ELEC 341: Systems and Control

Lecture 16

Frequency response

Course roadmap

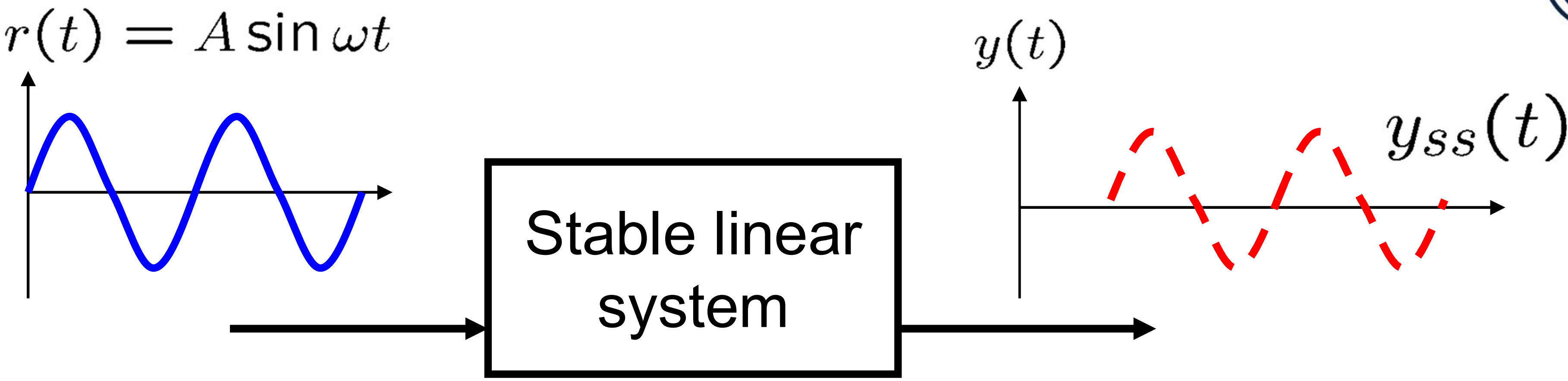
Modeling	*Analysis*	*Design*
✓ Laplace transform	✓ Stability	Design specs
✓ Transfer function	✓ • Routh-Hurwitz	✓ Root locus
Models for systems	• Nyquist	Frequency domain
✓ • Electrical	✓ Time response	✓ PID & Lead-lag
✓ • Electromechanical	✓ • Transient	Design examples
✓ • Mechanical	✓ • Steady state	
✓ Linearization, delay	► Frequency response	
	• Bode plot	

Matlab simulations

Summary up to now & *Topics from now on*

- **Modeling:** How to represent systems with transfer functions (s-domain).
- **Analysis:** How to extract time-response information from s-domain.
  - Steady-state error depends on TF evaluated at $s = 0$.
  - Stability and transient depends on pole locations.
  - ***Frequency responses contain all this information.***
- **Design:** How to obtain "**satisfactory**" closed-loop system.
  - Poles can be placed by the root-locus technique.
  - ***System's frequency responses can be shaped in Bode plot.***

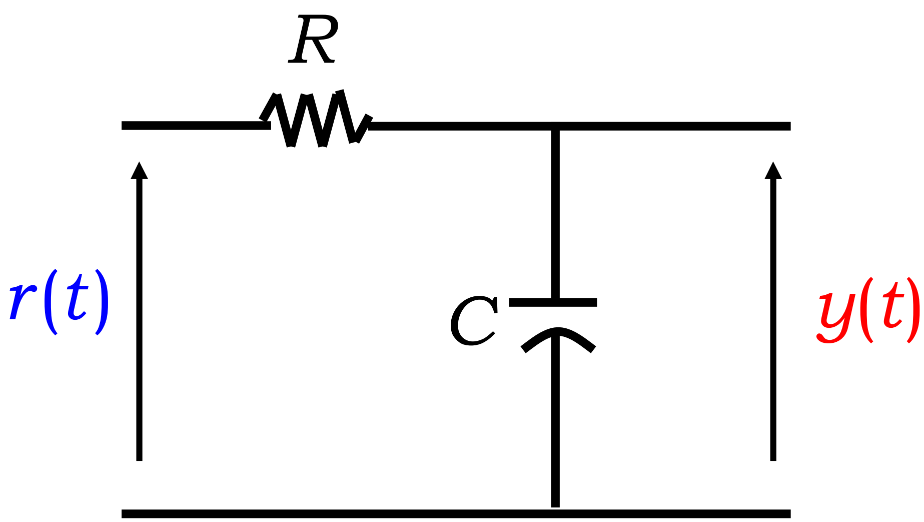
What is frequency response?



- We would like to analyze a system property by applying a *sinusoidal input* $r(t)$ and observing a response $y(t)$.
- Steady state response $y_{ss}(t)$ (after transient dies out) of a system to sinusoidal inputs is called ***frequency response***.

Example 1

- RC circuit



- Input a sinusoidal voltage $r(t)$
- What is the output voltage $y(t)$?

