# ELEC 341: Systems and Control

## Lecture 17

## Bode plot




See very important examples below (to be continued on the next slide):

### Difference between equation form for Bode & Nyquist plot:

**Example 1 (given as a final exam question):**

a) Find $|G(j\omega)|$ & $\angle G(j\omega)$ for Bode plot.
b) " " " " for Nyquist plot.

$$G(s) = \frac{4e^{-2s}}{s(s+6)(s^2+3s+4)}$$

a) **Method 1:** (Convert to 1st factor form)

$$s^2+3s+4 = \left(s-(-1.5+1.3228j)\right)\left(s-(-1.5-1.3228j)\right) = (s+1.5-1.3228j)(s+1.5+1.3228j) \rightarrow$$

$$G(j\omega) = \frac{4e^{-2j\omega}}{(j\omega)(j\omega+6)(1.5+(\omega-1.3228)j)(1.5+(\omega+1.3228)j)} \rightarrow$$

$$|G(j\omega)| = \frac{4}{\omega\sqrt{\omega^2+36}\sqrt{(\omega-1.3228)^2+2.25}\sqrt{(\omega+1.3228)^2+2.25}}$$

$$\angle G(j\omega) = \left\{0^\circ + \left(\frac{-2\omega\times 180^\circ}{\pi}\right)\right\} - \left\{\tan^{-1}\left(\frac{\omega}{0}\right) + \tan^{-1}\left(\frac{\omega}{6}\right) + \tan^{-1}\left(\frac{\omega-1.3228}{1.5}\right) + \ldots \right.$$
$$\left. \ldots + \tan^{-1}\left(\frac{\omega+1.3228}{1.5}\right)\right\} \rightarrow$$

$$\angle G(j\omega) = -114.59\omega - 90^\circ - \tan^{-1}\left(\frac{\omega}{6}\right) - \tan^{-1}\left(\frac{\omega-1.3228}{1.5}\right) - \tan^{-1}\left(\frac{\omega+1.3228}{1.5}\right)$$

In this method (for Bode plot), there is no need for any correction of angles provided that '$\omega$' is added to sth or substracted from sth. If '$\omega$' were alone, then we needed to correct for the quadrant.

| $\omega$ | $\angle G(j\omega)$ |
|---|---|
| 0.536 | $-179.94^\circ$ |
| 1.000 | $-259.05^\circ$ |
| 2.000 | $-427.61^\circ$ |
| 5.000 | $-847.21^\circ$ |

Both methods need angle correction.

page 1

**Method 2:**

$$G(s) = \frac{4e^{-2s}}{s(s+6)(s^2+3s+4)} = \frac{4e^{-2s}}{s^4+9s^3+22s^2+24s} \xrightarrow{s=j\omega}$$

$$\rightarrow G(j\omega) = \frac{4e^{-2j\omega}}{(j\omega)^4+9(j\omega)^3+22(j\omega)^2+24(j\omega)}$$

$$\angle G(j\omega) = \angle e^{-2j\omega} - \tan^{-1}\left(\frac{24\omega-9\omega^3}{\omega^4-22\omega^2}\right) = \frac{-2\omega\times180^\circ}{\pi} - \tan^{-1}(\ldots)$$

$$\rightarrow \angle G(j\omega) = \{-114.59\omega\}^\circ - \left\{\tan^{-1}\left(\frac{24\omega-9\omega^3}{\omega^4-22\omega^2}\right)\right\}^\circ = -114.59\omega - \tan^{-1}\left(\frac{N}{D}\right)$$

eqn (1) — corrected $\tan^{-1}\left(\frac{N}{D}\right)$

**Important Notes:**

For $\tan^{-1}(\ldots)$, we need to calculate numerator and denominator separately, and then correct for the quadrant.

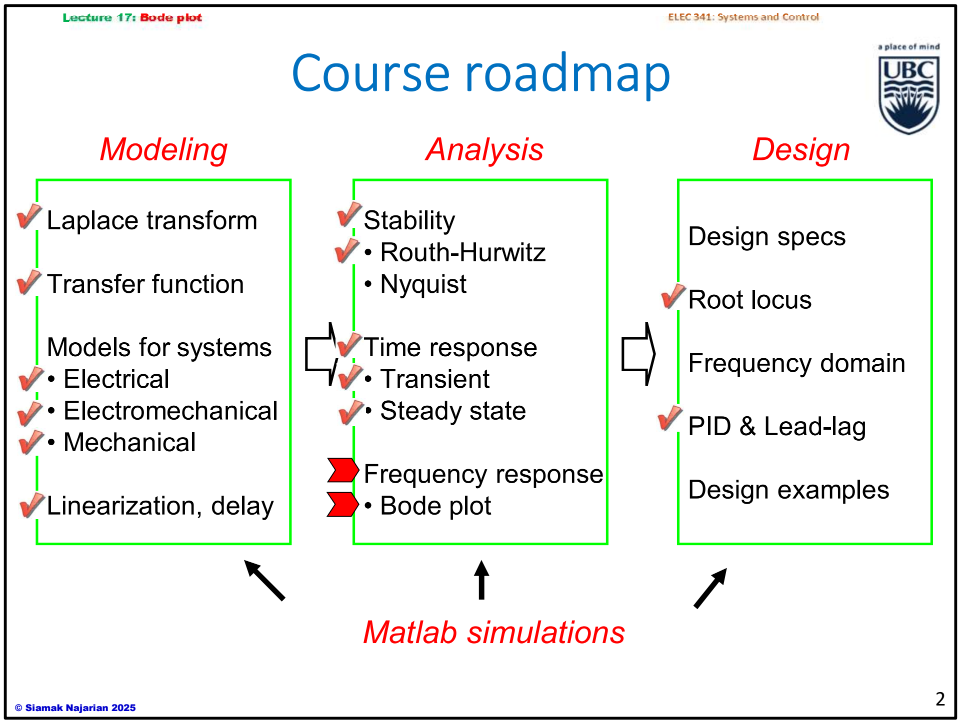
Example: Find $\angle G(j\omega)$ at $\omega = 0.536$ rad/s.

$$\angle G(j\times0.536) = -(114.59\times0.536) - \tan^{-1}\left(\frac{24\times0.536-9(0.536)^3}{(0.536)^4-22(0.536)^2}\right)$$
$$= -61.42 - \tan^{-1}\left(\frac{-11.478}{-6.237}\right)$$
$$= -61.42 - \{180^\circ - 61.48^\circ\} \rightarrow \angle G(j\times0.536) = -179.94^\circ$$

For $|G(j\omega)|$ use the following eqn:

$$|G(j\omega)| = \frac{4}{\sqrt{(\omega^4-22\omega^2)^2+(24\omega-9\omega^3)^2}}$$

page 2




Continuation from the previous slide:

Example ):

$$G(s) = \frac{6(s+2)}{(s-3)(s+4)} = \frac{6s+12}{s^2+s-12}, \quad s = j\omega \rightarrow$$

$$G(j\omega) = \frac{6(j\omega)+12}{(j\omega)^2+(j\omega)-12} \rightarrow G(j\omega) = \frac{6\omega j+12}{-\omega^2+\omega j-12} \rightarrow$$

$$\boxed{G(j\omega) = \frac{(6\omega)j+12}{(\omega)j+(-12-\omega^2)}}$$

$$\angle G(j\omega) = \angle\{(6\omega)j+12\} - \angle\{(\omega)j+(-12-\omega^2)\} \rightarrow$$

$$\boxed{\angle G(j\omega) = \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{-12-\omega^2}\right)}$$

b) Nyquist Plot: Convert into a+bj form.

