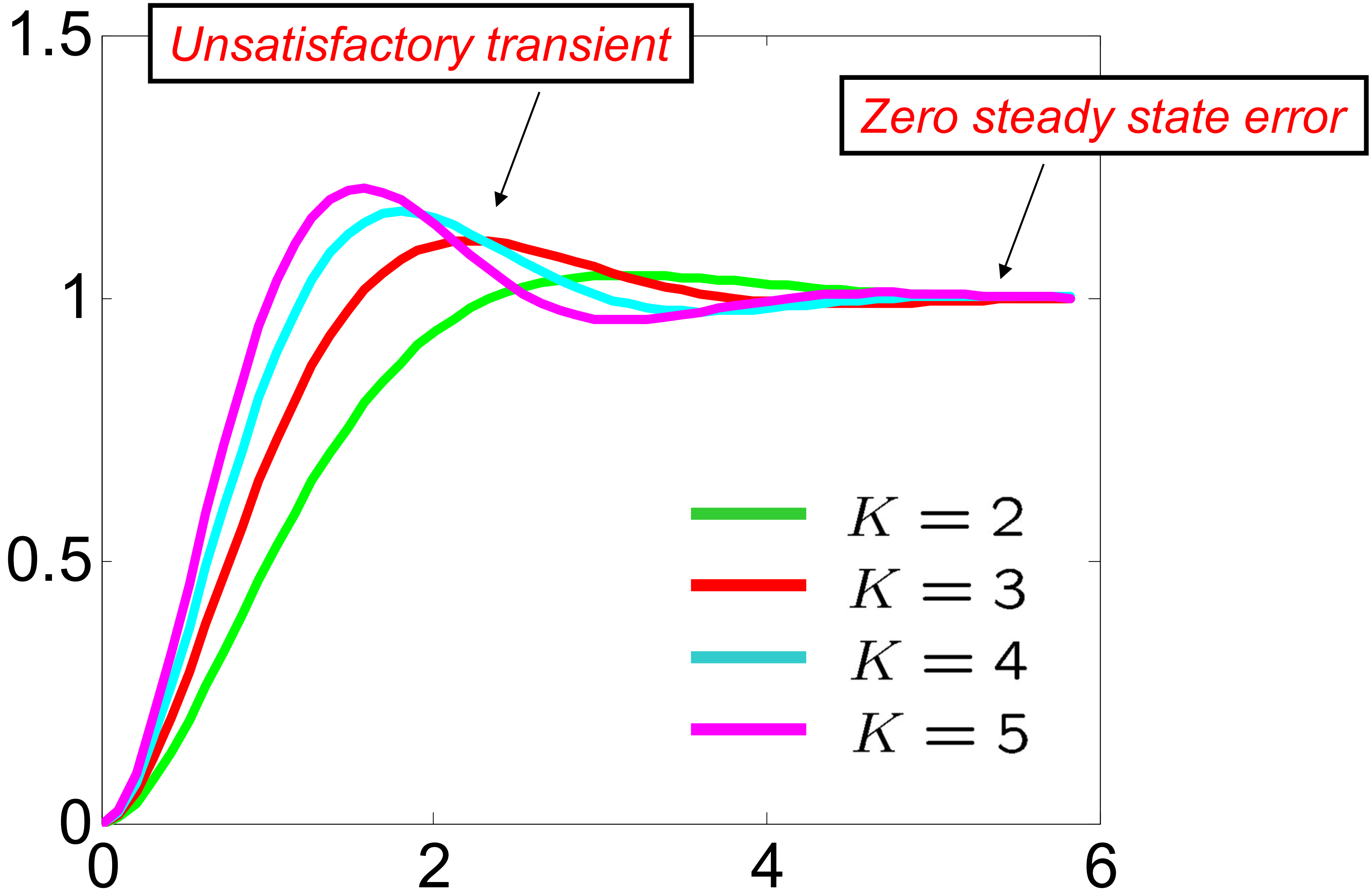
- Root locus for gain controller $C(s) = K$



- **Step 1:** Draw the allowable region using both settling time and overshoot constraints.
- **Step 2:** Draw the root locus diagram.
- **Step 3:** Find the *overlap region* of the root locus and the allowable region.

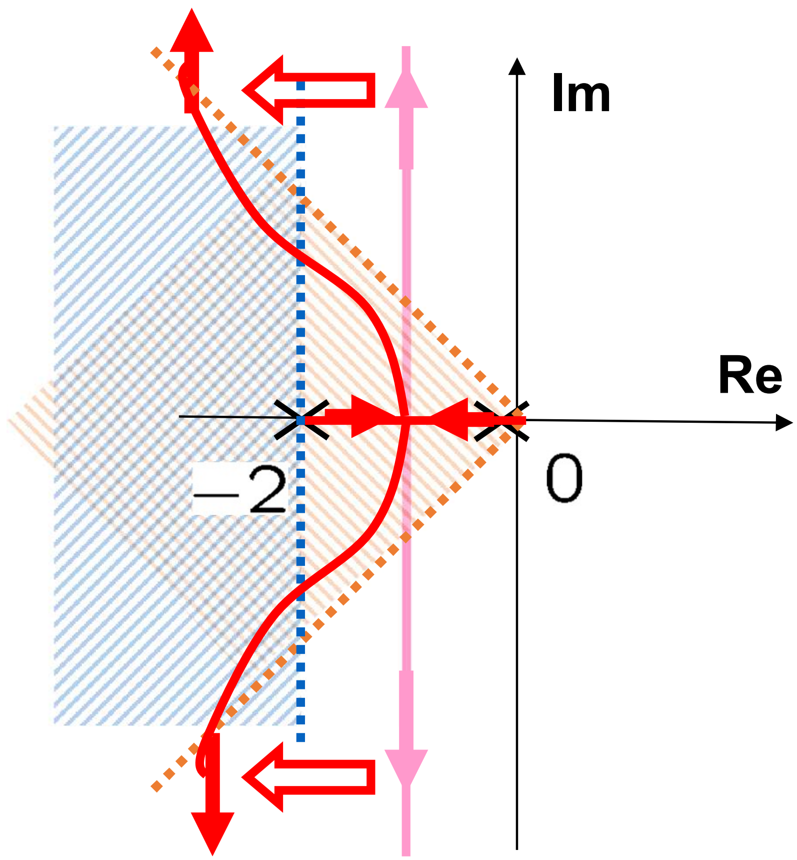
We cannot achieve the design specs with the gain feedback controller!

Example 1: Step responses for gain controllers $C(s) = K$



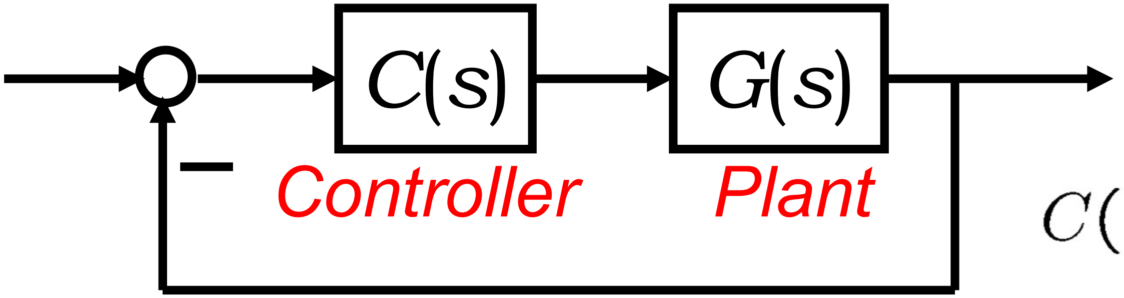
Example 1 (cont'd)

- We **reshape** the root locus so that it passes through the allowable region.



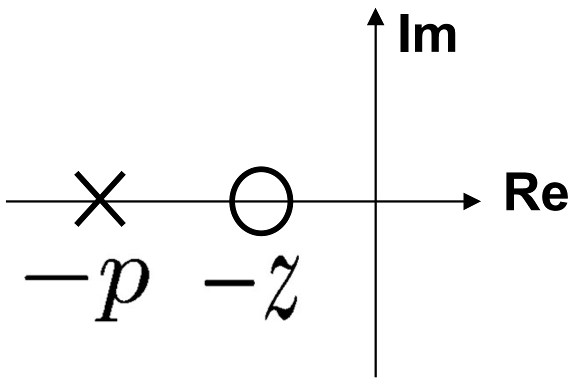
Pink ("Before" Design): Represents the original system design
Red ("After" Design): Represents the improved system design

Lead and lag compensators

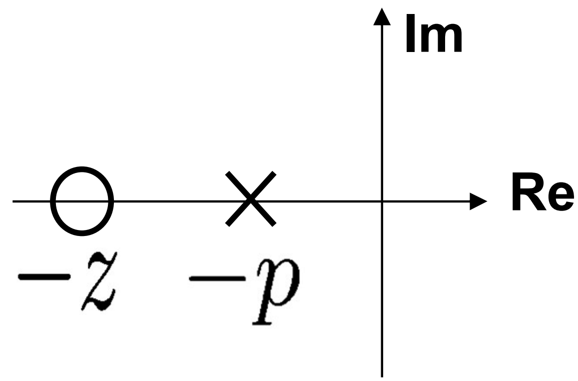


$$C(s) = K\frac{s+z}{s+p}, \quad (z>0, p>0)$$

- Lead compensator



- Lag compensator



The reason why these are called "lead" and "lag" will be explained in frequency response approach (later in this course).

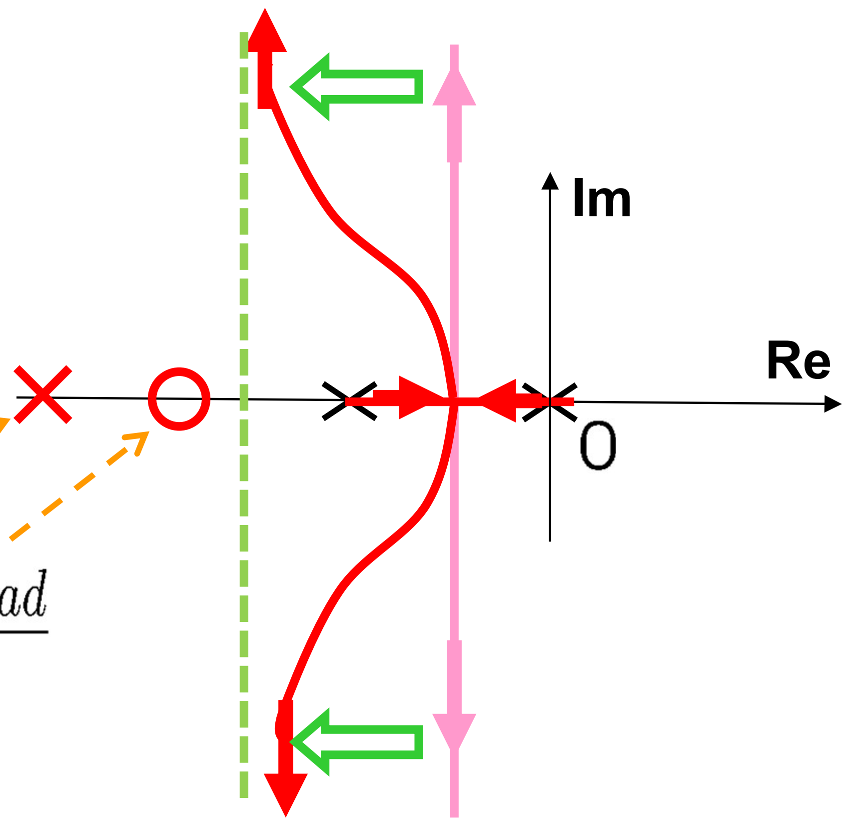
Reshaping root locus by lead compensators

- **Lead compensators** move the intersection of asymptotes (the centroid) to the left.
  - With $C(s) = K$ only:

$$\frac{\sum \text{pole} - \sum \text{zero}}{r}$$

  - With a lead $C(s)$:

$$\frac{\sum \text{pole} - \sum \text{zero} + \text{pole}_{lead} - \text{zero}_{lead}}{r}$$