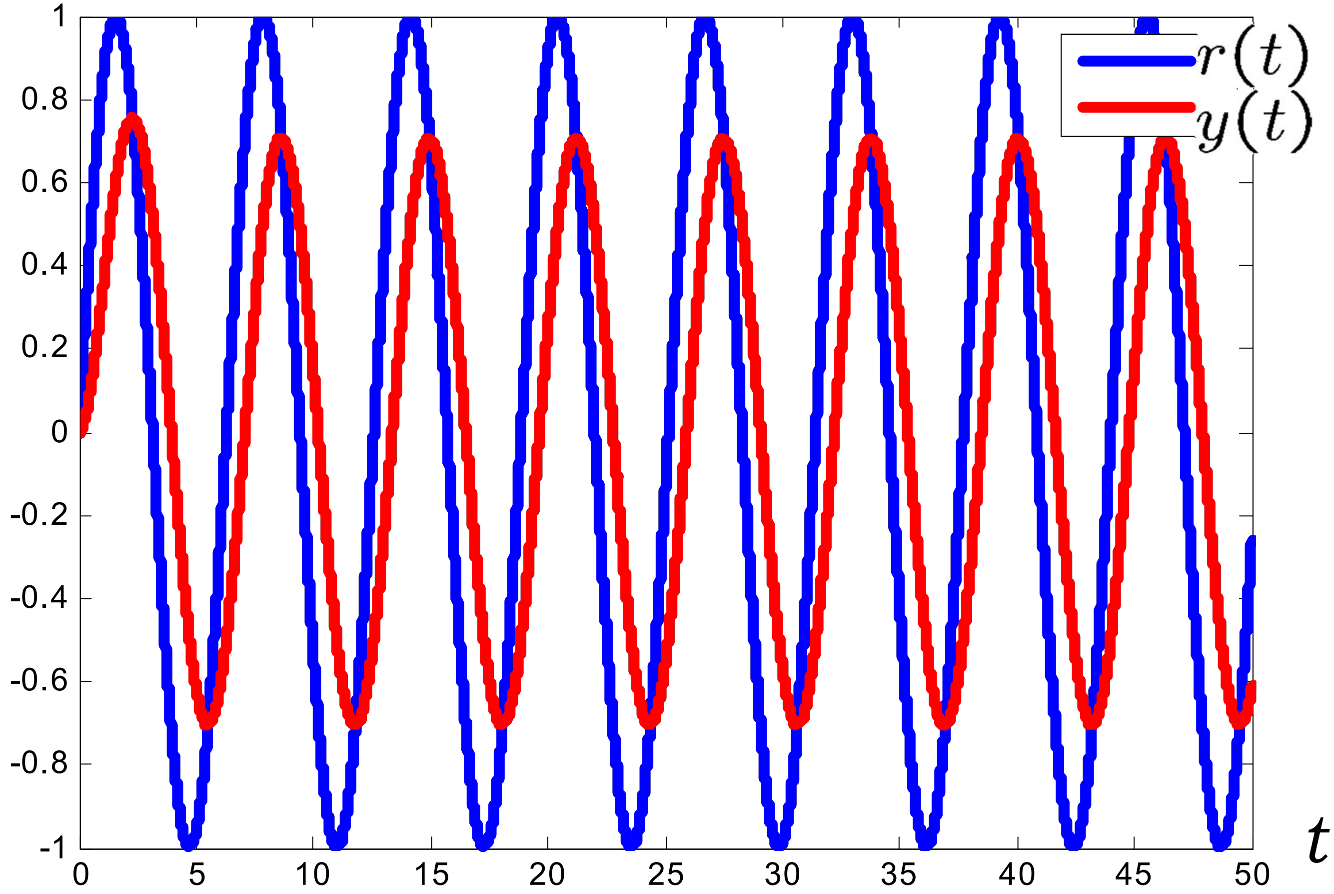
$$G(s) = \frac{Y(s)}{R(s)} \quad \Rightarrow \quad G(s) = \frac{1}{RCs+1}$$

Example 1 (cont'd)

- TF $(R = C = 1)$

$$G(s) = \frac{1}{s+1}$$

- $r(t) = \sin(t)$



At steady-state, $r(t)$ and $y(t)$ have the same frequency, but different amplitude and phase!

Example 1 (cont'd)

- Derivation of $y(t)$:

$$Y(s) = G(s)R(s) = \frac{1}{s+1} \cdot \frac{1}{s^2+1} = \frac{1}{2}\left(\frac{1}{s+1} + \frac{-s+1}{s^2+1}\right)$$

Partial fraction expansion

- Inverse Laplace:

$$y(t) = \frac{1}{2}\left(e^{-t} - \cos t + \sin t\right)$$

Approaches 0 as t goes to infinity.

$$\Rightarrow \quad y_{ss}(t) = \frac{1}{2}(-\cos t + \sin t) = \frac{1}{\sqrt{2}}\sin(t - 45^\circ)$$

$$\Rightarrow \quad \boxed{y_{ss}(t) = \frac{1}{\sqrt{2}}\sin(t - 45^\circ)}$$

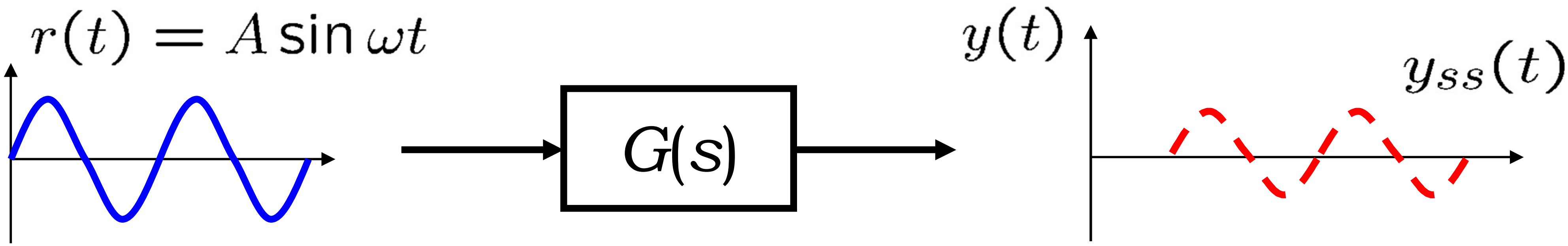
Trig identity:

$a.\sin(t) + b.\cos(t) = (\sqrt{a^2 + b^2}\ ).\sin(t + \theta)$

where $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Response to sinusoidal input

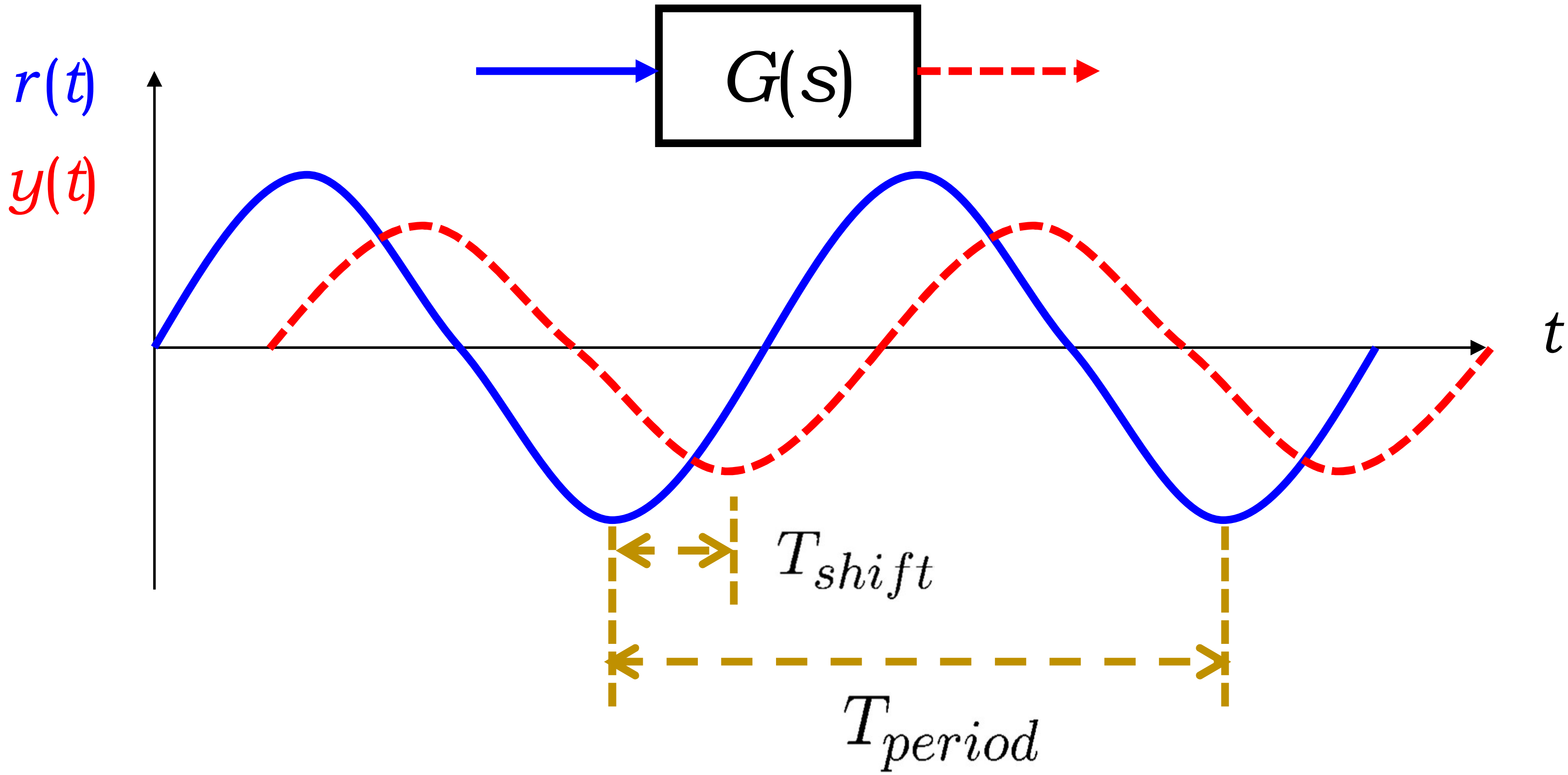
- What is the steady state output of a stable linear system when the input is sinusoidal?



- Steady state output is $y_{ss}(t) = A\,|G(j\omega)|\,\sin(\omega t + \angle G(j\omega))$
  - Frequency is the same as the input frequency ω
  - Amplitude is that of input (A) multiplied by $|G(j\omega)|$ (Gain)
  - Phase shift is $\angle G(j\omega)$

Phase Shift is also called **Phase Angle**.