$$e^{-2j\omega} = \cos(2\omega) - j\sin(2\omega)$$

$$G(j\omega) = Re + Im \cdot j; \quad Re = \frac{\omega[-88\cos(2\omega)+36\omega\sin(2\omega)+4\omega^2\cos(2\omega)] - 96\sin(2\omega)}{\omega(16+\omega^2+\omega^4)(36+\omega^2)}$$

$$Im = \frac{\omega[88\sin(2\omega)+36\omega\cos(2\omega)-4\omega^2\sin(2\omega)] - 96\cos(2\omega)}{\omega(16+\omega^2+\omega^4)(36+\omega^2)}$$

$$\angle G(j\omega) = \tan^{-1}\left(\frac{Im}{Re}\right), \quad |G(j\omega)| = \sqrt{Re^2 + Im^2}$$

| Point | $\omega$ | Re | Im | Uncorrected $\angle G(j\omega)$ | Corrected $\angle G(j\omega)$ |
|---|---|---|---|---|---|
| A | 0.536 | -0.3062 | -0.0003 | $\lvert 0.05^\circ \rvert$ | $+180.05^\circ$ or $-179.95^\circ$ |
| B | 1.000 | -0.0294 | 0.1522 | $\lvert 79.06^\circ \rvert$ | $+100.94^\circ$ or $-259.06^\circ$ |
| C | 2.000 | 0.0201 | -0.0487 | $\lvert 67.61^\circ \rvert$ | $-67.61^\circ$ or $+292.39^\circ$ |
| D | 5.000 | 0.0024 | -0.0032 | $\lvert 53.13^\circ \rvert$ | $+233.13^\circ$ or $-126.87^\circ$ |

Great! You're providing a clear quadrant-based rule set for computing phase angles manually — very helpful. Let's apply those rules to compute the **phase angle** of

$$G(s) = \frac{6(s+2)}{(s-3)(s+4)}$$

evaluated at $s = j\omega$, for three values of $\omega$: **1.01, 2.00, and 3.01.**

### Step-by-step strategy:

Let's define:

- $s = j\omega$
- So $G(j\omega) = \frac{6(j\omega+2)}{(j\omega-3)(j\omega+4)}$

We'll find the angle of each factor, apply your quadrant rules, and compute the net phase as:

$$\angle G(j\omega) = \angle(j\omega+2) - \angle(j\omega-3) - \angle(j\omega+4)$$

### Evaluate each at:

**1.** $\omega = 1.01$

Term 1: $j1.01 + 2 = (2, 1.01)$ → Quadrant I

$$\angle_1 = +\tan^{-1}(|1.01/2|) \approx \tan^{-1}(0.505) \approx 26.74^\circ$$

Term 2: $j1.01 - 3 = (-3, 1.01)$ → Quadrant II

$$\angle_2 = +[180 - \tan^{-1}(|1.01/3|)] \approx 180 - \tan^{-1}(0.337) \approx 180 - 18.60 = 161.40^\circ$$

Term 3: $j1.01 + 4 = (4, 1.01)$ → Quadrant I

$$\angle_3 = +\tan^{-1}(|1.01/4|) \approx \tan^{-1}(0.2525) \approx 14.13^\circ$$

Total angle:

$$\angle G(j1.01) = 26.74^\circ - 161.40^\circ - 14.13^\circ = \boxed{-148.79^\circ}$$

**2.** $\omega = 2.00$

Term 1: $j2 + 2 = (2, 2)$ → Quadrant I

$$\angle_1 = \tan^{-1}(1) = 45.00^\circ$$

Term 2: $j2 - 3 = (-3, 2)$ → Quadrant II

$$\angle_2 = 180 - \tan^{-1}(2/3) \approx 180 - 33.69 = 146.31^\circ$$

Term 3: $j2 + 4 = (4, 2)$ → Quadrant I

$$\angle_3 = \tan^{-1}(0.5) \approx 26.57^\circ$$

Total angle:

$$\angle G(j2.00) = 45.00^\circ - 146.31^\circ - 26.57^\circ = \boxed{-127.88^\circ}$$

**3.** $\omega = 3.01$

Term 1: $j3.01 + 2 = (2, 3.01)$ → Quadrant I

$$\angle_1 = \tan^{-1}(3.01/2) \approx \tan^{-1}(1.505) \approx 56.34^\circ$$

Term 2: $j3.01 - 3 = (-3, 3.01)$ → Quadrant II

$$\angle_2 = 180 - \tan^{-1}(3.01/3) \approx 180 - \tan^{-1}(1.003) \approx 180 - 45.04 = 134.96^\circ$$

Term 3: $j3.01 + 4 = (4, 3.01)$ → Quadrant I

$$\angle_3 = \tan^{-1}(3.01/4) \approx \tan^{-1}(0.7525) \approx 36.97^\circ$$

Total angle:

$$\angle G(j3.01) = 56.34^\circ - 134.96^\circ - 36.97^\circ = \boxed{-115.59^\circ}$$

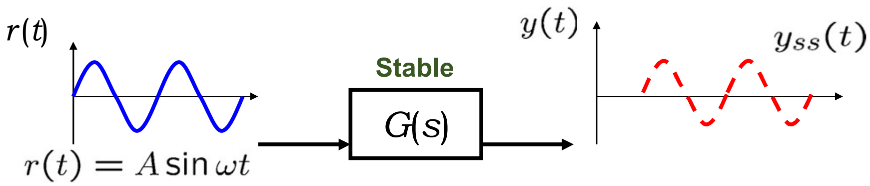
### Final Answers:

| $\omega$ | $\angle G(j\omega)$ |
|---|---|
| 1.01 | $\boxed{-148.79^\circ}$ |
| 2.00 | $\boxed{-127.88^\circ}$ |
| 3.01 | $\boxed{-115.59^\circ}$ |

# Frequency response (review)

- Steady state output $y_{ss}(t) = A\,|G(j\omega)|\,\sin(\omega t + \angle G(j\omega))$
  - Frequency is same as the input frequency $\omega$
  - Amplitude is that of input ($A$) multiplied by $|G(j\omega)|$
  - Phase shift is $\angle G(j\omega)$

Gain




- Frequency response function (FRF): $G(j\omega)$
- Bode plot: Graphical representation of $G(j\omega)$




**amplification** [amp-li-fi-**kay**-shen]

Great — let's work this out by plugging $s = j\omega$ directly into:

$$G(s) = \frac{6s + 12}{s^2 + s - 12}$$

We will evaluate the **phase angle of** $G(j\omega)$ for three values: $\omega = 1.01$, $2.00$, and $3.01$, using:

$$\angle G(j\omega) = \arg(6j\omega + 12) - \arg((j\omega)^2 + j\omega - 12)$$

✅ **Step-by-step Calculations**

🔘 **For** $\omega = 1.01$

**Numerator:**

$$6j(1.01) + 12 = 12 + j6.06 \Rightarrow \theta_1 = \arg(12 + j6.06) = \tan^{-1}(6.06/12) \approx \boxed{26.39^\circ}$$

**Denominator:**

$$(j1.01)^2 + j1.01 - 12 = -1.0201 + j1.01 - 12 = -13.0201 + j1.01 \Rightarrow \theta_2 = \arg(-13.0201 + j1.01)$$

This is in **Quadrant II**, so:

$$\theta_2 = \tan^{-1}(1.01/-13.0201) \approx 175.56^\circ$$

**Total Phase:**

$$\angle G(j1.01) = 26.39^\circ - 175.56^\circ = \boxed{-149.17^\circ}$$

🔘 **For** $\omega = 2.00$

**Numerator:**

$$6j(2.00) + 12 = 12 + j12 \Rightarrow \theta_1 = \arg(12 + j12) = \tan^{-1}(12/12) = \boxed{45.00^\circ}$$

**Denominator:**

$$(j2)^2 + j2 - 12 = -4 + j2 - 12 = -16 + j2 \Rightarrow \theta_2 = \arg(-16 + j2) \approx \tan^{-1}(2/-16) = 172.87^\circ$$

**Total Phase:**

$$\angle G(j2.00) = 45.00^\circ - 172.87^\circ = \boxed{-127.87^\circ}$$

🔘 **For** $\omega = 3.01$

**Numerator:**

$$6j(3.01) + 12 = 12 + j18.06 \Rightarrow \theta_1 = \tan^{-1}(18.06/12) \approx \boxed{56.20^\circ}$$

**Denominator:**

$$(j3.01)^2 + j3.01 - 12 = -9.0601 + j3.01 - 12 = -21.0601 + j3.01 \Rightarrow \theta_2 = \arg(-21.0601 + j3.01) \approx \tan^{-1}(3.01/-21.0601) = 171.85^\circ$$