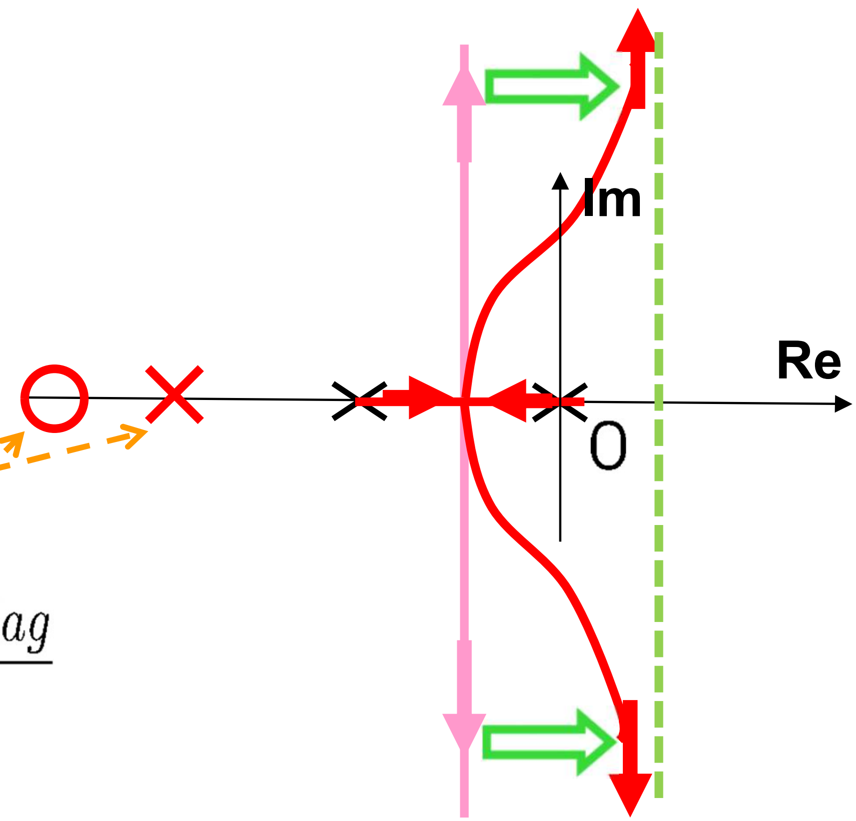
Reshaping root locus by lag compensators

- **Lag compensators** move the intersection of asymptotes (the centroid) to the right.
  - With $C(s) = K$ only:

$$\frac{\sum \text{pole} - \sum \text{zero}}{r}$$

  - With a lag $C(s)$:

$$\frac{\sum \text{pole} - \sum \text{zero} + \text{pole}_{lag} - \text{zero}_{lag}}{r}$$



Roles of lead & lag compensators

- Lead compensator
  - Improves transient response
  - Improves stability

$$C_{Lead}(s) = K\frac{s + z_{Lead}}{s + p_{Lead}}$$

- Lag compensator
  - Reduces steady state error

$$C_{Lag}(s) = \frac{s + z_{Lag}}{s + p_{Lag}}$$

- Lead-lag compensator
  - Takes into account both transient and steady state

$$C_{LL}(s) = C_{Lead}(s)C_{Lag}(s)$$

$$C_{LL}(s) = K\frac{s + z_{Lead}}{s + p_{Lead}}\frac{s + z_{Lag}}{s + p_{Lag}}$$

Lead compensator design

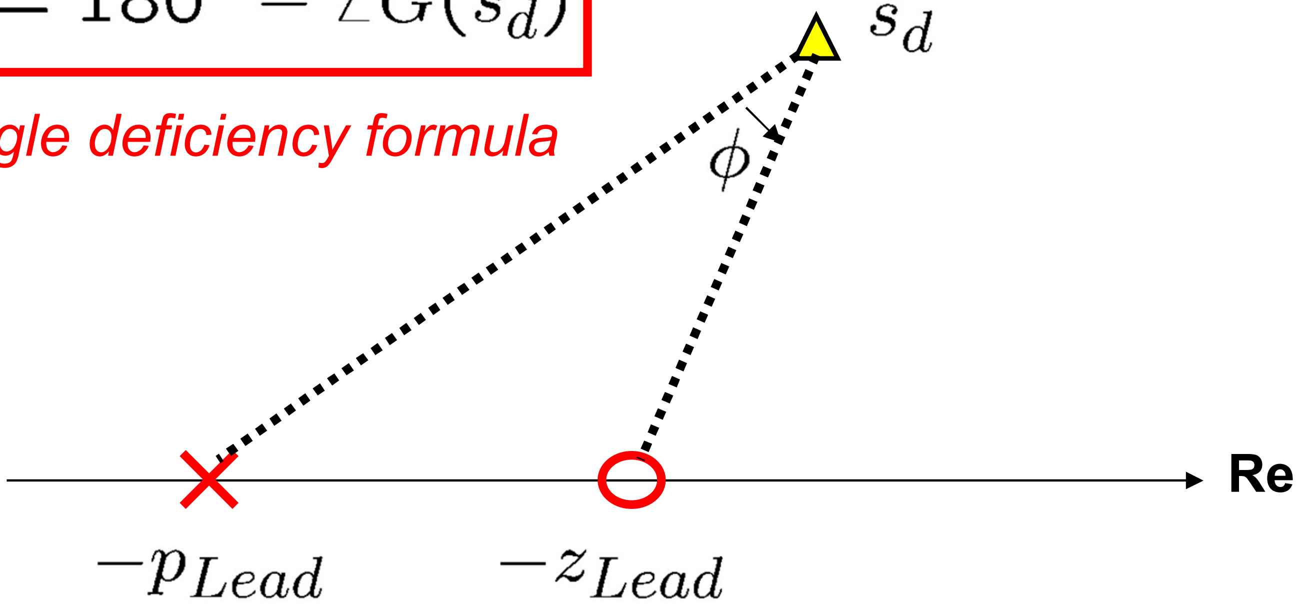
1. Select a desired pole in the allowable region (this is shown by s_d). We aim at reshaping RL to pass through this pole. In this course, s_d is given to you.

Lead compensator design

2. Select pole/zero in $C_{Lead}(s) = K\frac{s+z_{Lead}}{s+p_{Lead}}$ as

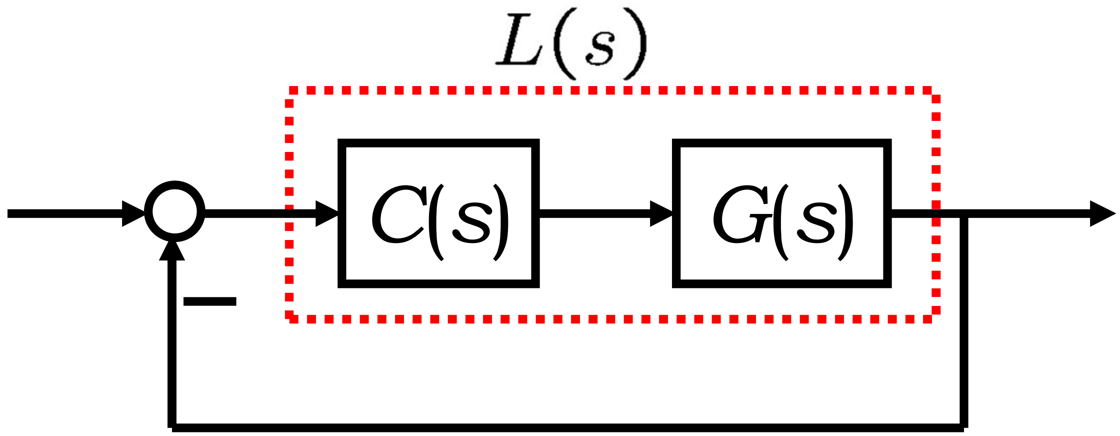
$$\phi = 180^\circ - \angle G(s_d)$$

Angle deficiency formula



Angle and magnitude conditions

- For an open-loop transfer function $L(s)$,



- A point s to be on root locus ←→ it satisfies
  - **Angle condition**

Odd number

$$\angle L(s) = 180^\circ \times (2k+1),\ \ k = 0, \pm 1, \pm 2, \ldots$$

- For a point on root locus, gain K is obtained by
  - **Magnitude condition**

$$|G(s)| = \frac{1}{K} \quad \text{or} \quad K = \frac{1}{|G(s)|}$$

How to find ϕ analytically

Follow the steps presented below:

Step 1: Calculate s_d using the following equation (if it is not given explicitly):

$$s_d = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

Step 2: Calculate $G(s_d)$.

Step 3: Calculate $\angle G^*(s_d)$.

Step 4: Convert $\angle G^*(s_d)$ to an angle from +Re-axis in CCW direction. Call this angle $\angle G(s_d)$.

Step 5: Calculate ϕ using the angle deficiency formula:

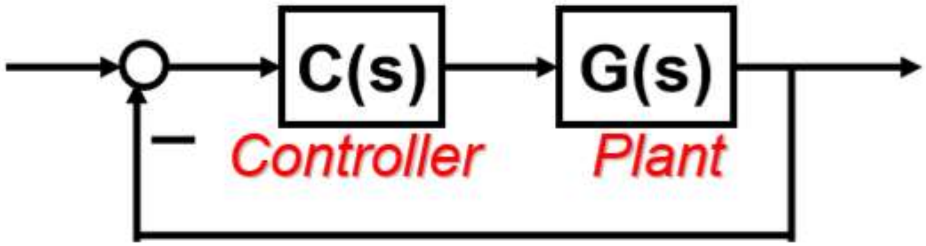
$$\phi = 180^\circ - \angle G(s_d)$$

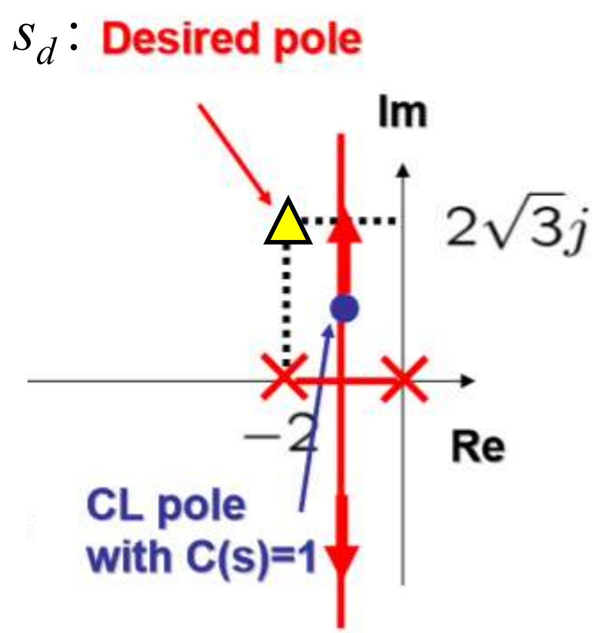
Example 2

Lead compensator design

- **Consider the system:**

$$G(s) = \frac{4}{s(s+2)}$$





- **Design specifications:**
  - Damping ratio $\zeta = 0.5$
  - Undamped natural frequency $\omega_n = 4$ rad/s

Example 2 (cont'd)

Lead compensator design (cont'd)

Evaluate G(s) at the desired pole.

$$s_d = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

$$s_d = -2+2\sqrt{3}j$$

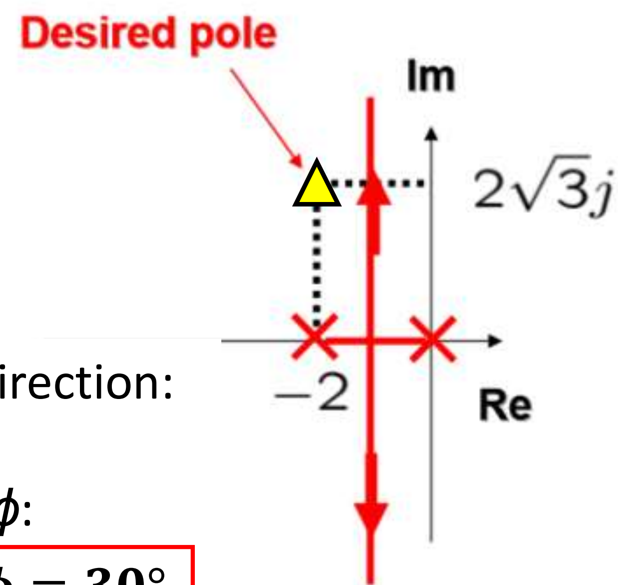
$$G(\underbrace{-2+2\sqrt{3}j}_{s_d}) = \frac{4}{(-2+2\sqrt{3}j)2\sqrt{3}j}$$

$$\angle G^*(s_d) = \angle 4 - \angle(-2+2\sqrt{3}j) - \angle 2\sqrt{3}j$$

$$= 0° - \left(\tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right)\right) - 90°$$

$$= 0° - (+120°) - 90° = \text{-}210° \rightarrow$$

$$\angle G^*(s_d) = \text{-}210°$$



Convert "-210°" to an angle from +Re-axis in CCW direction:

360° – 210° = +150° → $\angle G(s_d) = = +150°$

Now, use the **angle deficiency formula** to calculate ϕ:

$$\phi = 180° - \angle G(s_d) \rightarrow \phi = 180° - 150° \rightarrow \boldsymbol{\phi = 30°}$$

Example 2 (cont'd)

Lead compensator design

How to select pole and zero?