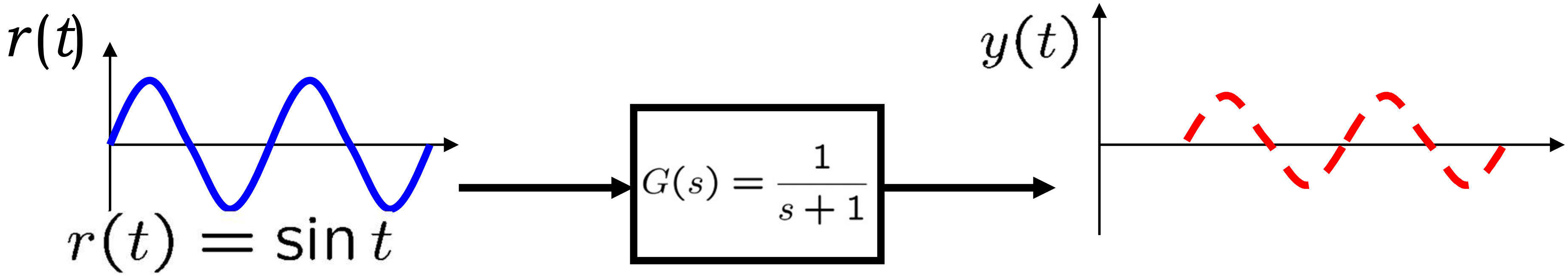
Phase shift



$$\frac{T_{shift}}{T_{period}} = \frac{-\angle G(j\omega)}{360°} \quad \Rightarrow \quad \boxed{\angle G(j\omega) = -\frac{T_{shift}}{T_{period}} \times 360°}$$

Example 2 (revisited)

- What is the steady state output of a stable linear system when the input is sinusoidal?



$r(t) = \sin t$

$A = 1 \; ; \omega = 1$

$y_{ss}(t) = A\,|G(j\omega)|\,\sin(\omega t + \angle G(j\omega))$

$$y_{ss}(t) = 1 \cdot \underbrace{\frac{1}{\sqrt{2}}}_{|G(j \cdot 1)|} \sin(t \underbrace{-45^\circ}_{\angle G(j \cdot 1)})$$

Gain:

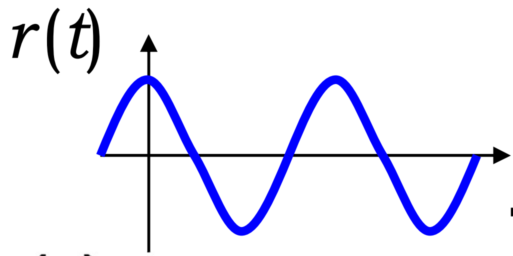
$$|G(j \cdot 1)| = \left|\frac{1}{j+1}\right| = \frac{1}{|j+1|} = \frac{1}{\sqrt{2}}$$

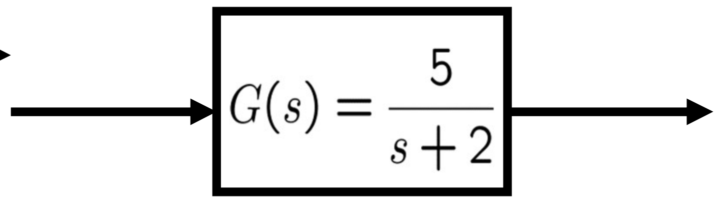
Phase:

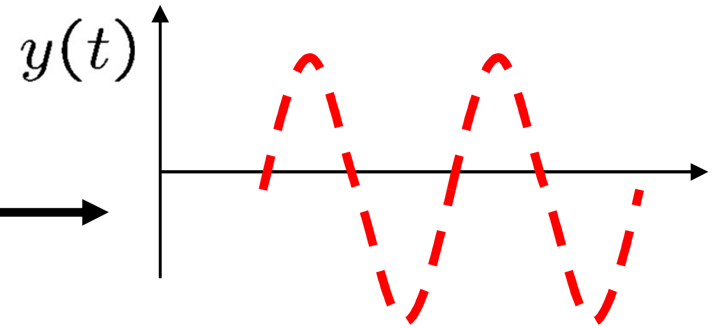
$$\angle G(j \cdot 1) = \angle \frac{1}{j+1} = \underbrace{\angle 1}_{=0^\circ} - \underbrace{\angle (j+1)}_{=45^\circ} = -45^\circ$$

Example 3

- What is the steady state output of a stable linear system when the input is sinusoidal?







$$r(t) = 7\cos 3t$$

$$A = 7 \quad ; \quad \omega = 3$$

$$y_{ss}(t) = 7 \cdot \underbrace{\frac{5}{\sqrt{13}}}_{|G(j\cdot 3)|} \cos(3t + \underbrace{\theta}_{\angle G(j\cdot 3)})$$

Gain

Phase

$$y_{ss}(t) = 9.7\cos(3t - 56.3^\circ)$$

Phase:

$$\angle G(j\cdot 3) = \angle \frac{5}{3j+2}$$

$$= \underbrace{\angle 5}_{= 0^\circ} - \underbrace{\angle (3j+2)}_{= \tan^{-1}\frac{3}{2}}$$

$$\theta = -56.3^\circ$$

Example 4: Frequency response function

- For a stable system $G(s)$, $G(jw)$ (w is positive) is called *frequency response function (**FRF**)*.
- For each w, FRF represents a complex number $G(jw)$, which has a gain and a phase.
- First order example:

$$G(s) = \frac{1}{s+1} \Rightarrow G(j\omega) = \frac{1}{j\omega + 1} = \textbf{FRF}$$

$$\Rightarrow \begin{cases} |G(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2}} \\ \angle G(j\omega) = \angle(1) - \angle(j\omega + 1) = -\tan^{-1}\omega \end{cases}$$

Example 4: First order FRF (cont'd)

- FRF $G(j\omega) = \dfrac{1}{j\omega + 1}$

frequency ω	gain $\lvert G(j\omega)\rvert$	phase $\angle G(j\omega)$
0	1	0°
0.5	0.894	-26.6°
1.0	0.707	-45°
$\vdots$	$\vdots$	$\vdots$
∞	0	-90°

- Two ways to represent a complex number (either gain-phase plot or real-imaginary parts plot), therefore, two different graphs can be used to represent FRF:
  - **Bode diagram (Bode plot)** (today's and next lecture)
  - **Nyquist diagram (Nyquist plot)** (in two lectures)

Example 5: Second order FRF

For the given $G(s)$, find the **FRF** magnitude and phase equations?