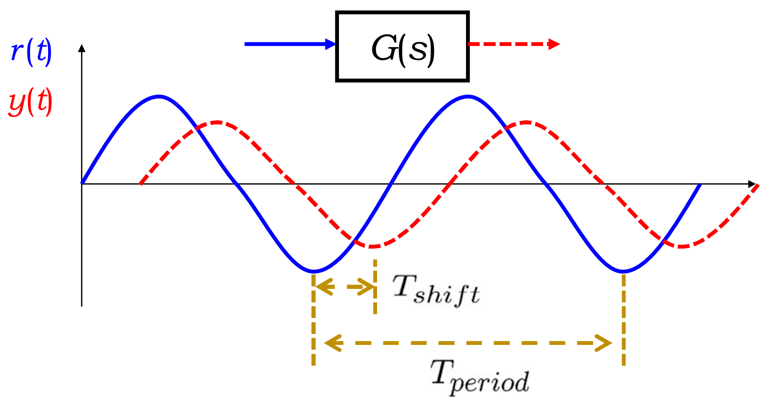
**Total Phase:**

$$\angle G(j3.01) = 56.20^\circ - 171.85^\circ = \boxed{-115.65^\circ}$$

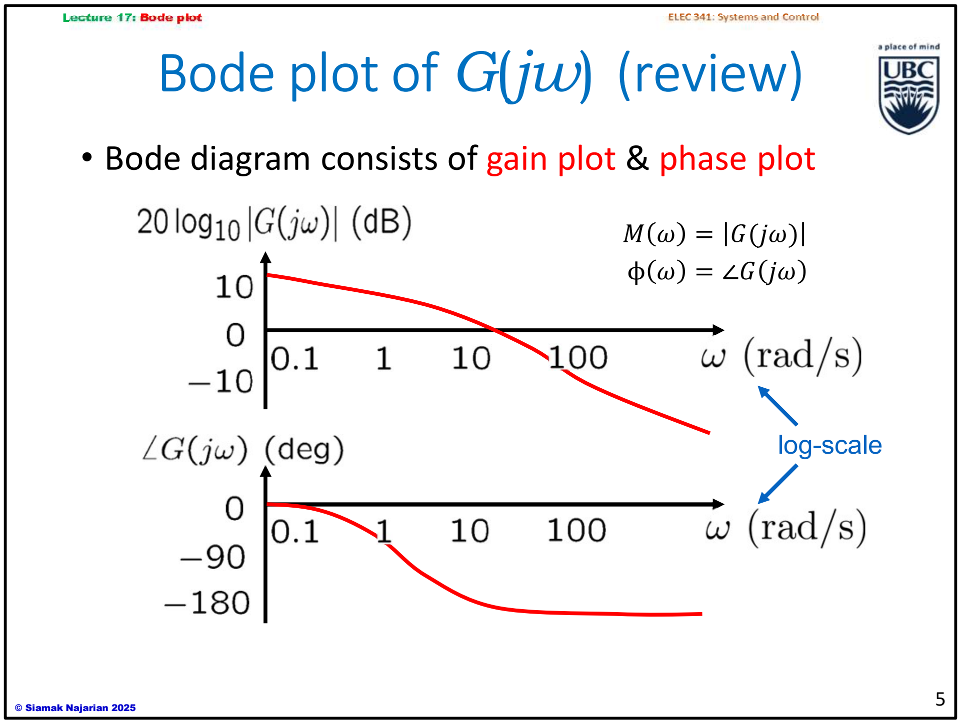
✅ **Final Answer Table**

| $\omega$ | $\angle G(j\omega)$ |
|---|---|
| 1.01 | $\boxed{-149.17^\circ}$ |
| 2.00 | $\boxed{-127.87^\circ}$ |
| 3.01 | $\boxed{-115.65^\circ}$ |

# Phase shift (review)




$$\frac{T_{shift}}{T_{period}} = \frac{-\angle G(j\omega)}{360°} \quad \Rightarrow \quad \angle G(j\omega) = -\frac{T_{shift}}{T_{period}} \times 360°$$




See below for how to extract data point from Bode Plot:

**[mag,phase,w]=bode(sys,w1);out2=[w1',20*log10(squeeze(mag)),squeeze(phase)]**

```
>> sys = tf (1,[1,1])

sys =

    1
  -----
  s + 1

Continuous-time transfer function.

>> wl=[0.01,0.1,1,2,3,5,10,100]

wl =

    0.0100     0.1000     1.0000     2.0000     3.0000     5.0000    10.0000   100.0000

>> [mag,phase,w]=bode(sys,wl);out2=[wl',20*log10(squeeze(mag)),squeeze(phase)]

out2 =

    0.0100   -0.0004   -0.5729
    0.1000   -0.0432   -5.7106
    1.0000   -3.0103  -45.0000
    2.0000   -6.9897  -63.4349
    3.0000  -10.0000  -71.5651
    5.0000  -14.1497  -78.6901
   10.0000  -20.0432  -84.2894
  100.0000  -40.0004  -89.4271
```

**Explain this MATLAB code, its commands, and what it does.[mag,phase,w]=bode(sys,w1);out2=[w1',20*log10(squeeze(mag)),squeeze(phase)]**

Great — let's break down and explain this MATLAB code line by line. The code works with the **Bode plot data** of a given system sys and processes it into a readable output format.

🔷 **Line 1:**

matlab

```
[mag, phase, w] = bode(sys, w1);
```

✅ **What it does:**

•This computes the **magnitude** and **phase** response of a linear system sys at specific frequencies w1.

🔍 **Details:**

•sys is a **LTI system object** — e.g., a transfer function (tf), state-space (ss), or zero-pole-gain (zpk) model.

•w1 is a **vector of frequency values (in rad/sec)** at which the Bode response is to be evaluated.

•bode(sys, w1) returns:

- mag: magnitude of the frequency response $|G(j\omega)|$
- phase: phase angle $\angle G(j\omega)$, in **degrees**
- w: the frequency vector (same as w1)

📌 Both mag and phase are returned as 3D arrays of size [1 × 1 × N] if sys is a single-input single-output (SISO) system, where N is the number of frequency points in w1.

🔷 **Line 2:**

matlab

```
out2 = [w1', 20*log10(squeeze(mag)), squeeze(phase)];
```

✅ **What it does:**

This line processes the raw Bode data into a matrix that combines:

1.Frequency values

2.Magnitude in decibels (dB)

3.Phase in degrees

🔍 **Breakdown:**

•w1': Transposes the frequency vector to make it a column vector.

•squeeze(mag): Removes singleton dimensions from mag (i.e., turns a 1×1×N array into a N×1 vector).

•20*log10(...): Converts the magnitude to **decibels** (standard for Bode magnitude plots).

•squeeze(phase): Also flattens the phase data to a column vector.

📌 out2 becomes an N × 3 matrix, where each row contains:

$[\omega_i, 20\log_{10}|G(j\omega_i)|, \angle G(j\omega_i)]$

✅ **Summary of What the Code Does:**

This MATLAB code:

•Evaluates the **frequency response** of an LTI system at custom frequencies w1

•Converts the results to **magnitude (in dB)** and **phase (in degrees)**

•Combines the frequency, magnitude, and phase into a **single output matrix out2**

This output can then be easily displayed, saved, or plotted using custom commands (instead of using the built-in Bode plot).

# Sketching Bode plot

- **Basic functions:**
  - Constant gain
  - Differentiator and integrator
  - Double integrator
  - First order system and its inverse
  - Second order system
  - Time delay
- **Product of basic functions:**
  1. Sketch Bode plot of each factor, and
  2. Add the Bode plots graphically.