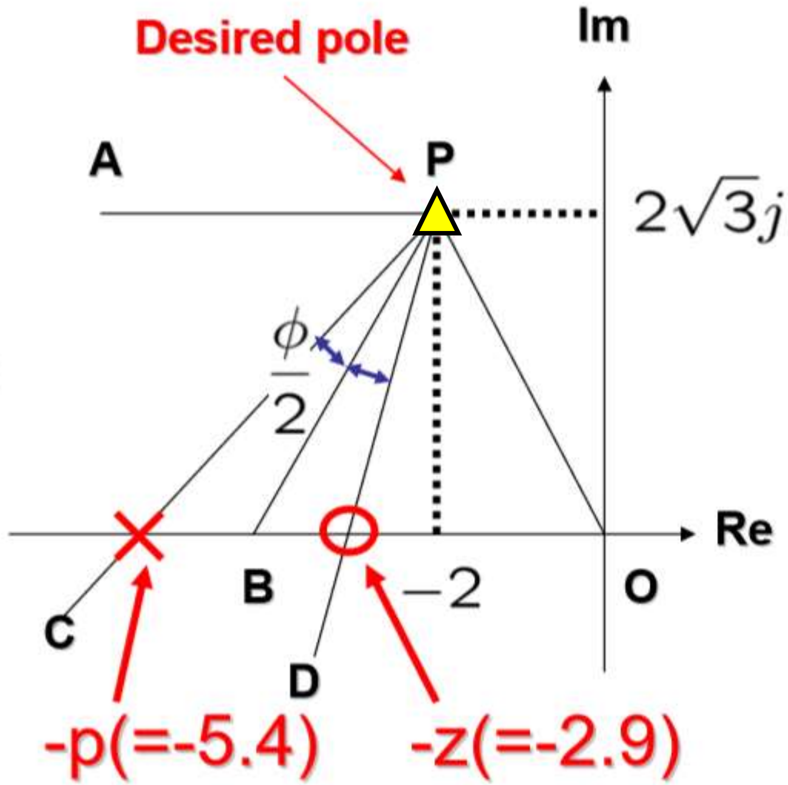
Method 1 (graphical):

- Draw horizontal line PA
- Draw line PO
- Draw bisector PB

$$\angle APB = \angle BPO = \frac{1}{2}\angle APO$$

- Draw PC and PD

$$\angle CPB = \angle BPD = \frac{\phi}{2}$$

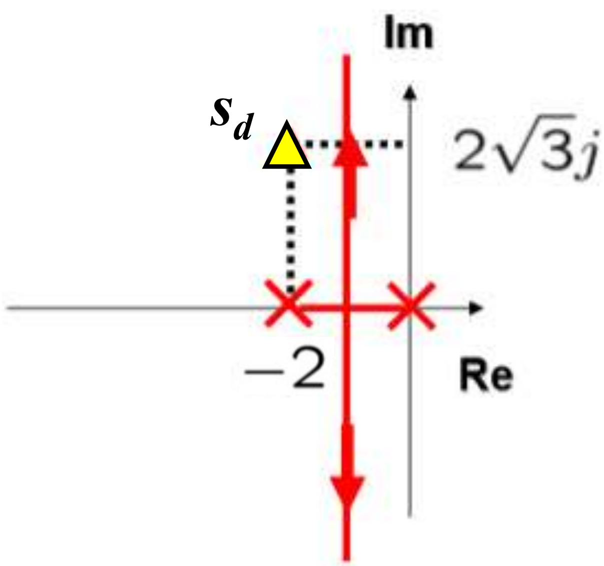


- Pole and zero of C(s) are shown in the figure.

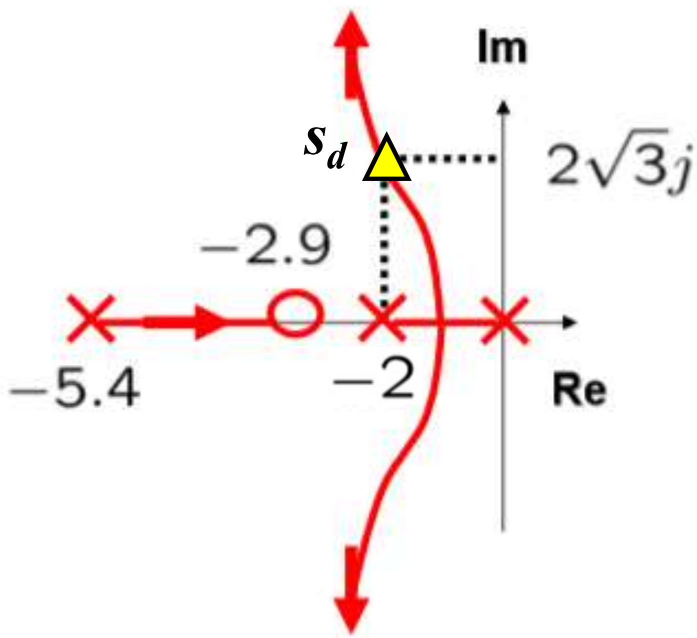
Example 2 (cont'd)

Comparison of root locus

$$C(s).G(s) = K.G(s)$$



$$C(s).G(s) = C_{Lead}.G(s) = K\frac{s + z_{Lead}}{s + p_{Lead}}G(s)$$



Improved stability!

Example 2 (cont'd)

How to design the gain K?

- Lead compensator

$$C(s) = K\frac{s+2.9}{s+5.4}$$

- Open loop transfer function

$$G(s)C(s) = K\frac{4(s+2.9)}{s(s+2)(s+5.4)} = K.\tilde{G}(s) = L(s)$$

Note: $|(s+2.9)|_{s_d} = |(-2+2\sqrt{3}j+2.9)|$
$= |(0.9+2\sqrt{3}j)| = \sqrt{(0.9)^2+(2\sqrt{3})^2}$
$= 3.579$

- Magnitude condition $K = \dfrac{1}{|\tilde{G}(s_d)|}$

$$K = \frac{1}{\left|\frac{4(s+2.9)}{s(s+2)(s+5.4)}\right|_{s_d=-2+2\sqrt{3}j}} \Rightarrow \boxed{K = 4.675}$$

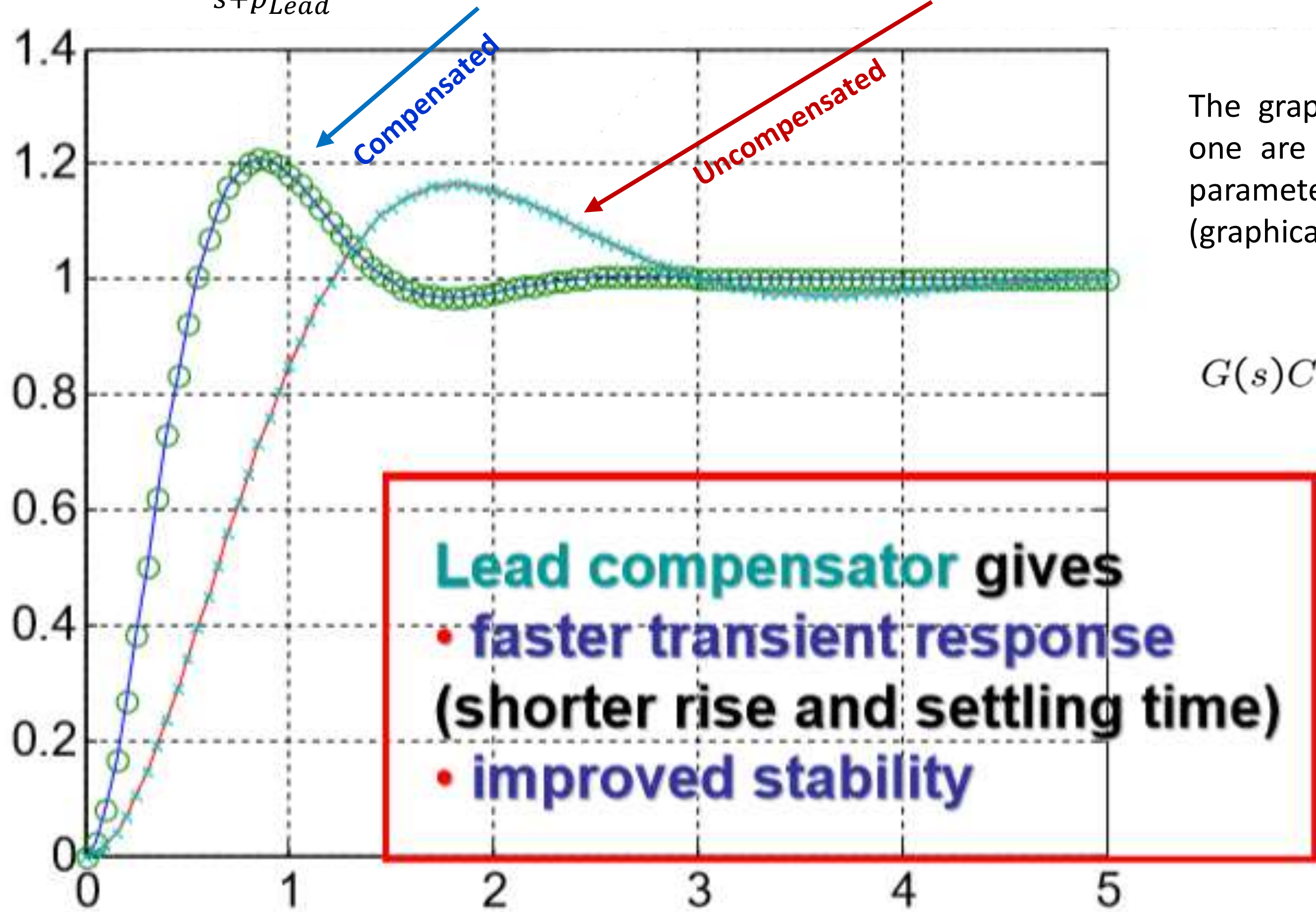
$$\Rightarrow \boxed{G(s)C(s) = \frac{4}{s(s+2)} \cdot \frac{4.675(s+2.9)}{s+5.4}}$$

Example 2 (cont'd)

Comparison of step responses

$C(s).G(s) = C_{Lead}.G(s) = K\frac{s+z_{Lead}}{s+p_{Lead}}G(s)$; For $\boldsymbol{K = 4.675}$

$C(s).G(s) = K.G(s)$; For $\boldsymbol{K = 1}$



The graphs in this slide and the next one are plotted based on the design parameters obtained through Method 1 (graphical method).

$$G(s)C(s) = \frac{18.7\,(s+2.9)}{s(s+2)(s+5.4)}$$

Example 2 (cont'd)

Error constants

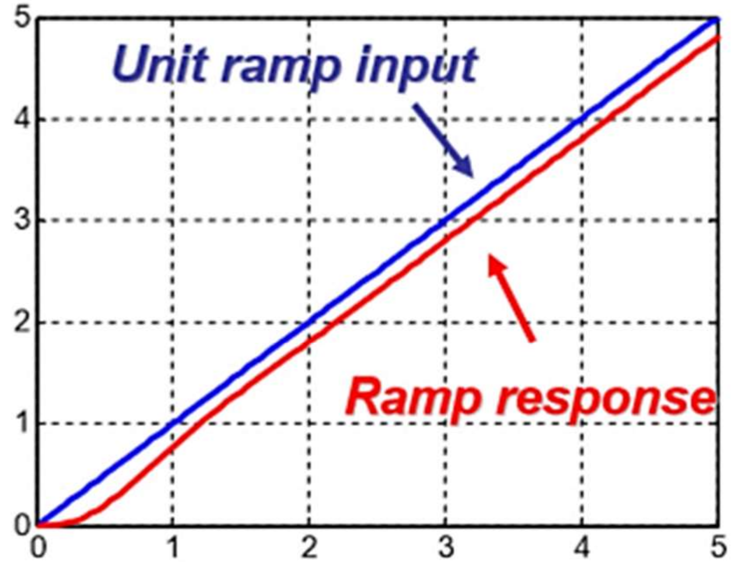
$$G(s)C(s) = \frac{4}{s(s+2)} \cdot \frac{4.675(s+2.9)}{s+5.4}$$

- Step-error constant

$$K_p = \lim_{s\to 0} G(s)C(s) = \infty$$

- Ramp-error constant

$$K_v = \lim_{s\to 0} sG(s)C(s) = 5.02$$



Lag compensator can be used to reduce steady-state error.

$$e_{ss} = \frac{1}{K_v} = \frac{1}{5.02} = 0.19$$

This will be shown soon.