- Second order system (**Method 1**):

$$G(s) = \frac{2}{s^2 + 3s + 2}$$

$$\Rightarrow G(j\omega) = \frac{2}{(j\omega)^2 + 3(j\omega) + 2} = \frac{2}{2 - \omega^2 + j \cdot 3\omega}$$

$$\Rightarrow \begin{cases} |G(j\omega)| = \dfrac{2}{\sqrt{(2-\omega^2)^2 + 9\omega^2}} \\ \angle G(j\omega) = \angle(2) - \angle(2 - \omega^2 + j \cdot 3\omega) \\ \qquad\quad\;\; = -\tan^{-1} \dfrac{3\omega}{2 - \omega^2} \end{cases}$$

Example 5: Second order FRF

- Second order system (**Method 2**):

$$G(s) = \frac{2}{s^2+3s+2} \Rightarrow G(s) = \frac{2}{(s+1)(s+2)} \to G(j\omega) = \frac{2}{(j\omega+1)(j\omega+2)}$$

$$\Rightarrow |G(j\omega)| = \frac{2}{\sqrt{\omega^2+1}\sqrt{\omega^2+4}}$$

$$\angle G(j\omega) = \angle 2 - \angle(j\omega+1) - \angle(j\omega+2) = 0^\circ - \tan^{-1}\left(\frac{\omega}{1}\right) - \tan^{-1}\left(\frac{\omega}{2}\right) \Rightarrow$$

$$\angle G(j\omega) = -\tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{2}\right)$$

$$\Rightarrow \begin{cases} |G(j\omega)| = \dfrac{2}{\sqrt{\omega^2+1}\sqrt{\omega^2+4}} \\ \angle G(j\omega) = -\tan^{-1}(\omega) - \tan^{-1}\left(\dfrac{\omega}{2}\right) \end{cases}$$

General Method for Computing Phase Angle

- After converting $G(s)$ to $G(j\omega)$, plug in the given value of omega and then compute the angles based on which quadrant the vector is located at. That is, you first need to ***determine the quadrant in which the point lies***, and then use the appropriate equation to find the angle based on that.

1st quadrant: just use $+\tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

2nd quadrant: $+\left\{180-\tan^{-1}\left(\left|\frac{y}{x}\right|\right)\right\}$

3rd quadrant: $-\left\{180-\tan^{-1}\left(\left|\frac{y}{x}\right|\right)\right\}$

4th quadrant: just use $-\tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

$\omega = 1$

$$G(s)=\frac{(s+1)(s-2)(-s-3)}{(s+5)(s-6)(-s-7)(-s+8)}$$

$$\angle G(j\omega)=\tan^{-1}(1)+\tan^{-1}\left(\frac{1}{-2}\right)+\tan^{-1}\left(\frac{-1}{-3}\right)-\tan^{-1}\left(\frac{1}{5}\right)-\tan^{-1}\left(\frac{1}{-6}\right)-\tan^{-1}\left(\frac{-1}{-7}\right)-\tan^{-1}\left(\frac{-1}{8}\right)$$

$$\angle G(j\omega)=\tan^{-1}(1)+\left(180-\tan^{-1}\left(\tfrac{1}{2}\right)\right)-\left(180-\tan^{-1}\left(\tfrac{1}{3}\right)\right)-\tan^{-1}\left(\tfrac{1}{5}\right)-\left(180-\tan^{-1}\left(\tfrac{1}{6}\right)\right)+\left(180-\tan^{-1}\left(\tfrac{1}{7}\right)\right)+\tan^{-1}\left(\tfrac{1}{8}\right)$$

$$\angle G(j\omega)=34.02^\circ$$

Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- Models for systems
  - ✓ Electrical
  - ✓ Electromechanical
  - ✓ Mechanical
- ✓ Linearization, delay

Analysis

- ✓ Stability
  - ✓ Routh-Hurwitz
  - Nyquist
- ✓ Time response
  - ✓ Transient
  - ✓ Steady state
- ➤ Frequency response
  - ➤ Bode plot