See below for a very important note on calculating phase plot:

In Matlab: **ATAN2(y,x)**

**1st quadrant:** just use $+\tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

**2nd quadrant:** $+\left\{180-\tan^{-1}\left(\left|\frac{y}{x}\right|\right)\right\}$

**3rd quadrant:** $-\left\{180-\tan^{-1}\left(\left|\frac{y}{x}\right|\right)\right\}$

**4th quadrant:** just use $-\tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

Example:

$G(s)=\frac{6(s+2)}{(s-3)(s+4)} \xrightarrow{s=j\omega} G(j\omega)=\frac{6(j\omega+2)}{(j\omega-3)(j\omega+4)}$

$\rightarrow \angle G(j\omega)=\angle(j\omega+2)-\angle(j\omega-3)-\angle(j\omega+4)$

$=\tan^{-1}\left(\frac{\omega}{2}\right)-\left\{180^\circ-\tan^{-1}\left(\frac{\omega}{3}\right)\right\}-\tan^{-1}\left(\frac{\omega}{4}\right)$

$\rightarrow \angle G(j\omega)=\tan^{-1}\left(\frac{\omega}{2}\right)-180^\circ+\tan^{-1}\left(\frac{\omega}{3}\right)-\tan^{-1}\left(\frac{\omega}{4}\right)$

e.g.: $\omega=3 \rightarrow \angle G(j3)=56.3^\circ-135^\circ-36.86^\circ$

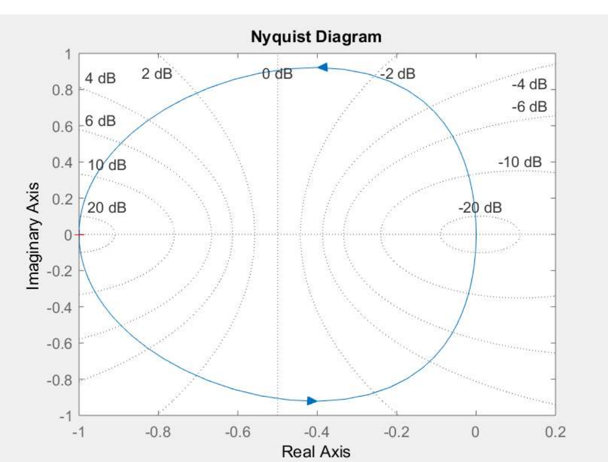
$\angle G(j3)=-115.56^\circ$ I verified this using Matlab

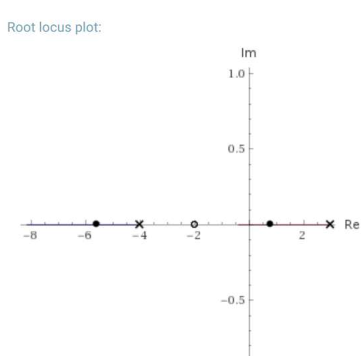
**Matlab Output:**

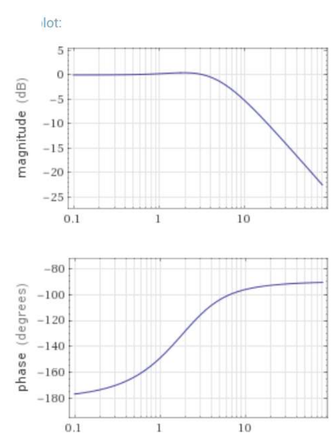
```
>> (atan2(3,2)-atan2(3,-3)-atan2(3,4))*180/pi

ans =

 -115.5600
```



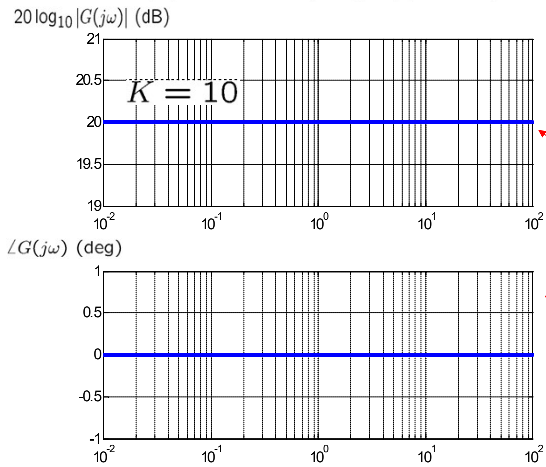



# Bode plot of a constant gain

$$G(s) = K \Rightarrow |G(j\omega)| = K,\ \angle G(j\omega) = 0^\circ,\ \forall\omega$$

(for all)



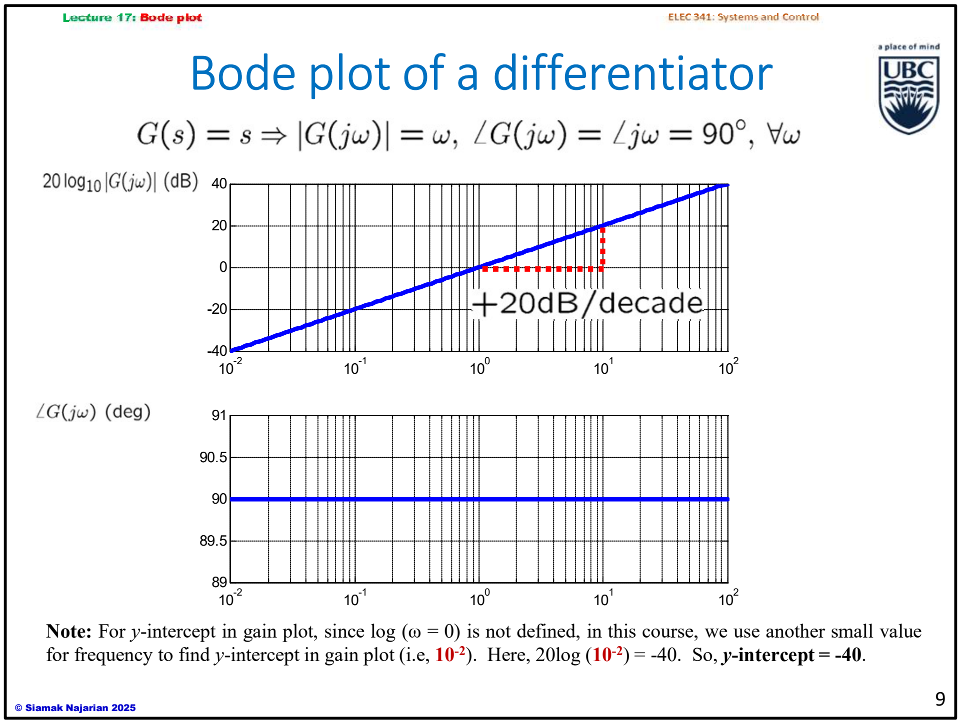

| $K$ | $20\log_{10}K$ |
|---|---|
| 100 | 40 dB |
| 10 | 20 dB |
| 2 | ≈ 6 dB |
| 1 | 0 dB |
| 0.1 | −20 dB |
| 0.01 | −40 dB |




Intersection of line of K = 10 and the gain plot is 20 log 10 = 20×1 = 20.