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Example 2 (cont'd)
Lead compensator design

Method 2 (analytical):

Important Note: Method 2 should be used in your tests and other class activities and not Method 1 (the graphical method). The graphical method was presented just for demonstration purposes. Below is a summary for **Method 2**:

Step 1: Take z_{lead} = |real part of s_d|.

Step 2: Find p_{lead} using the following relation (ϕ is assumed to be already calculated analytically using the angle deficiency formula):

$$\angle \frac{s_d + z_{lead}}{s_d + p_{lead}} = \phi$$

Step 3: Substitute for the parameters in the C_{lead} relation:

$$C_{lead} = K\frac{s + z_{lead}}{s + p_{lead}}$$

Step 4: Find K using the following relation:

$$K = \frac{1}{|\tilde{G}(s_d)|}$$

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Example 2 (cont'd)
Lead compensator design

Method 2 (analytical):

Step 1:

$s_d = -2+2\sqrt{3}j \quad \rightarrow \quad z_{lead}$ = |real part of s_d| $= 2$

Step 2:

$$\angle\frac{s_d+z_{lead}}{s_d+p_{lead}} = \phi \rightarrow \angle\frac{s_d+2}{s_d+p_{lead}} = 30° \rightarrow \angle\frac{-2+2\sqrt{3}j+2}{-2+2\sqrt{3}j+p_{lead}} = 30° \rightarrow p_{lead} = 4$$

Step 3:

$$C_{lead} = K\frac{s+z_{lead}}{s+p_{lead}} = K\frac{s+2}{s+4}$$

Step 4:

$$K = \frac{1}{|\tilde{G}(s_d)|} = 4.00$$

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Example 2 (cont'd)
Lead compensator design

Details of the Solution:

Angle deficiency formula:

$$\angle\frac{s_d + z_{lead}}{s_d + p_{lead}} = \phi \quad \Rightarrow \quad \angle\left(\frac{s_d + 2}{s_d + p_{lead}}\right) = 30^\circ$$

Substitute $s_d = -2 + 2\sqrt{3}j$:

Let's compute the phase angle condition:

$$\angle(s_d + 2) = \angle(0 + 2\sqrt{3}j) = 90^\circ$$

$$\angle(s_d + p_{lead}) = \tan^{-1}\left(\frac{2\sqrt{3}}{-2 + p_{lead}}\right)$$

So:

$$90^\circ - \tan^{-1}\left(\frac{2\sqrt{3}}{-2 + p_{lead}}\right) = 30^\circ$$

$$\Rightarrow \tan^{-1}\left(\frac{2\sqrt{3}}{-2 + p_{lead}}\right) = 60^\circ$$

$$\Rightarrow \frac{2\sqrt{3}}{-2 + p_{lead}} = \tan 60^\circ = \sqrt{3} \approx 1.732$$

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Example 2 (cont'd)
Lead compensator design

Solve for p_{lead}:

$$1.732 = \frac{2\sqrt{3}}{-2 + p_{lead}}$$

Multiply both sides:

$$1.732(-2 + p_{lead}) = 2\sqrt{3}$$

$$-3.464 + 1.732 \cdot p_{lead} = 2\sqrt{3}$$

$$1.732 \cdot p_{lead} = 2\sqrt{3} + 3.464$$

$$\Rightarrow p_{lead} = 4$$

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Example 2 (cont'd)

Lead compensator design

Now, we have:

$$L(s) = K.\tilde{G}(s) \Rightarrow L(s) = K\frac{4(s+2)}{s(s+2)(s+4)}$$

Now solve for gain K:

Given:

$$s_d = -2 + 2\sqrt{3}j$$

We are told:

$$|\tilde{G}(s_d)| = \left|\frac{4(-2+2\sqrt{3}j+2)}{(-2+2\sqrt{3}j)(-2+2\sqrt{3}j+2)(-2+2\sqrt{3}j+4)}\right|$$

Simplify numerator:

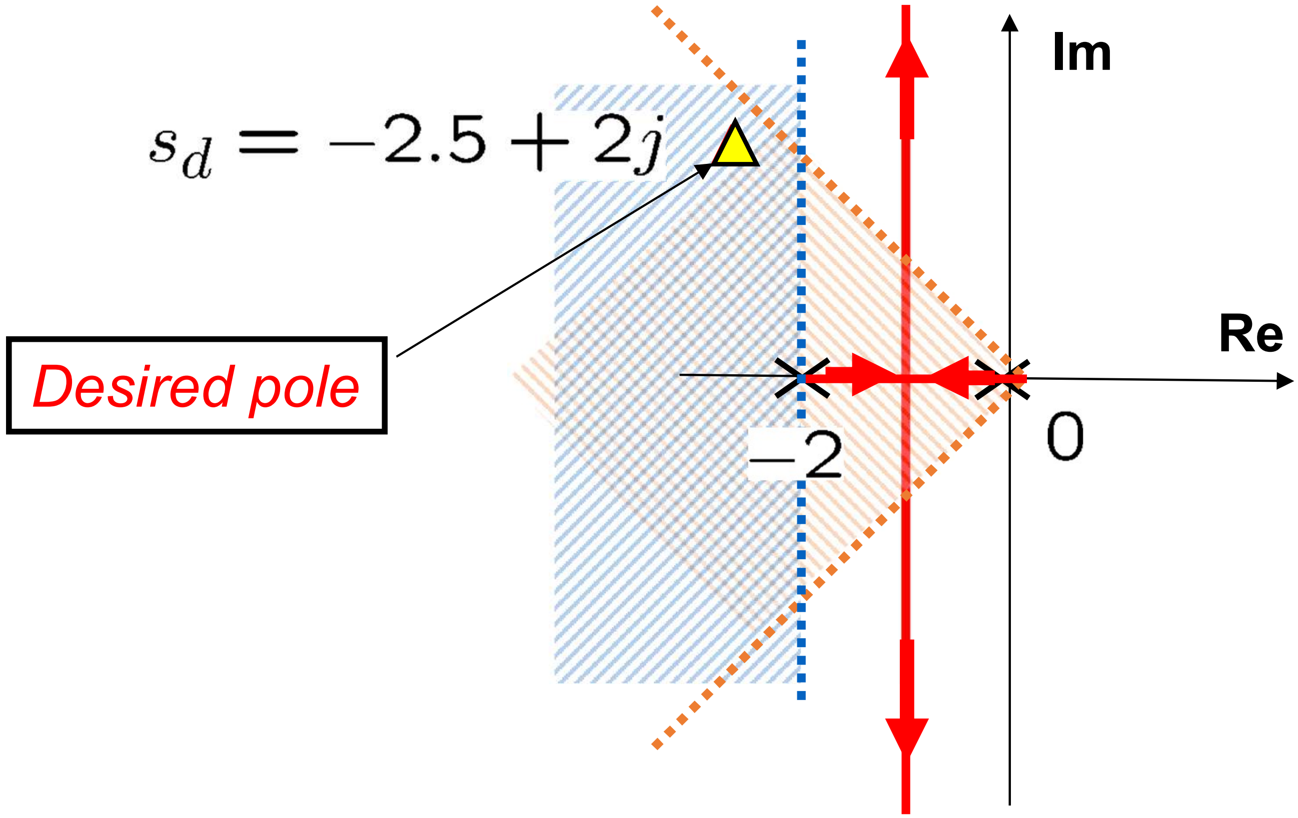
$$|-2+2\sqrt{3}j+2| = |2\sqrt{3}j| = 2\sqrt{3}$$

So:

$$|\tilde{G}(s_d)| = \frac{4\cdot 2\sqrt{3}}{4\cdot 2\sqrt{3}\cdot 4} = \frac{8\sqrt{3}}{32\sqrt{3}} = \frac{1}{4} \quad \Rightarrow \quad K = \frac{1}{|\tilde{G}(s_d)|} = \frac{1}{\frac{1}{4}} = \boxed{4.00}$$

Example 3

Design a lead compensator controller by using the given desired pole in the allowable region.



$$G(s) = \frac{1}{s(s+2)}$$

We aim at reshaping RL to pass through the desired pole.

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Example 3: One possible C_{Lead}

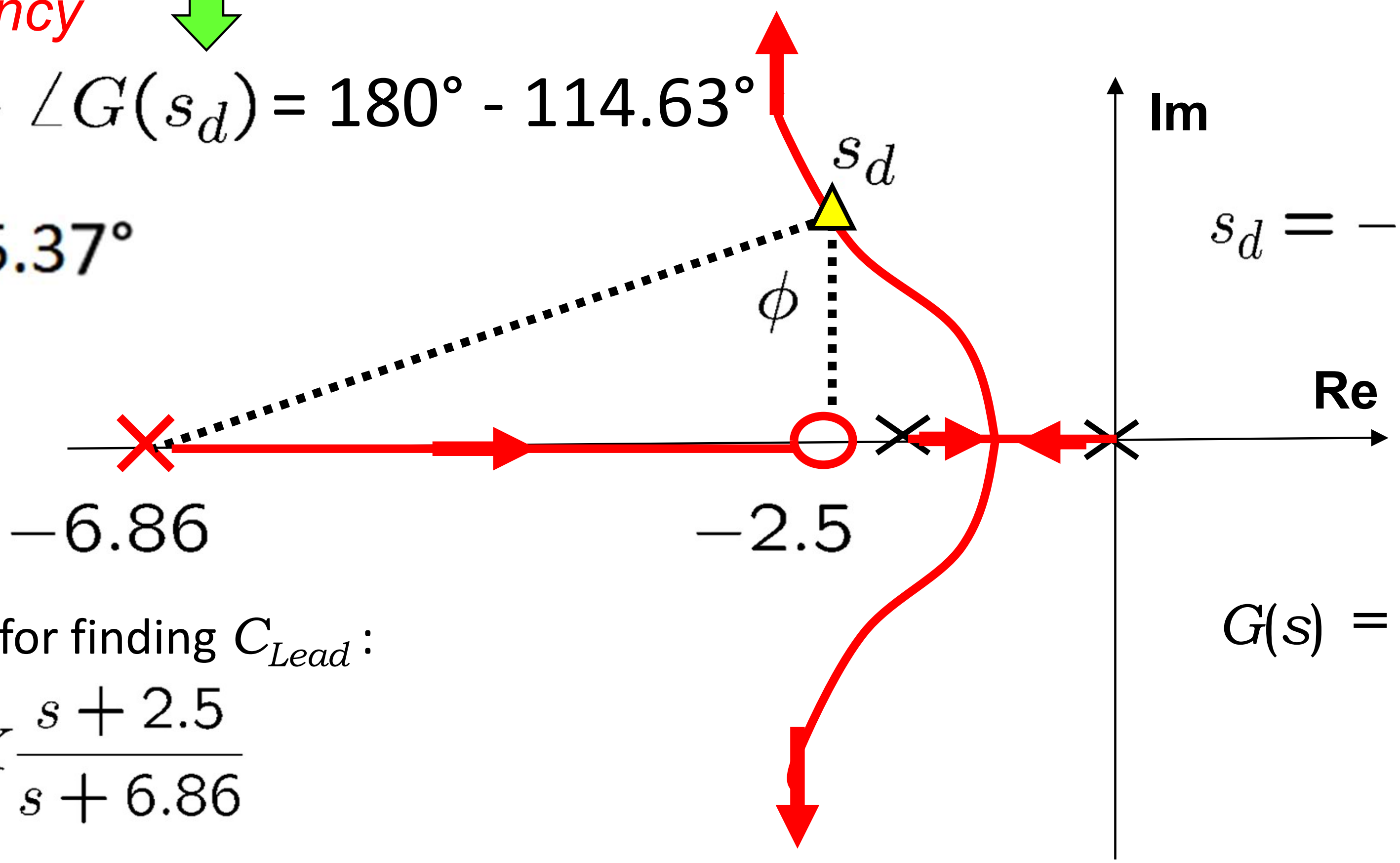
$$\angle G^{*}(s_d) = \angle \frac{1}{s_d(s_d+2)} = \text{-245.37°} \Rightarrow 360° - 245.37° = 114.63°$$