Design

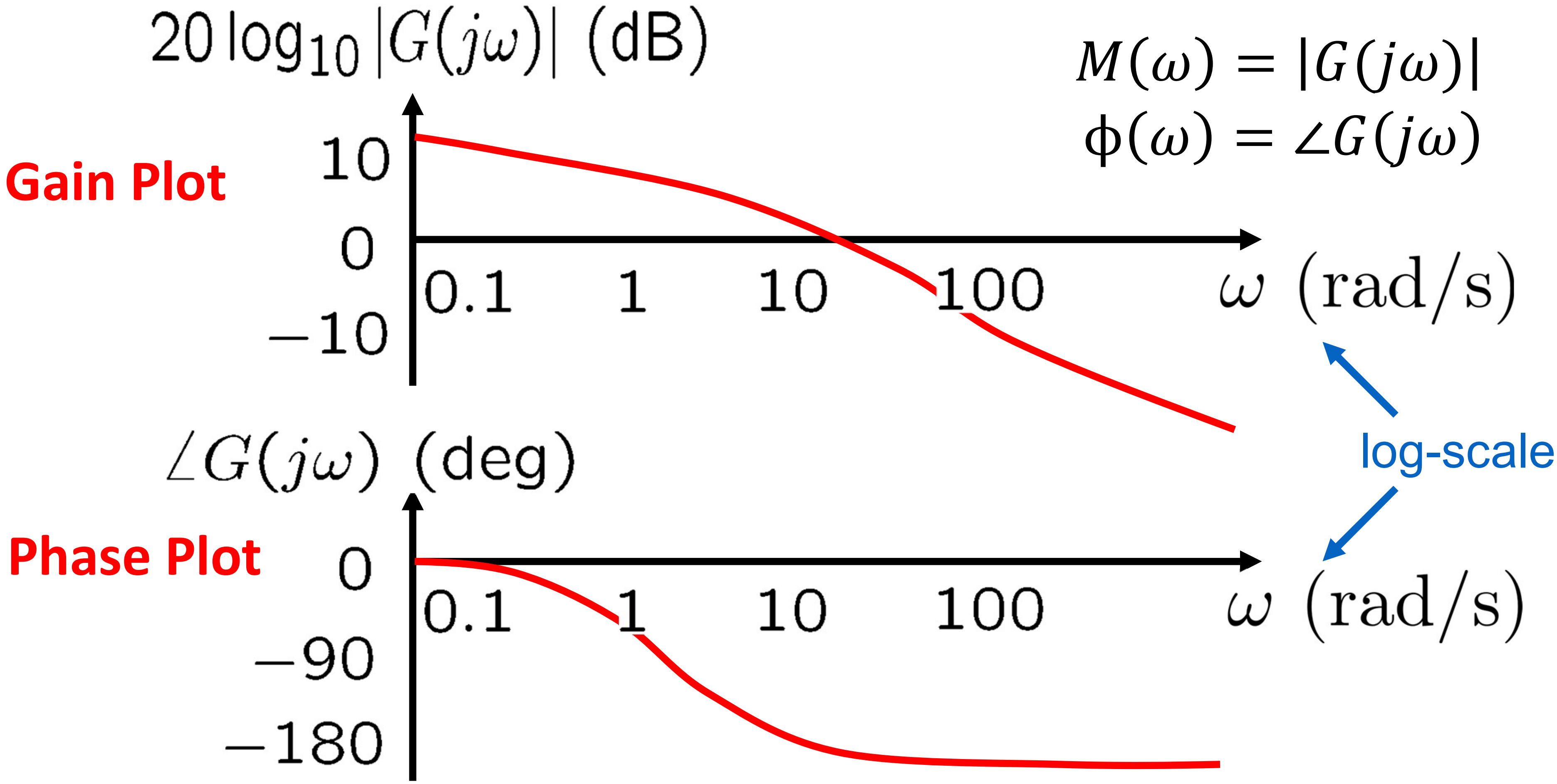
- Design specs
- ✓ Root locus
- Frequency domain
- ✓ PID & Lead-lag
- Design examples