# Sketching Bode plot

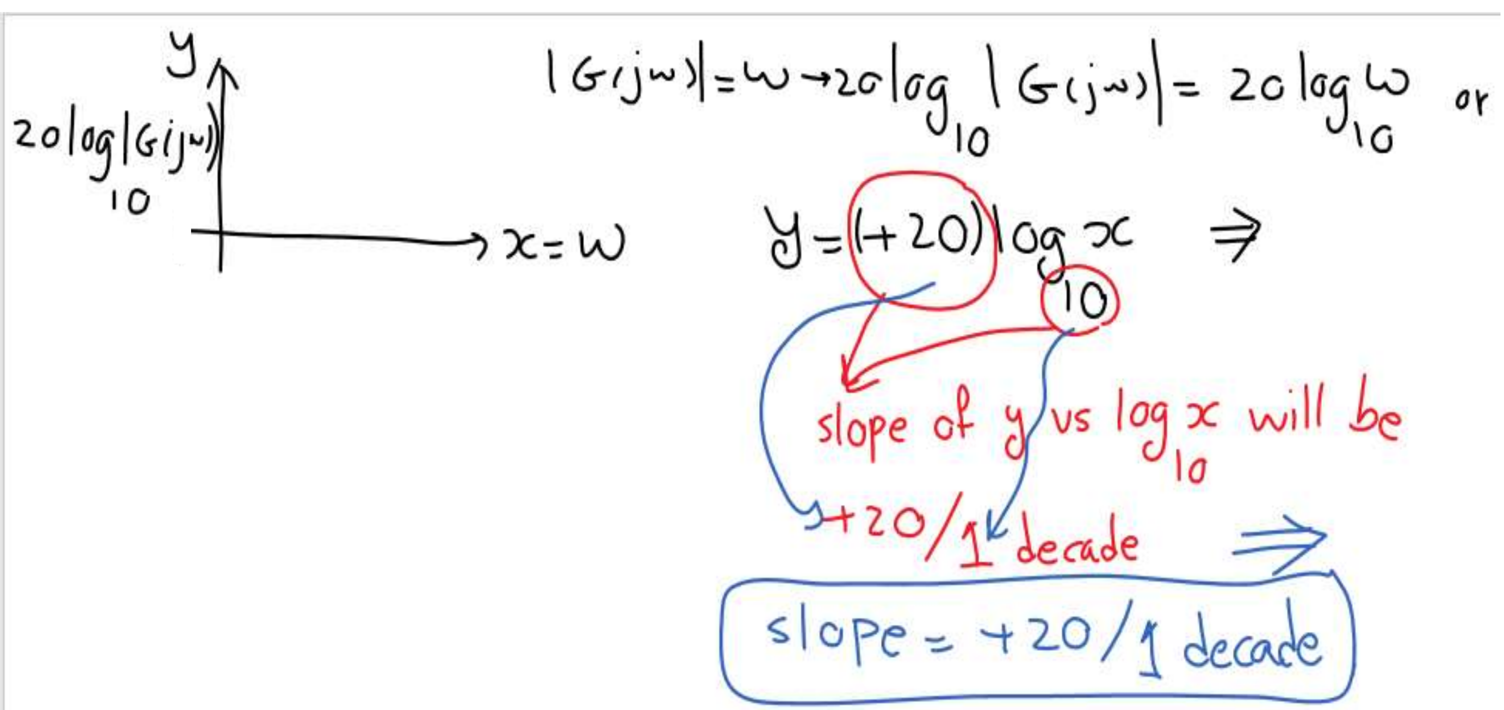
- **Basic functions:**
  - Constant gain
  - Differentiator and integrator
  - Double integrator
  - First order system and its inverse
  - Second order system
  - Time delay
- **Product of basic functions:**
  1. Sketch Bode plot of each factor, and
  2. Add the Bode plots graphically.

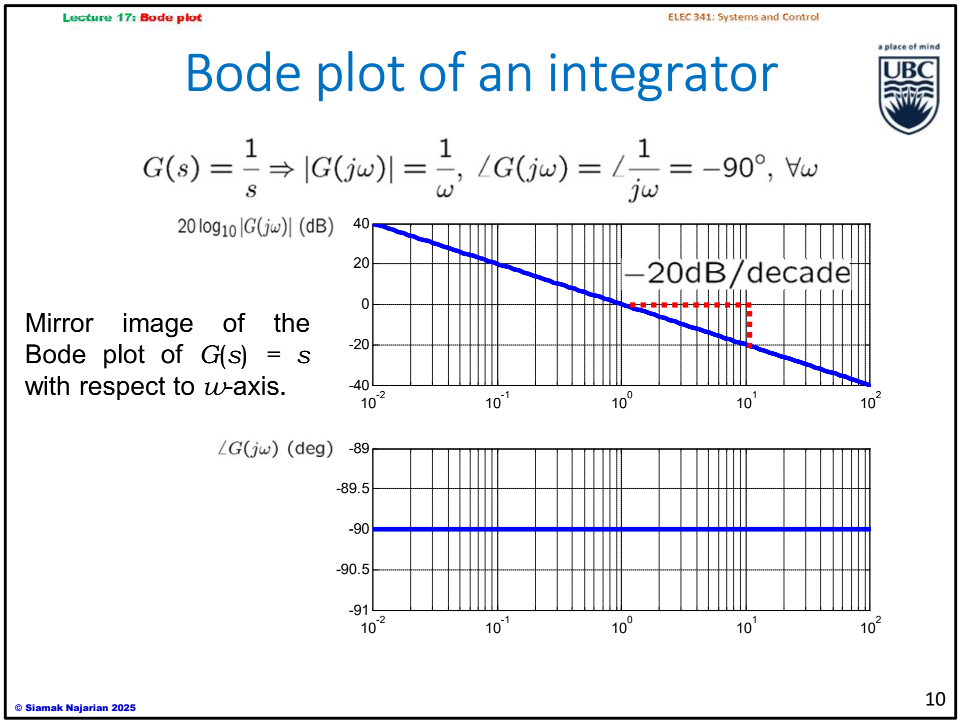



"decade" here means "per 10 rad/s".

**Important note about y-intercept in gain plot:**
Since log (ω = 0) is not defined, we use another small value for frequency, such as, $10^{-2}$ to find y-intercept in gain plot. Here, 20log (0.01) = -40. So, **y-intercept = -40**.

See below:






"Mirror image of the Bode plot of *G*(*s*) = *s* with respect to $\omega$-axis." This actually means that you put the mirror at right angle with respect to the gain of 0 (i.e., perpendicular to the plane of the page).

See below:

$$|G(j\omega)| = \frac{1}{\omega} \longrightarrow 20\log|G(j\omega)| = 20\log\frac{1}{\omega} =$$

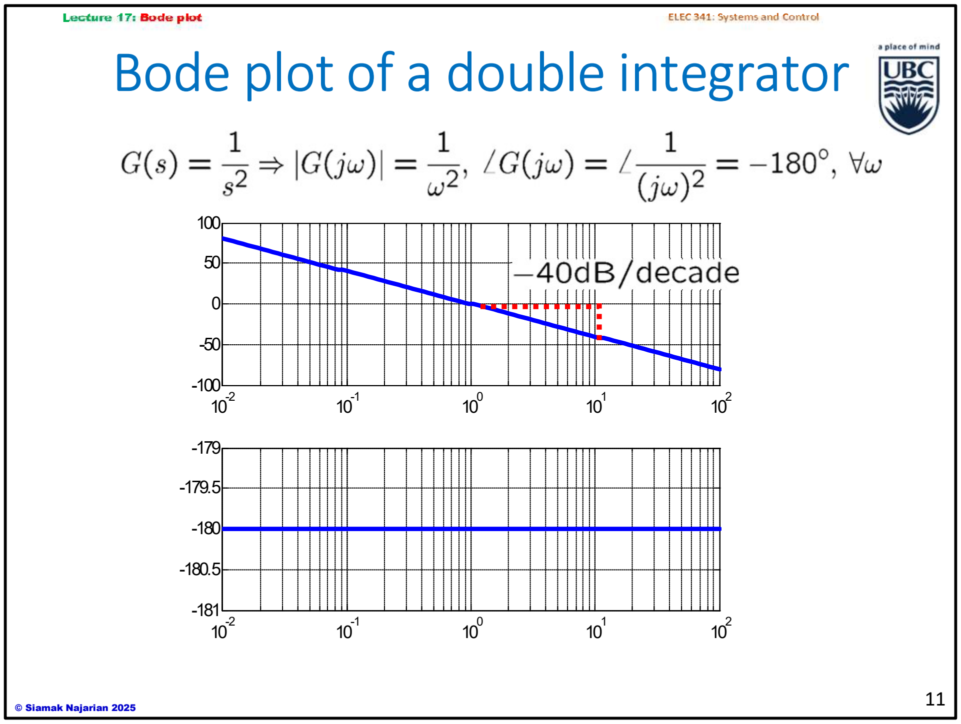
$$= 20(\log 1 - \log\omega)$$

$$= 20(0 - \log\omega)$$

$$= -20\log_{10}\omega \Rightarrow$$

$$20\log|G(j\omega)| = -20\log_{10}\omega$$

−20 / 1 decade




See below:

$$G(s) = \frac{1}{s^2} \xrightarrow{s=j\omega} G(s) = \frac{1}{s \cdot s} = \frac{1}{(j\omega)(j\omega)} \longrightarrow$$

$$\angle G(j\omega) = \angle 1 - \angle(j\omega) - \angle(j\omega)$$

$$= 0^\circ - 90^\circ - 90^\circ \longrightarrow \boxed{\angle G(j\omega) = -180^\circ}$$

$$|G(j\omega)| = \left| \frac{1}{(j\omega)^2} \right| = \frac{1}{\omega^2} \rightarrow 20\log_{10}|G(j\omega)| = 20\log_{10}\frac{1}{\omega^2}$$