$$\Rightarrow \angle G(s_d) = 114.63°$$

Angle deficiency

$$\phi = 180° - \angle G(s_d) = 180° - 114.63°$$

$$\Rightarrow \phi = 65.37°$$



$$s_d = -2.5 + 2j$$

$$G(s) = \frac{1}{s(s+2)}$$

Using Method 2 for finding C_{Lead} :

$$C_{Lead}(s) = K\frac{s+2.5}{s+6.86}$$

Example 3: Design of pole/zero in C_{Lead}

- Lead compensator:

$$C_{Lead}(s) = K\frac{s+2.5}{s+6.86}$$

- Open-loop transfer function:

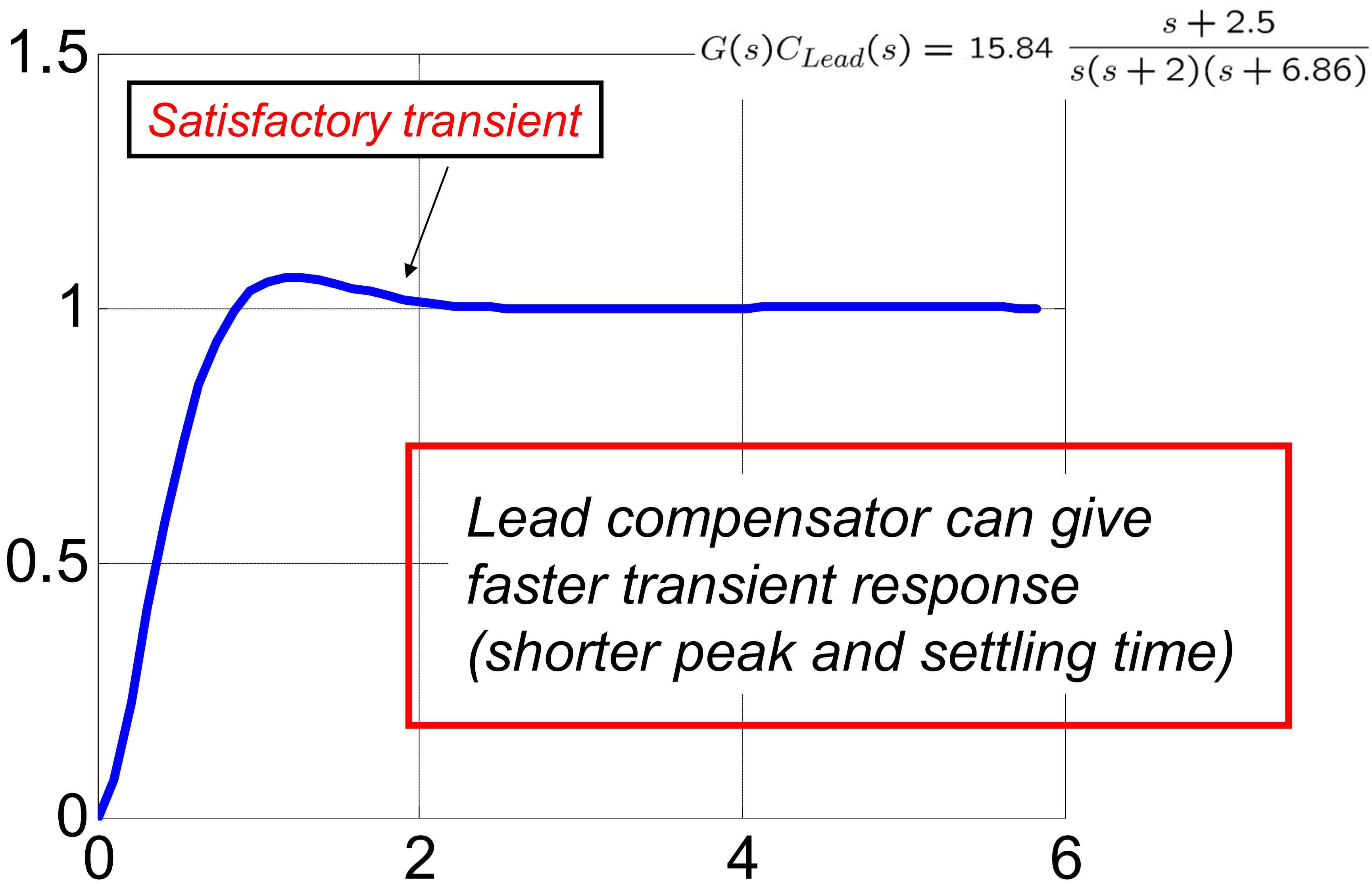
$$G(s)C_{Lead}(s) = K\frac{s+2.5}{s(s+2)(s+6.86)} = L(s)$$

- Gain computation:

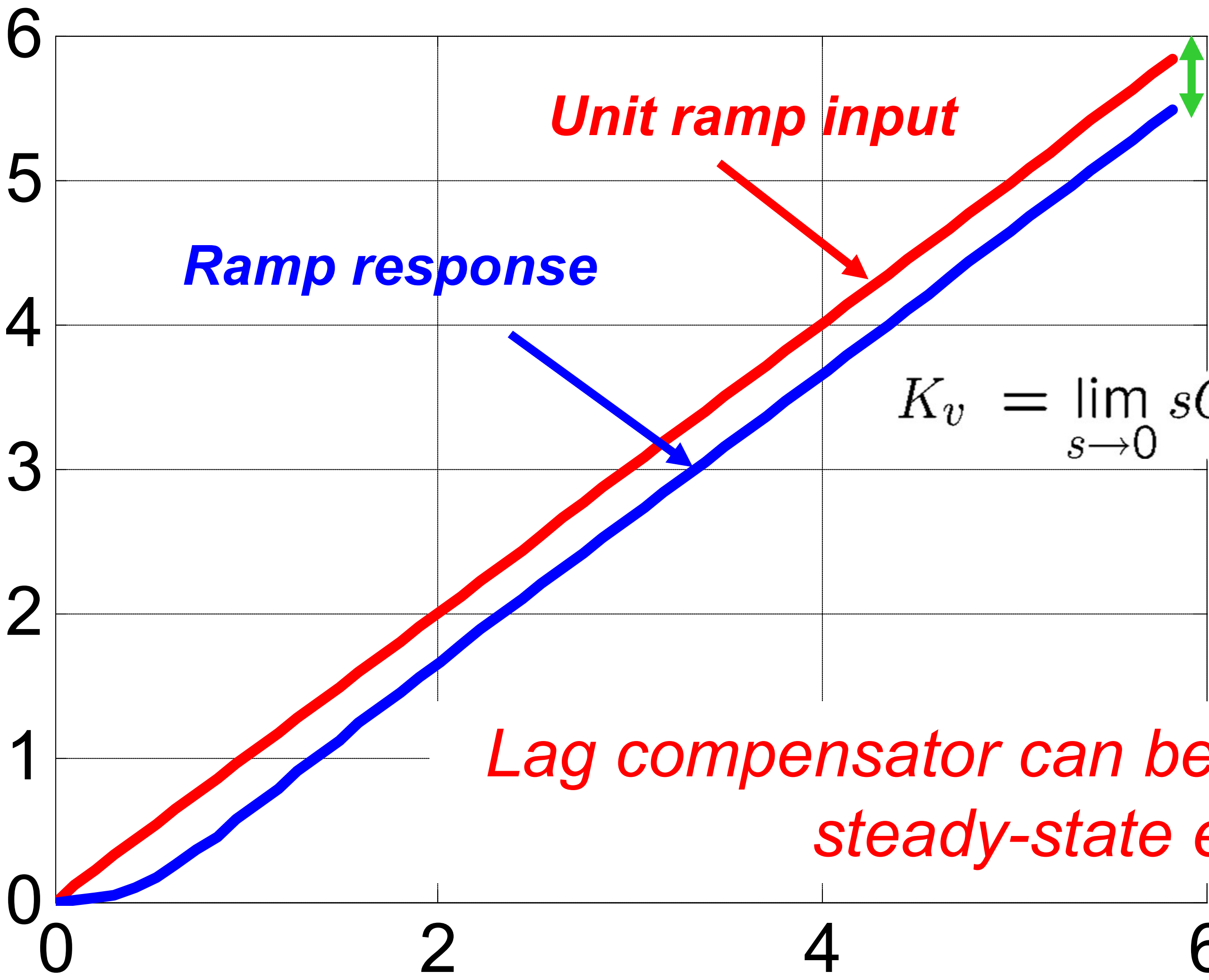
$$K = \frac{1}{|\tilde{G}(s_d)|} = 15.84 \quad \Rightarrow$$

$$G(s)C_{Lead}(s) = 15.84\frac{s+2.5}{s(s+2)(s+6.86)}$$

Example 3: Step response with C_{Lead}



Example 3: Steady state error for $r(t) = tu(t)$



$$e_{ss} = \frac{1}{K_v} \approx 0.35$$

$$K_v = \lim_{s \to 0} sG(s)C_{Lead}(s) = 2.89$$

Lag compensator can be used to reduce steady-state error.