Matlab simulations

Bode plot (Bode diagram) of $G(j\omega)$

- Bode diagram consists of gain plot & phase plot

$$M(\omega) = |G(j\omega)|$$
$$\phi(\omega) = \angle G(j\omega)$$



Bode plot of a 1st order system

- TF

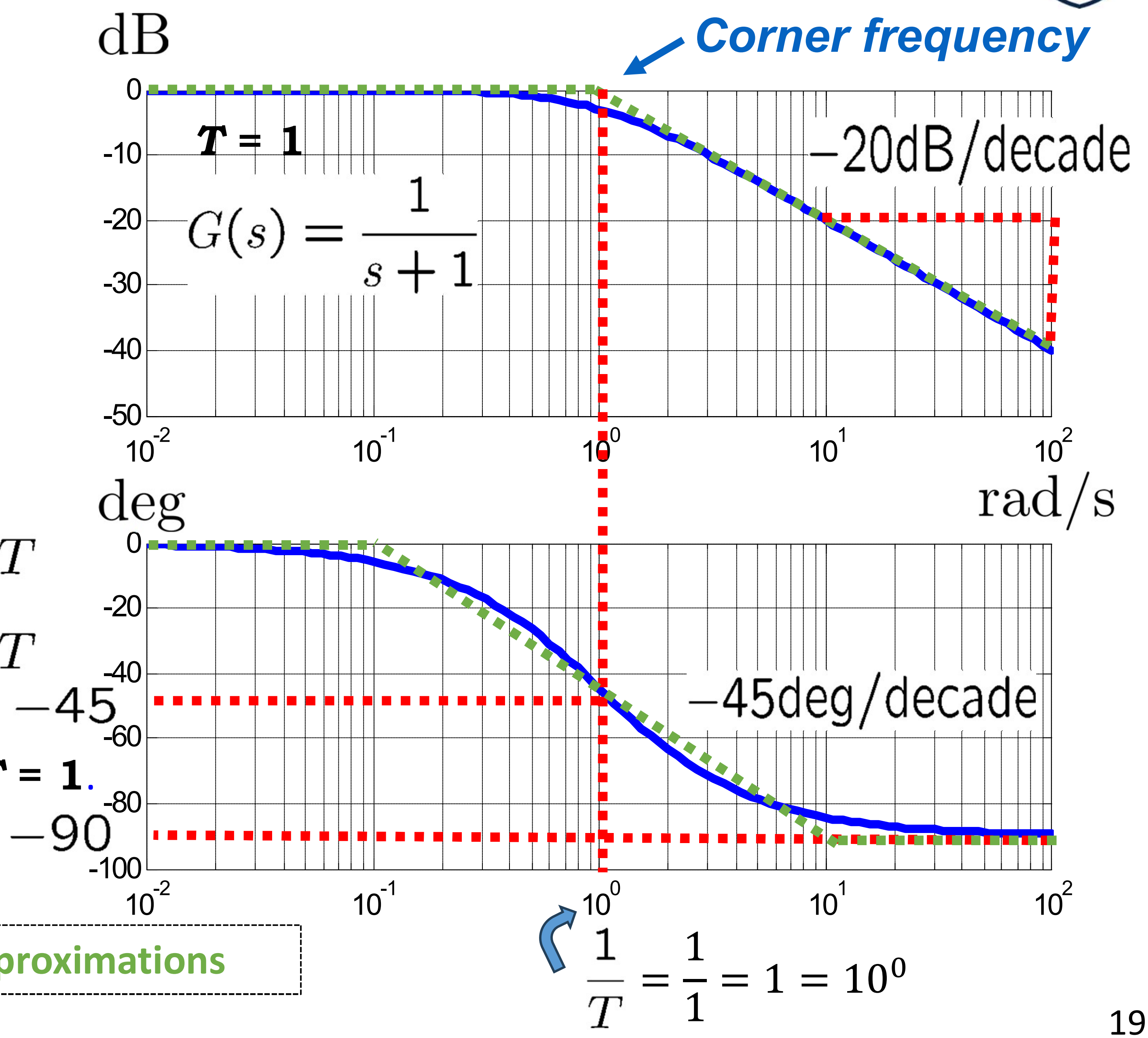
$$G(s) = \frac{1}{Ts+1}$$

FRF

$$G(j\omega) = \frac{1}{j\omega T+1}$$

$$\approx \begin{cases} 1 & \text{if } 1 \gg \omega T \\ \frac{1}{j\omega T} & \text{if } 1 \ll \omega T \end{cases}$$

The shown Bode plot is for $T = 1$.



"-----" are straight line approximations

Effect of T on Bode plot (first order system)