$$= -20\log_{10}\omega^2 = -40\log_{10}\omega \rightarrow$$

$$\rightarrow 20\log_{10}|G(j\omega)| = (-40)\log_{10}\omega = y$$

slope = −40 / decade

$$@\ \omega = 10^{-2} \rightarrow y = -40\log_{10}10^{-2}$$

$$\hookrightarrow \boxed{y = +80}$$

# Sketching Bode plot

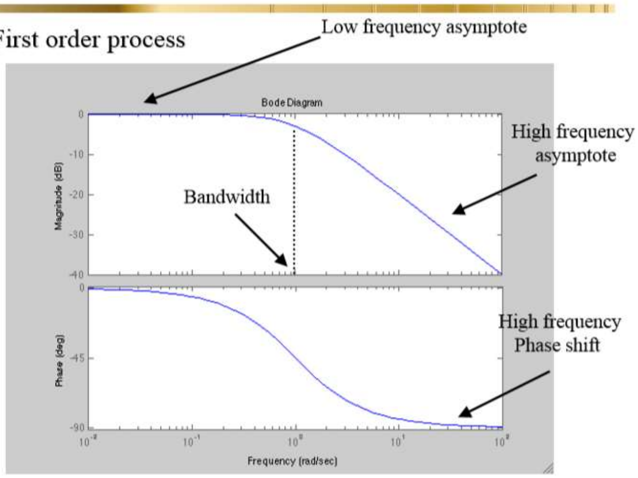
- **Basic functions:**
  - Constant gain
  - Differentiator and integrator
  - Double integrator
  - First order system and its inverse
  - Second order system
  - Time delay
- **Product of basic functions:**
  1. Sketch Bode plot of each factor, and
  2. Add the Bode plots graphically.




**Excellent link for plotting Bode diagram (especially, the straight-line approximation): See below:**
https://www.bing.com/videos/search?q=how+to+graph+bode+asymptotes&&view=detail&mid=B28D905B293A2D860AB7B28D905B293A2D860AB7&&FORM=VDRVRV

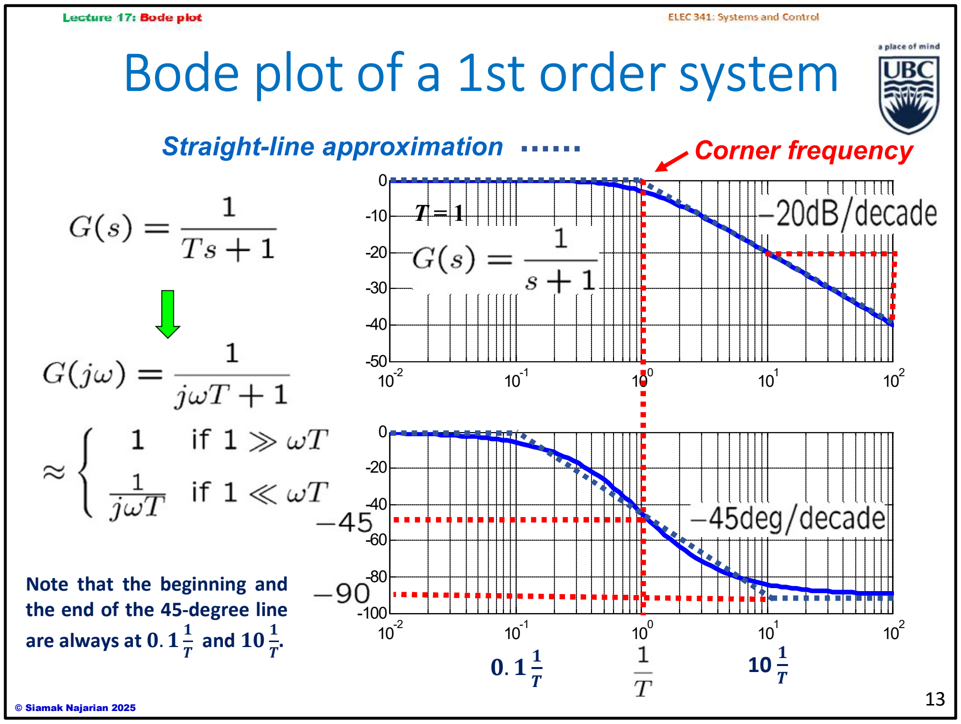
## Bode Plots

- First order process




## Bode Plots

- Filters
  - *Pass band* is the range of frequencies where the signals pass through the system at the same degree of amplification
  - *Low pass filter* is a dynamical system with a pass band in the low frequeny range
  - *High pass filter* is a dynamical system with a pass band in the high frequency range
  - *Band pass filter* is a dynamical system with a pass band over a certain range of frequencies
  - *Bandwidth* is the width of the frequency interval over the pass band of the filter




The above Bode plot is for $T = 1$. The gain plot is approximated by two straight lines while the phase plot is approximated by three straight lines. **Note that the beginning and the end of the 45-degree line are always at $0.1\frac{1}{T}$ and $10\frac{1}{T}$.**

For 1 >> ωT, we have a constant gain. For 1 << ωT, we have an integrator gain.

**Note:** Other names for **corner frequency** are **break-point frequency** and **break frequency**.
See below:

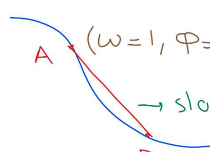
Prove that for $G(s) = \frac{1}{Ts+1}$ $(T=1)$, slope $\angle G(j\omega)$ vs $\omega$ is equal to $-45°/1$ decade and slope $|G(j\omega)|$ vs $\omega$ is $-20°/1$ decade:

$G(j\omega) = \frac{1}{j\omega+1}$ ; $|G(j\omega)| = \left|\frac{1}{j\omega+1}\right| = \frac{1}{\sqrt{\omega^2+1}} \rightarrow 20\log|G(j\omega)| = 20\log\frac{1}{\sqrt{\omega^2+1}}$

$= 20(0 - \log\sqrt{\omega^2+1}) = -20\log(\sqrt{\omega^2+1})$ → $-20°/1$ decade

$\angle G(j\omega) = \angle\frac{1}{j\omega+1} = \angle 1 - \angle(j\omega+1) = 0° - \tan^{-1}(\omega)$ →

$\angle G(j\omega) = -\tan^{-1}\omega$ ; $\begin{cases} \text{for } \omega = 1 \rightarrow \angle G(j\omega) = -45° \\ \text{for } \omega = 10 \rightarrow \angle G(j\omega) = -\tan^{-1}(10) = -84.2° \end{cases}$



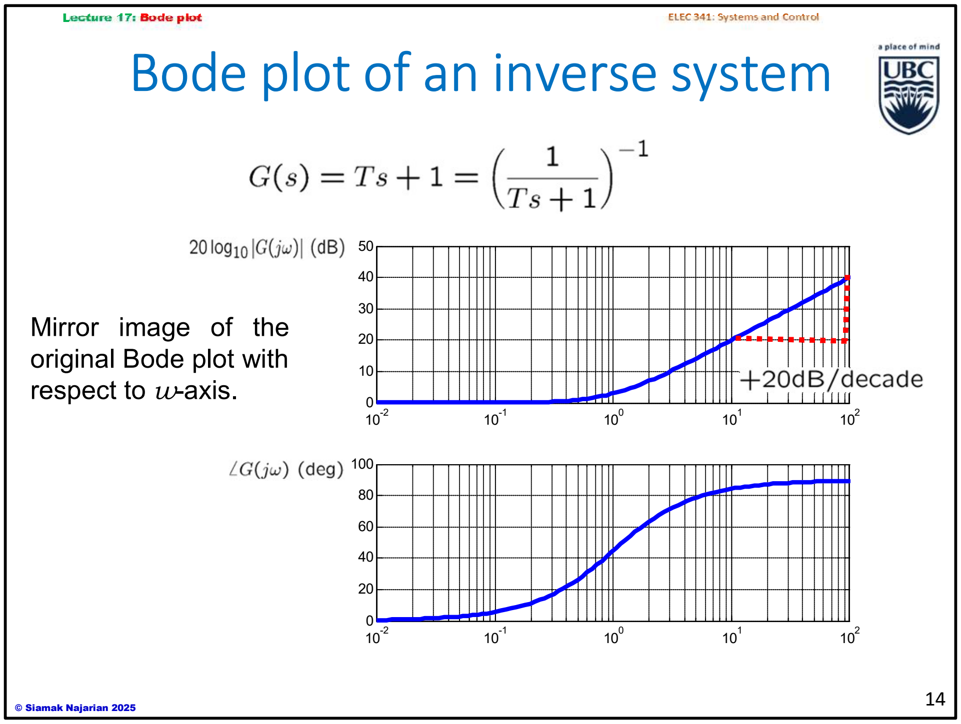

A $(\omega = 1, \varphi = -45°)$ → slope AB = ? ⇒

B $(\omega = 10, \varphi = -84.2°)$

Slope of line AB $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-45°) - (-84.2°)}{(1 - 10)} = \frac{-4.35°}{1 \text{ rad/s}}$