Roles of lead & lag compensators

- Lead compensator
  - Improves transient response
  - Improves stability

$$C_{Lead}(s) = K\frac{s+z_{Lead}}{s+p_{Lead}}$$

- Lag compensator
  - Reduces steady state error

$$C_{Lag}(s) = \frac{s+z_{Lag}}{s+p_{Lag}}$$

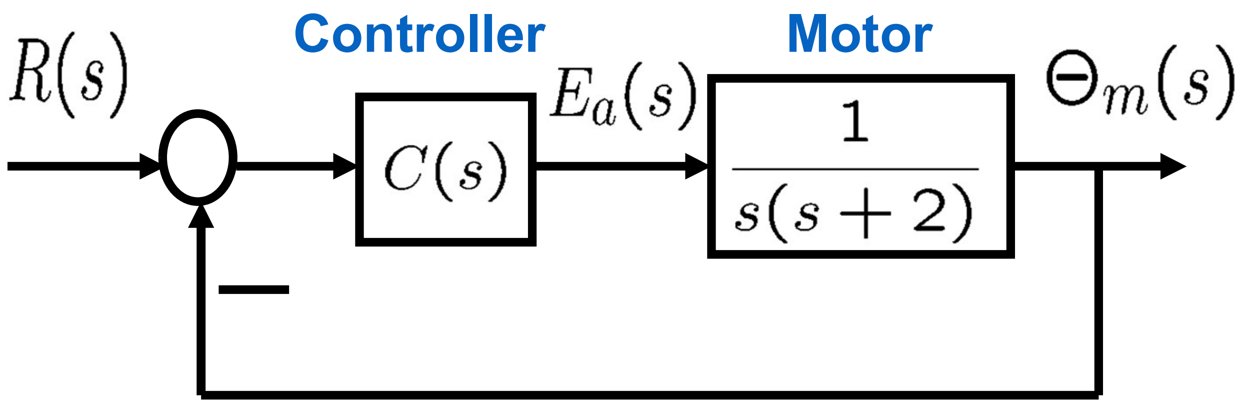
- Lead-lag compensator
  - Takes into account both transient and steady state.

$$C_{LL}(s) = C_{Lead}(s)C_{Lag}(s)$$

$$C_{LL}(s) = K\frac{s+z_{Lead}}{s+p_{Lead}}\frac{s+z_{Lag}}{s+p_{Lag}}$$

Example 4 (revisited)

- A feedback system is given as below:

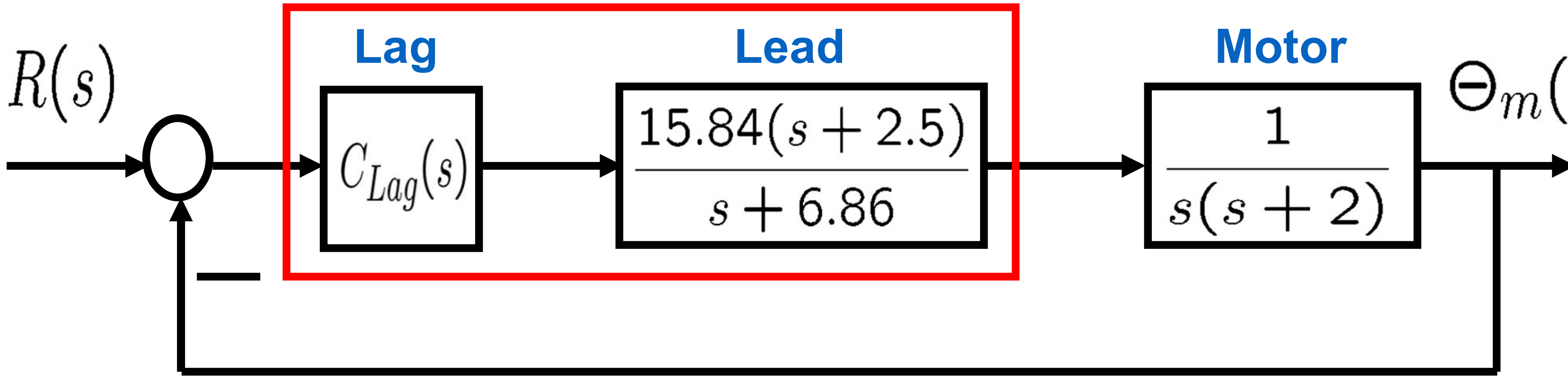


- Design specifications:
  - 2% settling time at most 2 seconds
  - Overshoot at most 4.32%
  - Assume that we have already designed a lead compensator with the following TF:

$$C_{Lead}(s) = 15.84 \frac{s+2.5}{s(s+2)(s+6.86)}$$

  - Steady state error:
    - Zero for unit step $r(t) = u(t)$
    - At most 0.05 for unit ramp $r(t) = tu(t)$

Example 4: After C_{Lead} design



- For a designed C_{Lead}, we need to design C_{Lag} so that
  - Steady state error for unit ramp $r(t)$ is reduced (e_{ss} should be less than 0.05). Here, p_{Lag} is given as 0.005.
  - Transient is maintained. That is, we do not want to lose the satisfactory transient property achieved by the lead compensator.

Solution:

$$e_{ss} < 0.05;\quad e_{ss} = \frac{R}{K_v} \to \frac{1}{K_v} < 0.05 \to \boldsymbol{K_v > 20}$$

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Example 4: Design of $C_{Lag}(s) = \frac{s + z_{Lag}}{s + p_{Lag}}$

- We would like to reduce the steady state error, i.e., to increase ramp-error constant:

$$K_v = \lim_{s \to 0} sG(s)C_{Lead}(s)C_{Lag}(s) = 2.89 \times \frac{z_{Lag}}{p_{Lag}} \xrightarrow{K_v > 20} 2.89 \times \frac{z_{Lag}}{p_{Lag}} > 20 \Rightarrow \frac{z_{Lag}}{p_{Lag}} > 6.92$$

$$\Rightarrow z_{Lag} > 6.92\, p_{Lag}$$

Take, for example, $\boxed{z_{Lag} = 10 p_{Lag}}$

In your tests, just round up 6.92 (or any other similar number) to its next nearest integer number (here, 7) and take z_{Lag} = (integer number)p_{Lag}.

- Also, we want to maintain the transient property,

$$\boxed{|p_{Lag}| : \text{small}}$$

In your tests, I will give you p_{Lag} and it is considered a known parameter.

- We still want our RL to pass through s_d, despite modifying our $L(s)$.

$$\left.\begin{array}{r} 1 + G(s_d)C_{Lead}(s_d) = 0 \\ C_{Lag}(s_d) \approx 1 \end{array}\right\} \Rightarrow 1 + G(s_d)C_{Lead}(s_d)C_{Lag}(s_d) \approx 0$$

Why should p_{Lag} be small?

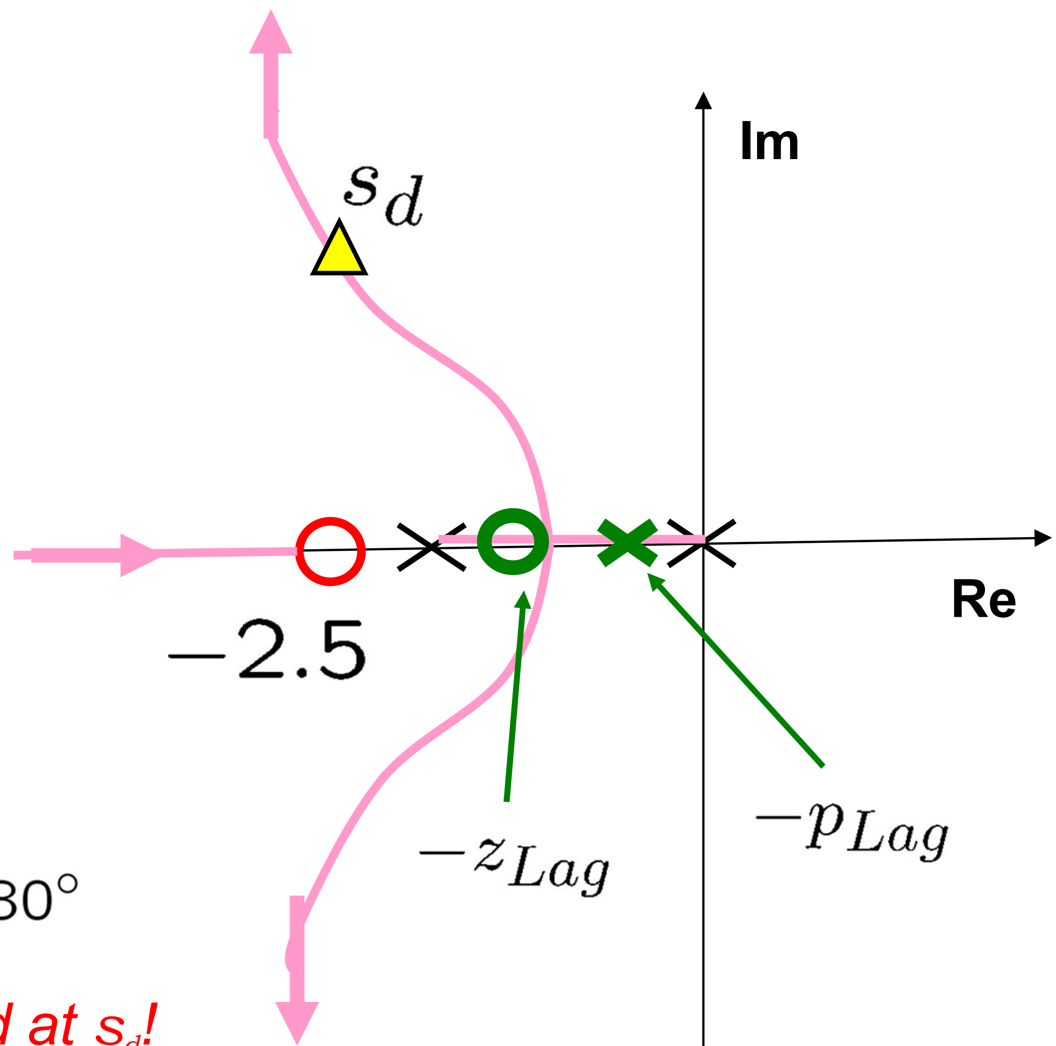
p_{Lag} : small

$$z_{Lag} = 10p_{Lag}$$

$$C_{Lag}(s_d) = \frac{s_d + z_{Lag}}{s_d + p_{Lag}} \approx 1$$

$$\angle G(s_d)C_{Lead}(s_d)C_{Lag}(s_d) \approx 180^\circ$$

Angle condition: almost satisfied at s_d!



Example 4: C_{Lag} design

- In design projects, we assume a small p_{Lag} and use trial-and-error! Here, p_{Lag} will be given to you.

$$p_{Lag} = 0.005 \Rightarrow z_{Lag} = 0.05$$

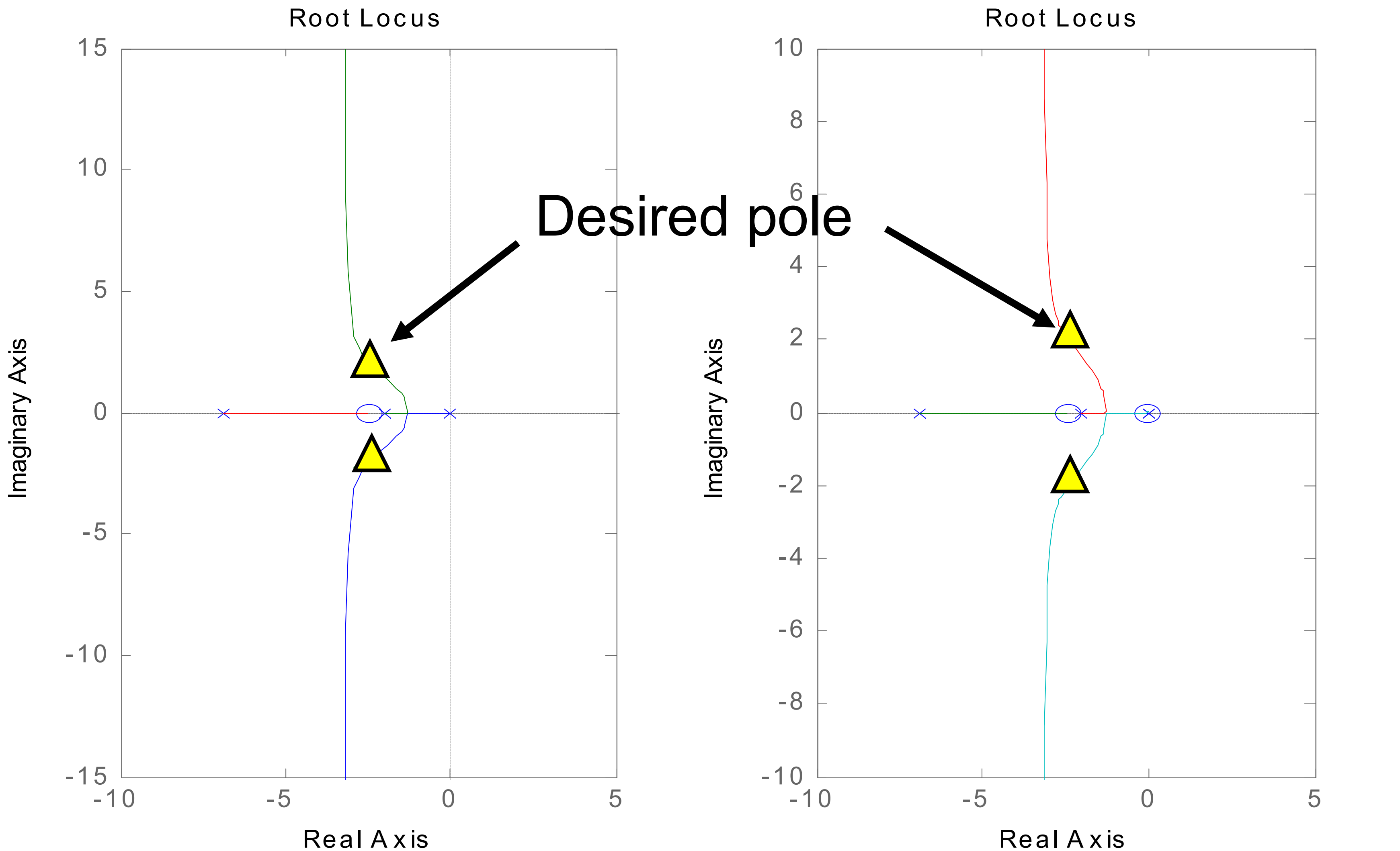
$$\Rightarrow C_{Lag}(s) = \frac{s + 0.05}{s + 0.005}$$

$$\Rightarrow C_{LL}(s) = \underbrace{\frac{15.84(s + 2.5)}{s + 6.86}}_{C_{Lead}(s)} \times \underbrace{\frac{s + 0.05}{s + 0.005}}_{C_{Lag}(s)}$$