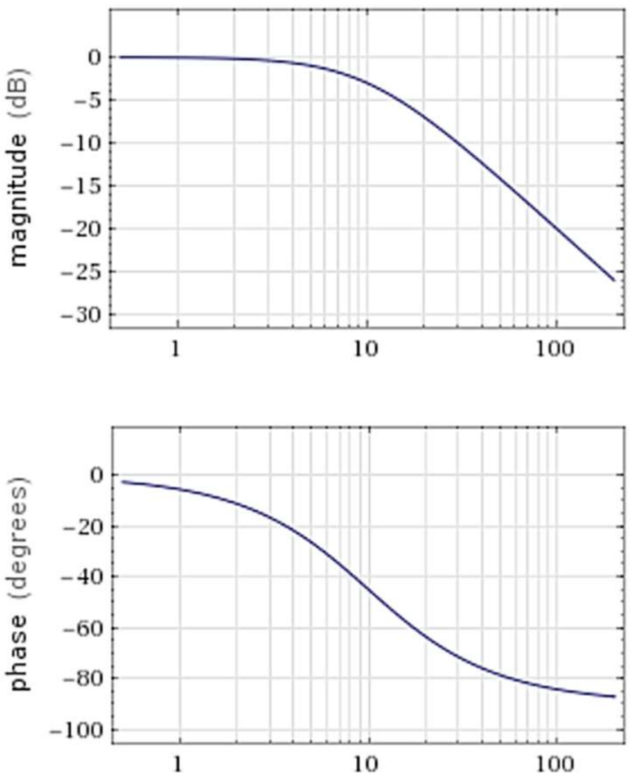
a place of mind

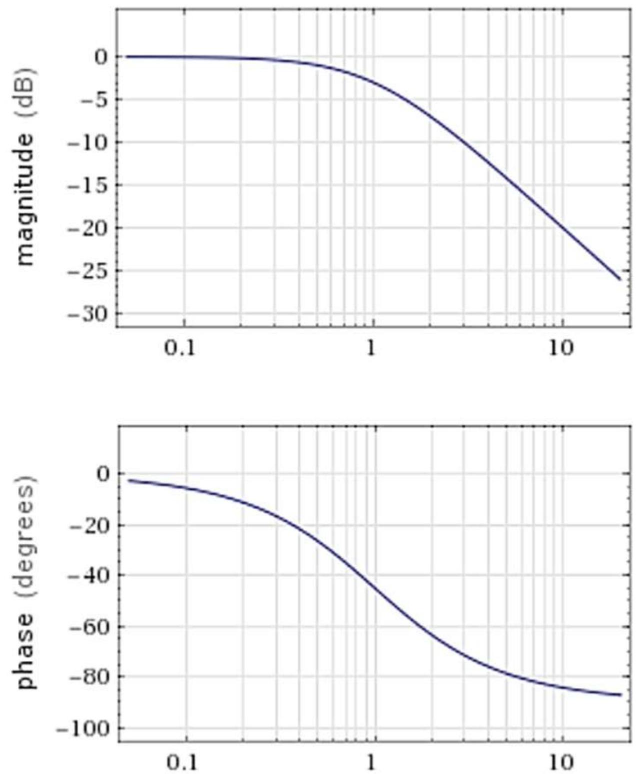
UBC

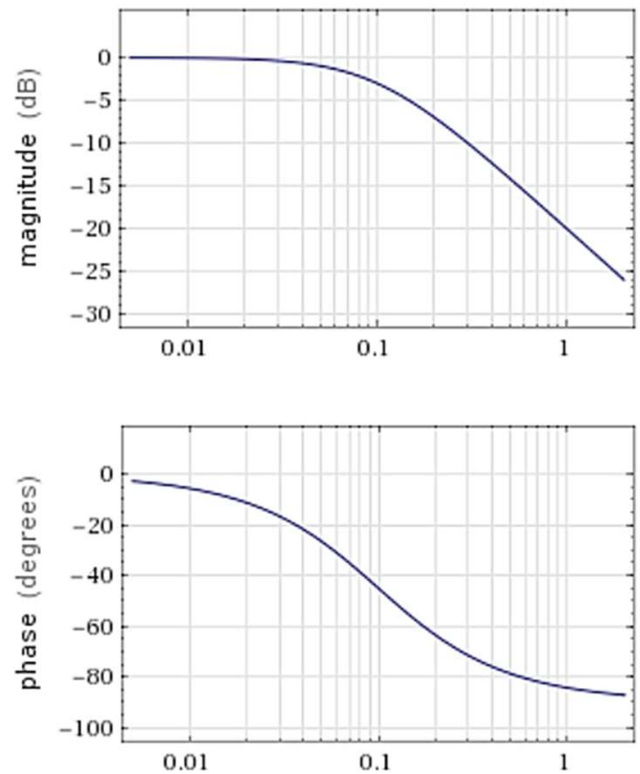
$$G(s) = \frac{1}{0.1s + 1}$$

$$G(s) = \frac{1}{s + 1}$$

$$G(s) = \frac{1}{10s + 1}$$







Remarks on Bode diagram

- A **Bode diagram** shows the **gain** and **phase shift** of a system's transfer function $G(j\omega)$ as a function of input frequency. It characterizes how the system modifies the amplitude and phase of sinusoidal inputs at different frequencies.
- Bode diagram is very useful and important in analysis and design of control systems.
- The shape of Bode plot contains information of CL stability, time responses, and much more!
- It can also be used for **system identification**.
  - Given FRF experimental data, we can obtain a transfer function that matches the data (see next slide).
- MATLAB command for bode plot is "bode(sys)".

System identification

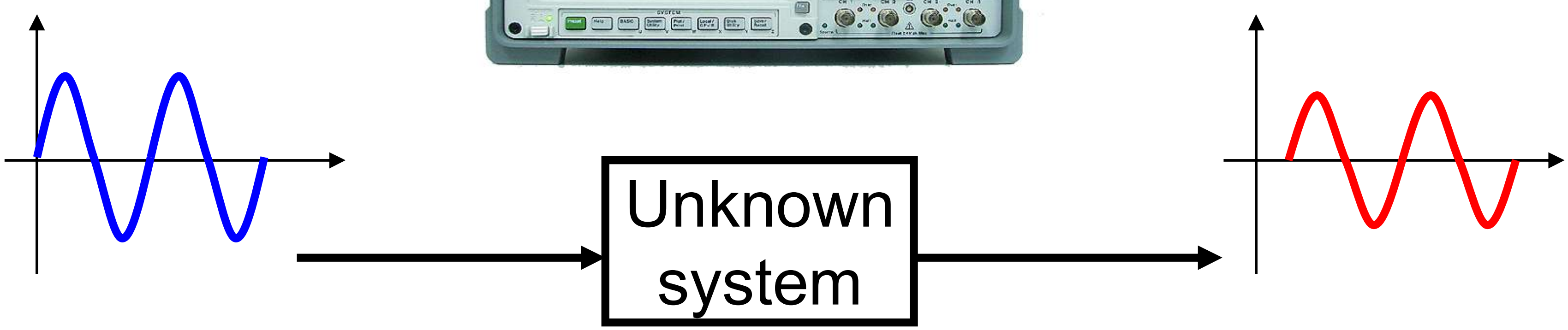
- Sweep frequencies of sinusoidal signals and obtain FRF data (i.e., gain and phase).
- Select $G(s)$ so that $G(jw)$ fits the FRF data.

1. Generate sine signals

2. Sweep frequencies

3. Collect FRF data

4. Select $G(s)$



Why deg(den) ≥ deg(num)?

- All the transfer functions we encountered so far have the property $\deg(\text{den}) \geq \deg(\text{num})$

Ex: $\dfrac{1}{Ms^2 + Bs + K}$; $\dfrac{K}{Ts+1}$; $K\dfrac{s+z}{s+p}$

- What if deg(num) is larger than deg(den)?
  - Then, $$|G(j\omega)| \to \infty \text{ as } \omega \to \infty$$
  - However, there is no such system in reality that has increasing gain as input frequency increases to infinity.
- That is why all the transfer function needs to meet

$$\deg(\text{den}) \geq \deg(\text{num})$$

Why deg(den) ≥ deg(num)?

Strictly proper transfer function:

- In control theory, a **strictly proper transfer function** is a transfer function where the degree of the numerator is less than the degree of the denominator, i.e., **deg(den) > deg(num)**.
  - For instance, $K\frac{s+1}{s^2+1}$

Proper transfer function:

- In control theory, a **proper transfer function** is a transfer function in which the degree of the numerator does not exceed the degree of the denominator, i.e., **deg(den) ≥ deg(num)**.
  - For instance, $K\frac{s+z}{s+p}$

Summary

- Frequency response
  - Steady state response to a sinusoidal input
  - For a linear stable system, a sinusoidal input generates a sinusoidal output with same frequency but different amplitude and phase.
- Bode plot is a graphical representation of frequency response function.
- Next
  - How to sketch Bode plots