$= \frac{-4.35°}{1 \text{ rad/s}} \times \frac{10}{10} = \frac{-43.5°}{10 \text{ rad/s}} \approx \frac{-45°}{1 \text{ decade}}$ ⇒

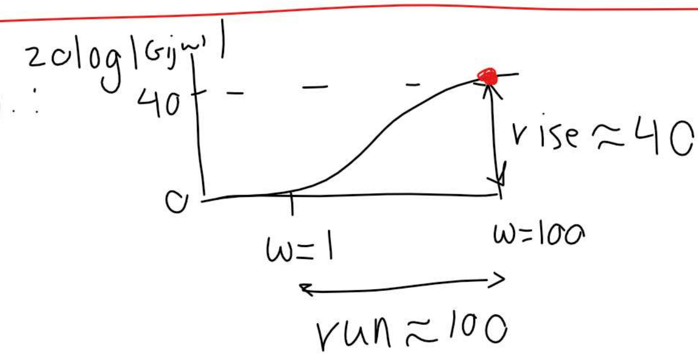
slope $\simeq \frac{-45°}{1 \text{ decade}}$




See below:

## What if we read the slope randomly?

e.g.:




$\Rightarrow \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{40}{100} =$

$= \frac{40}{10 \times 10} = \frac{40}{1\text{ decade} \times 1\text{ decade}}$

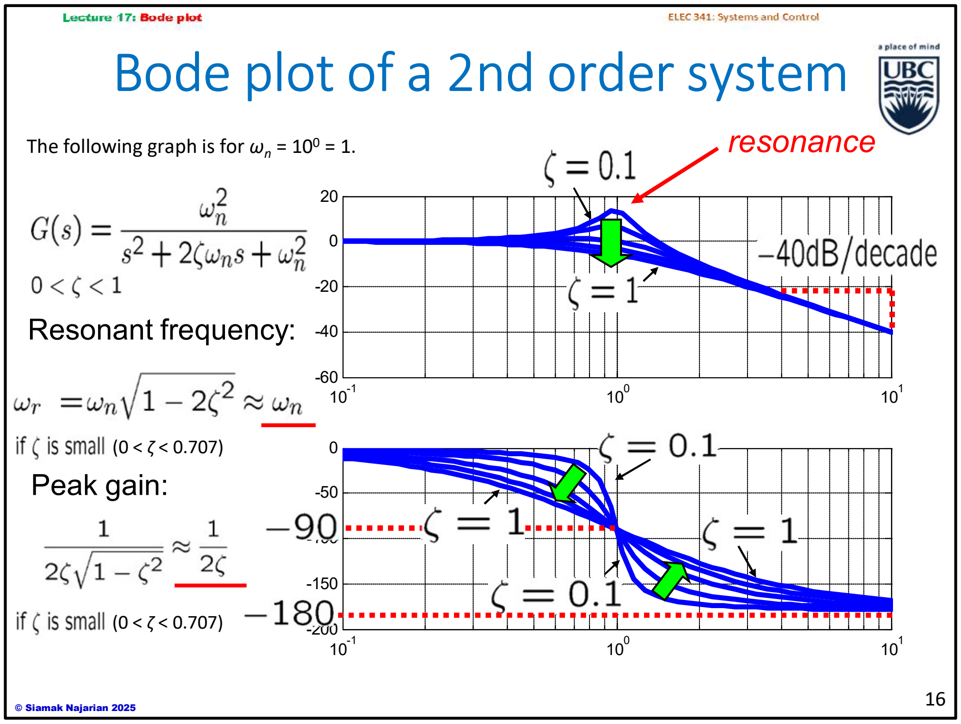
(10 → 1 decade; 10 → 1 decade)

$= \frac{\cancel{40}^{\,20}}{\cancel{2}_{\,1}\text{ decade}} = \frac{20}{1\text{ decade}} \Rightarrow$

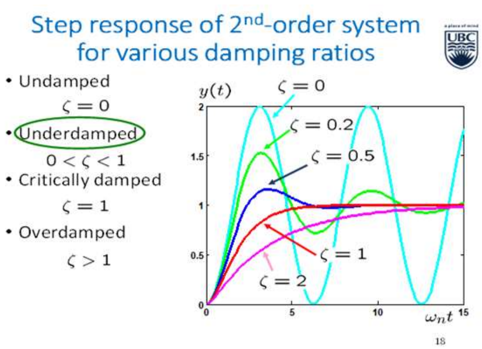
$\Rightarrow \boxed{\text{slope} \approx \frac{20}{1\text{ decade}}}$ ↳ this is the same as what we got before

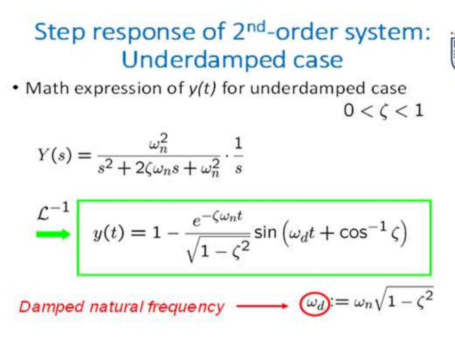
# Sketching Bode plot

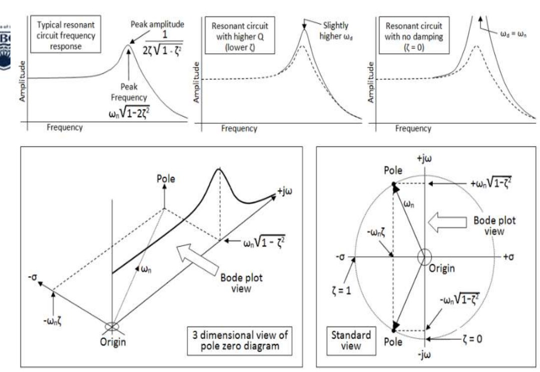
- **Basic functions:**
  - Constant gain
  - Differentiator and integrator
  - Double integrator
  - First order system and its inverse
  - Second order system
  - Time delay
- **Product of basic functions:**
  1. Sketch Bode plot of each factor, and
  2. Add the Bode plots graphically.




See below for review of damping ratio effect. In the above graph, we assumed that $\omega_n = 10^0 = 1$. Myself: For small ζ, we have **Peak Gain (PG) ∝ PO ∝ $\frac{1}{\zeta}$**



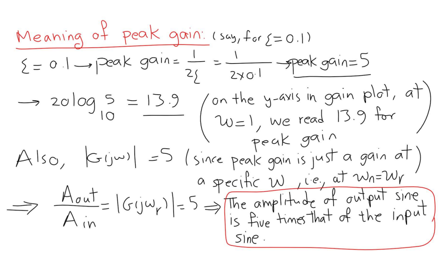



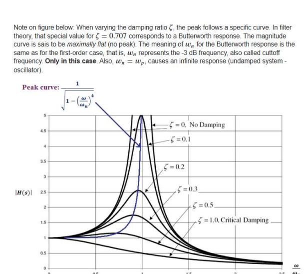



https://electronics.stackexchange.com/questions/112521/resonant-frequency-from-bode-plot

**Resonant Frequency from Bode plot**

If we have a transfer function that shows no peaking in the magnitude bode plot (Starting from a flatline and then rolling off), does this mean that there is no resonant frequency? Or do we consider the point at which the curve begins to roll off the resonant frequency? I understand that resonant frequency is the location at which we have the maximum value so I'm assuming that there isn't a resonant frequency in this case but I wanted to be sure. My answer applies to higher-than-1st-order systems. There will always be a resonant point even if you can't see it. You need to understand how "poles" work. Even if there doesn't appear to be a resonance in the bode plot there will be a "pole" that is present and this pole represents the resonant frequency even though the "dampening" is causing it not to appear in the bode plot. See below:



A **resonant frequency** is the natural vibrating frequency of an object and denoted as **'f' with a subscript zero ($f_0$).** When an object is in equilibrium with acting forces and could keep vibrating for a long time under perfect conditions, this phenomenon is **resonance**. In our daily life example of a resonant frequency is a pendulum.