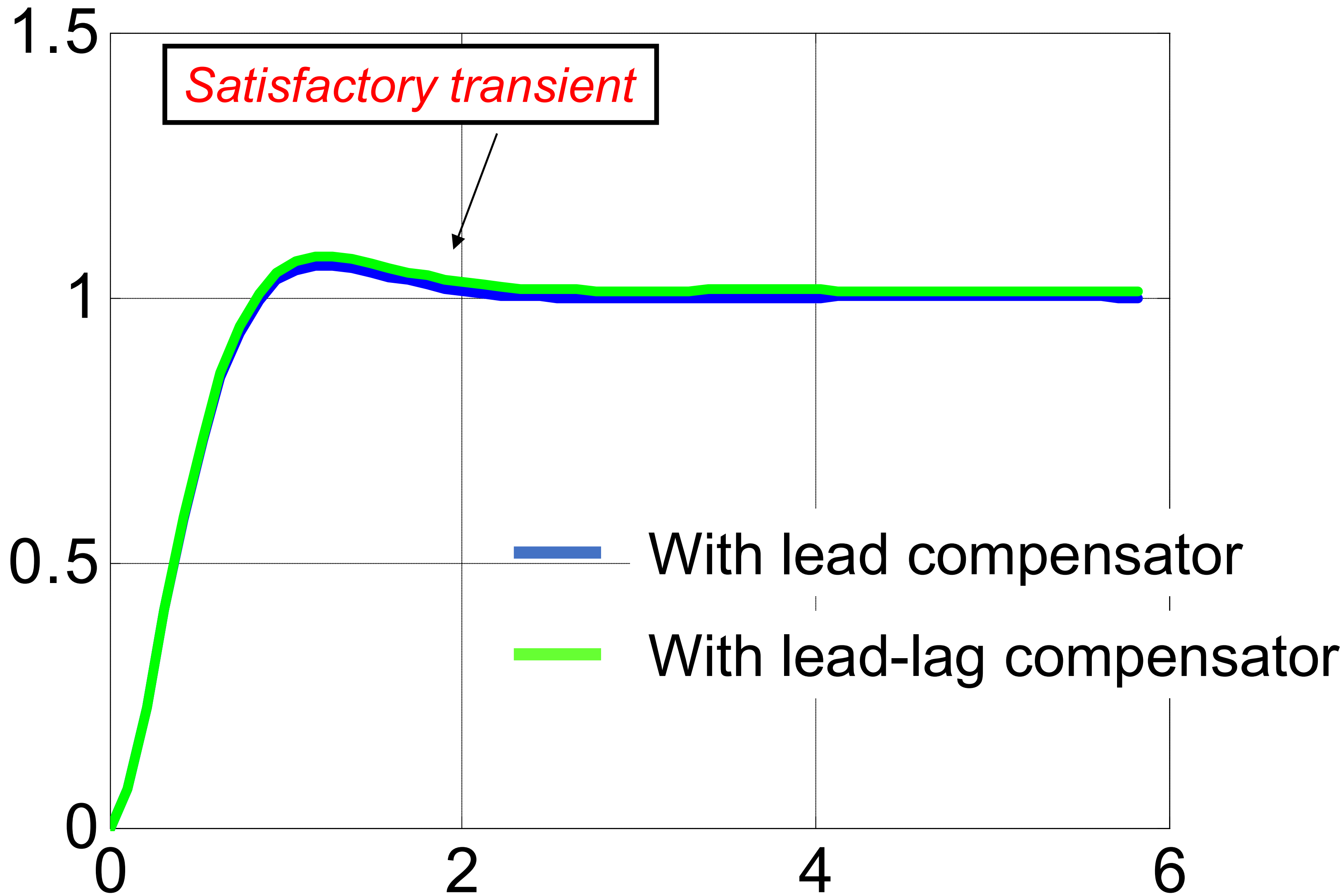
Example 4: Root locus

With lead compensator

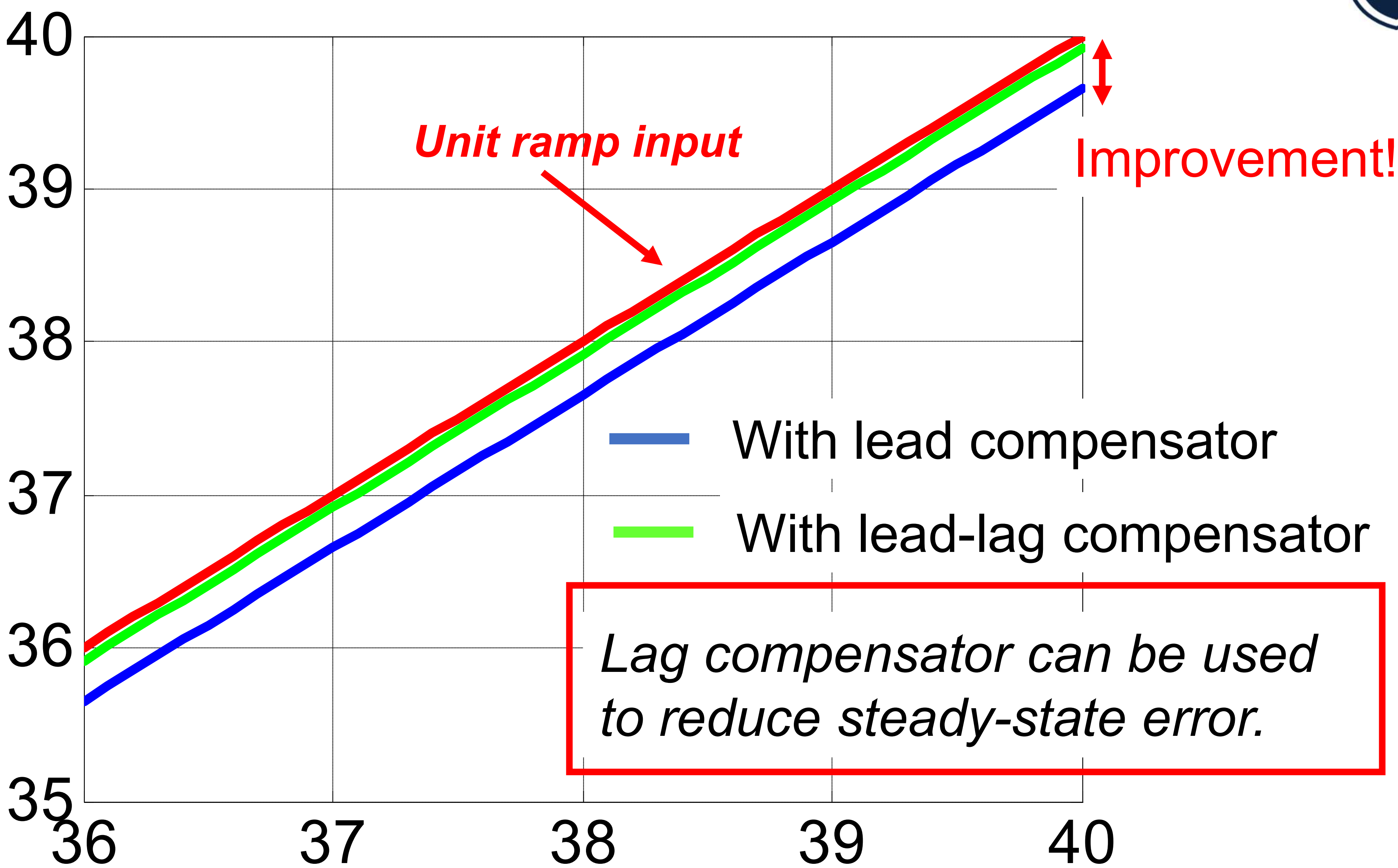
With lead-lag compensator



Example 4: Comparison of step responses



Example 4: Comparison of ramp responses



Lag compensator can be used to reduce steady-state error.

Example 5 (Example 2, revisited)

If we only use a lead compensator.

Error constants

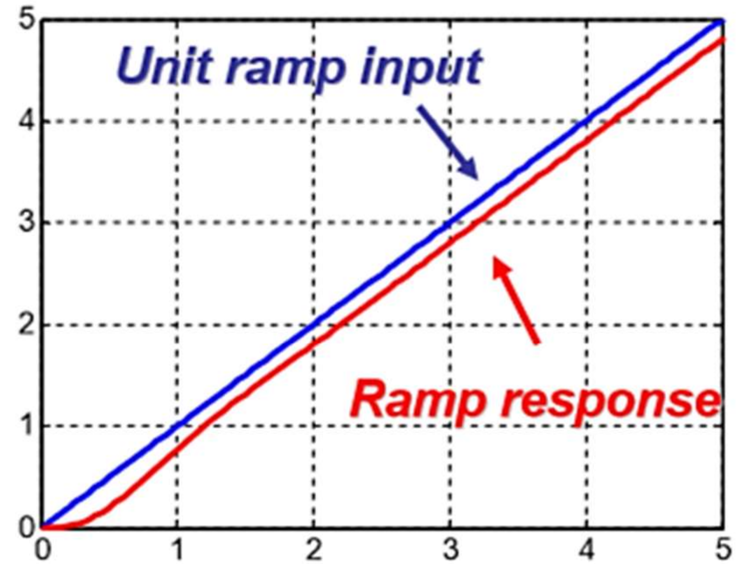
$$G(s)C(s) = \frac{4}{s(s+2)} \cdot \frac{4.675(s+2.9)}{s+5.4}$$

- Step-error constant

$$K_p = \lim_{s \to 0} G(s)C(s) = \infty$$

- Ramp-error constant

$$K_v = \lim_{s \to 0} sG(s)C(s) = 5.02$$



Lag compensator can be used to reduce steady-state error.

$$e_{ss} = \frac{1}{K_v} = \frac{1}{5.02} = 0.19$$

a place of mind UBC

Example 5

- Let us design a lead-lag compensator so that we can have a lower steady-state error when the input is a unit ramp function. We would like e_{ss} to be less than 0.02.

$$e_{ss} < 0.02; \quad e_{ss} = \frac{R}{K_v} \rightarrow \frac{1}{K_v} < 0.02 \rightarrow \boldsymbol{K_v > 50}$$

$$K_v = \lim_{s \to 0} sG(s)C_{Lead}(s)C_{Lag}(s) = 5.02 \cdot \frac{z_{Lag}}{p_{Lag}} \xrightarrow{K_v > 50} \frac{z_{Lag}}{p_{Lag}} > 9.96$$

$$\rightarrow \boldsymbol{z_{Lag} = 10 p_{Lag}}$$

- We still want our RL to pass through s_d, despite modifying our $L(s)$.

$$\left.\begin{matrix} 1 + G(s_d)C_{Lead}(s_d) = 0 \\ C_{Lag}(s_d) \approx 1 \end{matrix}\right\} \Rightarrow 1 + G(s_d)C_{Lead}(s_d)C_{Lag}(s_d) \approx 0$$

Example 5

- Let us say p_{Lag} is given as 0.025:

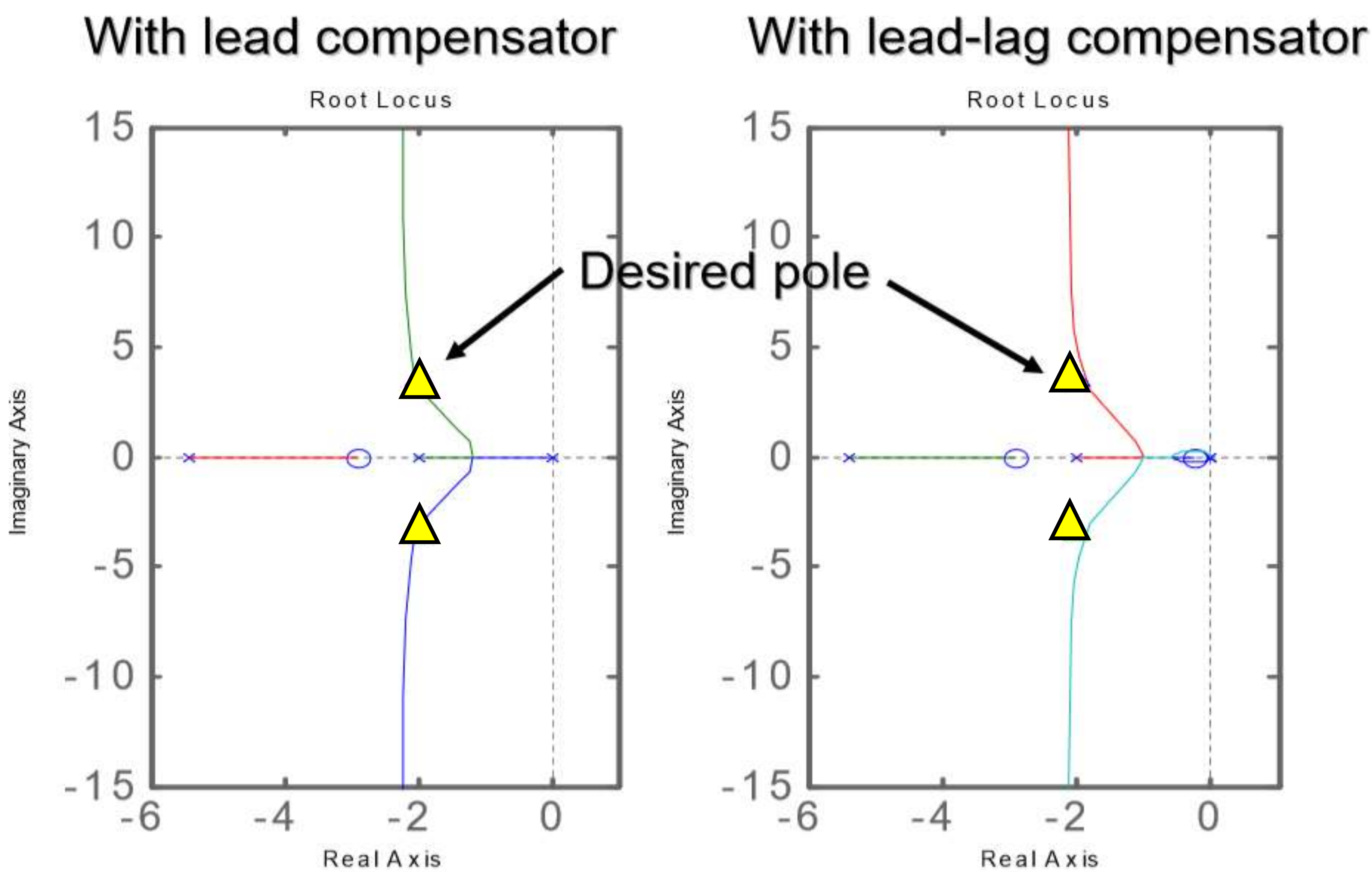
$z_{Lag} = 10p_{Lag} = (10)(0.025) = 0.25$ ⇒ $z_{Lag} = 0.25$

- Lead-lag controller

$$C_{LL}(s) = 4.675\frac{s+2.9}{s+5.4}\cdot\frac{s+0.25}{s+0.025}$$

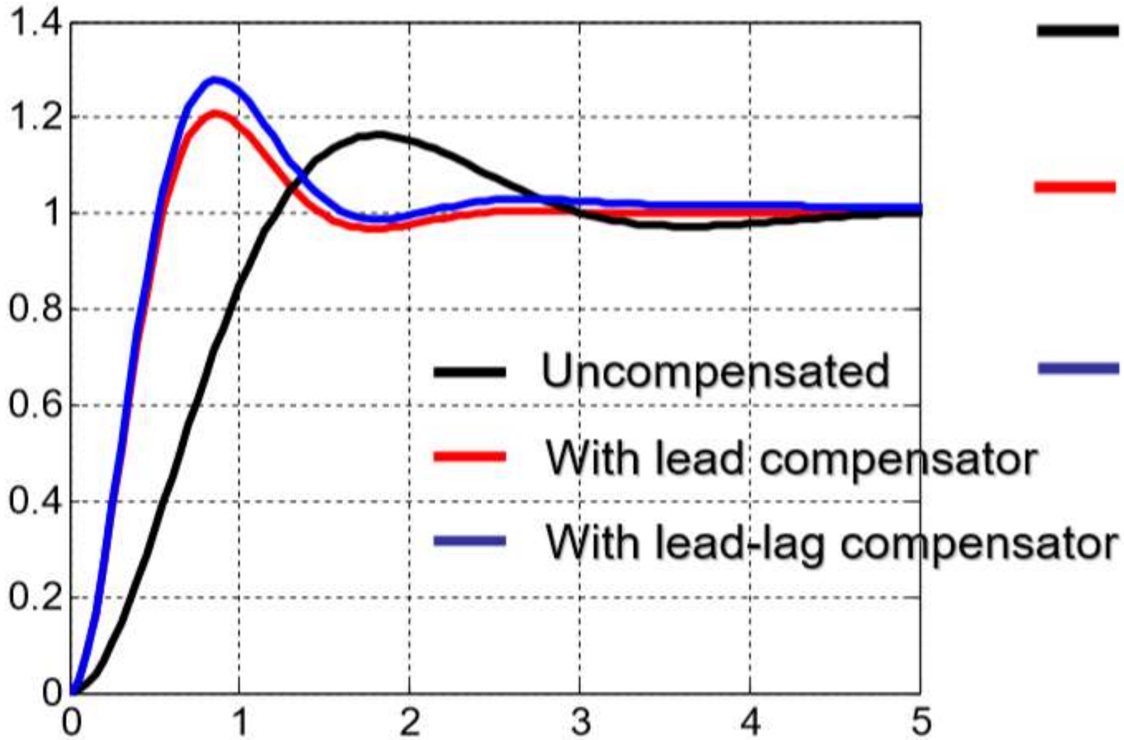
Example 5

Root locus



Example 5

Comparison of step responses



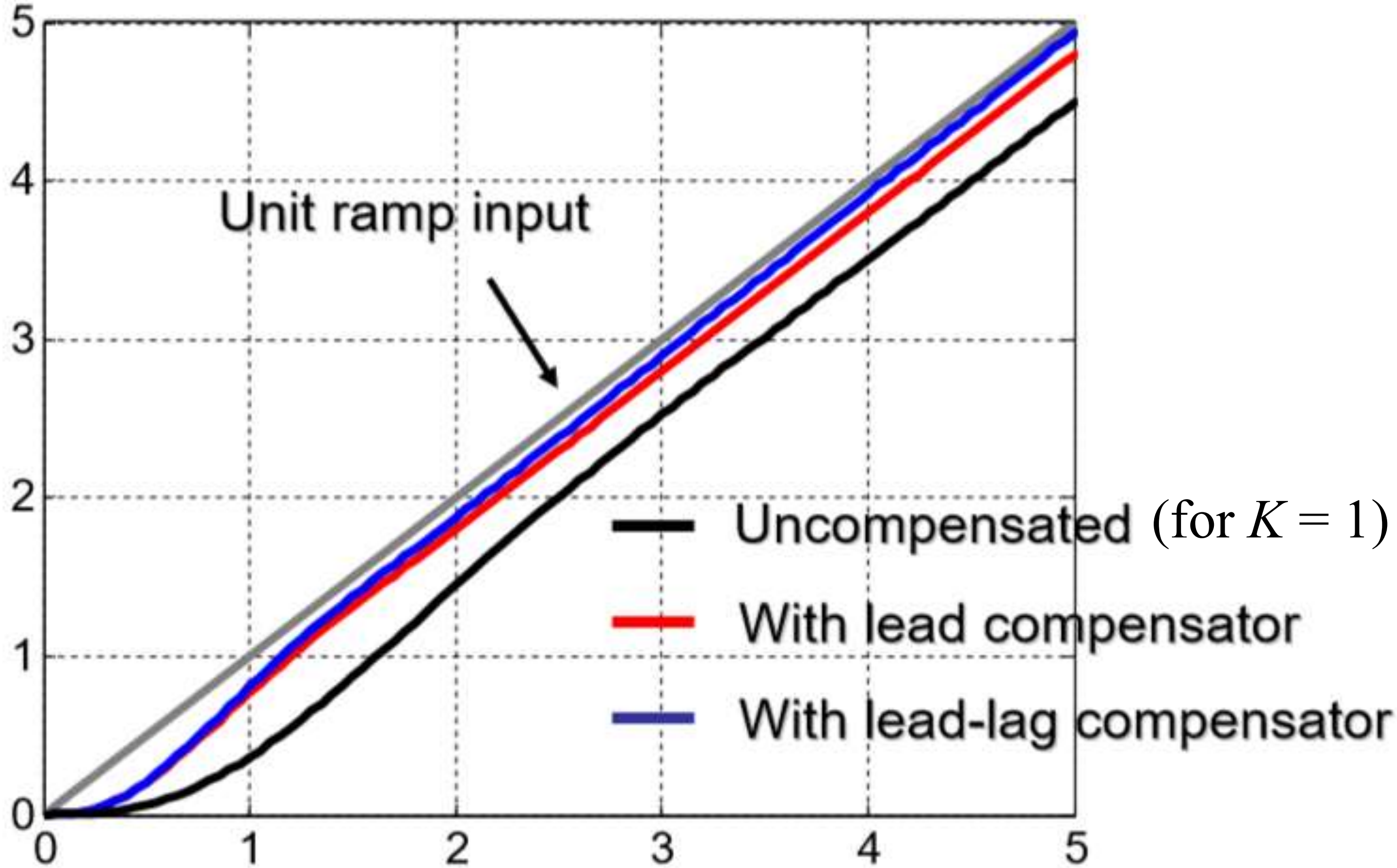
$$G(s)C(s) = \frac{4}{s(s+2)} \cdot K = \frac{4}{s(s+2)} \cdot 1$$

$$G(s)C(s) = \frac{4}{s(s+2)} \cdot \frac{4.675(s+2.9)}{s+5.4}$$

$$G(s)C(s) = \frac{4}{s(s+2)} \cdot \frac{4.675(s+2.9)}{s+5.4} \cdot \frac{s+0.25}{s+0.025}$$

Example 5

Comparison of ramp responses



Remarks on controller design

- Existence of a satisfactory controller is generally unknown before the design.
- When a satisfactory controller exists, such controller is not unique.
- Controller design methods are not unique either.
  - Different references explain controller design in a different way.
  - Different control engineers design controllers in a different way.

Summary

- Controller design based on root locus
  - Lead compensator **to improve transient response**.
  - Lag compensator **to improve steady state error**.
  - Lead-lag compensator **to improve both transient and steady state responses**.
- Next
  - Lead-lag compensator in Matlab and PID Controller design