**In simple terms and from a practical engineering point of view, what does it mean and what is the implications of a resonant frequency and the high amplitude we see at the resonant frequency?**

Great question — and let's keep it simple and practical from an engineering point of view.

🔊 **What Is Resonant Frequency?**

Imagine you're pushing someone on a swing. If you time your pushes just right — matching the swing's natural rhythm — the swing goes **higher and higher** with little effort. That rhythm is like the **resonant frequency**.

In engineering terms:

•**Resonant frequency** is the frequency at which a system **naturally wants to vibrate or oscillate**.

•When an input (like a force, voltage, sound wave, etc.) hits that frequency, the system **responds with a much larger amplitude** than at other frequencies.

⚙️ **What Causes the High Amplitude?**

At resonance:

•Energy input **adds up** efficiently — you're "in sync" with the system.

•Very little energy is lost, so the system builds up a **large response** (vibration, voltage, current, etc.).

•This is why you often see a **sharp peak** in frequency response graphs at the resonant frequency.

🧰 **Practical Engineering Implications**

✅ **Useful Applications**

**1.Tuning Circuits** (like radios): Use resonance to **select specific frequencies** and filter out the rest.

**2.MRI Machines**: Use resonance in human tissues to produce clear images.

**3.Vibration-based Sensors**: Detect changes in mass or stiffness based on shifts in resonant frequency.

❌ **Potential Problems**

**1.Mechanical Structures** (e.g., bridges, buildings, aircraft): If exposed to vibrations at their resonant frequency, they can **shake violently** or even **collapse** (like Tacoma Narrows Bridge).

**2.Circuit Design**: Resonance can lead to **unwanted high voltages or currents**, damaging components.

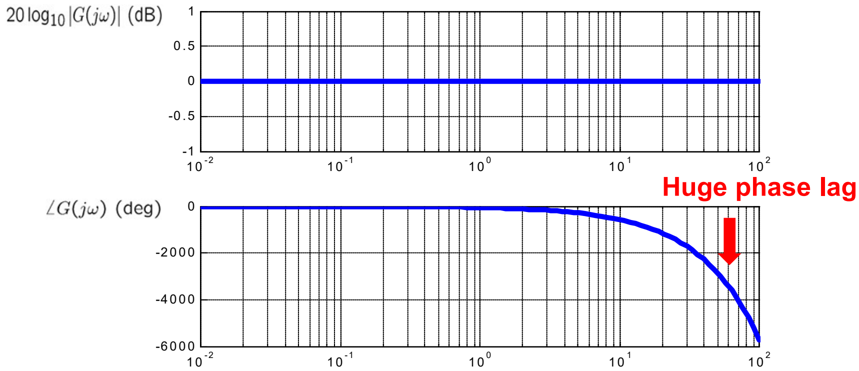
📌 **Bottom Line**

Resonance is like hitting a system's "sweet spot" — it can **supercharge** its response.

That can be **very useful** or **very dangerous**, depending on the context.

# Bode plot of a time delay

$$G(s) = e^{-Ts} \Rightarrow |G(j\omega)| = 1, \forall\omega,\ \angle G(j\omega) = -\omega T\text{(rad)}$$




*The phase lag can cause instability of the closed-loop system, and thus, the difficulty in control.*

$$\angle G(j\omega) = -\omega T \times \frac{180^\circ}{\pi}\text{ (degrees)}$$





See below for the proof of the above results: **The above Bode plot is for T≈ 1.047.** **"Huge phase lag!"** because the angles are huge (order of a few thousands). See below:

Euler's Formula: $e^{+jx} = \cos x + j\sin x$ ; $e^{-jx} = \cos x - j\sin x$

$G(s) = e^{-Ts} \xrightarrow{s=j\omega} G(j\omega) = e^{-j(\omega T)} = \cos(\omega T) - j\sin(\omega T)$

$|G(j\omega)| = |e^{-j(\omega T)}| = |\cos(\omega T) - j\sin(\omega T)| = \sqrt{\cos^2(\omega T) + \sin^2(\omega T)} = 1$

$\rightarrow |G(j\omega)| = 1$

$\angle G(j\omega) = \tan^{-1}\left[\frac{-\sin(\omega T)}{\cos(\omega T)}\right] = \tan^{-1}[-\tan(\omega T)] = -\omega T \rightarrow$

$\rightarrow \angle G(j\omega) = -\omega T$

$\angle e^{-Ts} = -\frac{T.\omega \times 180^\circ}{\pi}$ (in degrees)

@ $\omega = 100 \rightarrow \angle e^{-Ts}$ (in degrees) $= -6000^\circ \Rightarrow$

$-6000^\circ = -\frac{T \times 100 \times 180^\circ}{\pi} \rightarrow T = 1.047$

**Pade Approximation:**

Root locus techniques assume a system has a set of (known) poles and zeros

$$G(s) = k\frac{z(s)}{p(s)}$$

Unfortunately, delays are not in this form

$$Delay(T) = e^{-sT}$$

One way around this problem is to use the Pade approximation.

First, rewrite the delay as a numerator and denominator term:

NDSU — Root Locus for Systems with Delays

$$e^{-sT} = \left(\frac{e^{\frac{-sT}{2}}}{e^{\frac{sT}{2}}}\right)$$

For the numerator and denominator, expand using a Taylors series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

This results in

$$e^{-sT} = \left(\frac{1 + \left(\frac{-sT/2}{1!}\right) + \frac{(-sT/2)^2}{2!} + \frac{(-sT/2)^3}{3!} + \frac{(-sT/2)^4}{4!} + \frac{(-sT/2)^5}{5!} + \ldots}{1 + \left(\frac{sT/2}{1!}\right) + \frac{(sT/2)^2}{2!} + \frac{(sT/2)^3}{3!} + \frac{(sT/2)^4}{4!} + \frac{(sT/2)^5}{5!} + \ldots}\right)$$

or

$$e^{-sT} = \left(\frac{1 - \left(\frac{T}{2}\right)s + \left(\frac{T^2}{8}\right)s^2 - \left(\frac{T^3}{48}\right)s^3 + \left(\frac{T^4}{384}\right)s^4 + \ldots}{1 + \left(\frac{T}{2}\right)s + \left(\frac{T^2}{8}\right)s^2 + \left(\frac{T^3}{48}\right)s^3 + \left(\frac{T^4}{384}\right)s^4 + \ldots}\right)$$

The more terms you add, the better the approximation.

Example: Find 'k' so that the following system has 20% overshoot for its step response: (A DC servo motor with a 1/2 second delay)

$$Y = \left(\left(\frac{100}{s(s+5)(s+10)}\right) \cdot e^{-0.5s}\right)U$$

Solution #1: Use a Pade approximation:

$$e^{-sT} = \left(\frac{1 - \left(\frac{T}{2}\right)s + \left(\frac{T^2}{8}\right)s^2 - \left(\frac{T^3}{48}\right)s^3 + \left(\frac{T^4}{384}\right)s^4 + \ldots}{1 + \left(\frac{T}{2}\right)s + \left(\frac{T^2}{8}\right)s^2 + \left(\frac{T^3}{48}\right)s^3 + \left(\frac{T^4}{384}\right)s^4 + \ldots}\right)$$

Plugging in T = 0.5 and using the first two terms results in

$$e^{-0.5s} \approx \left(\frac{1 - 0.25s + 0.0313s^2}{1 + 0.25s + 0.0313s^2}\right)$$

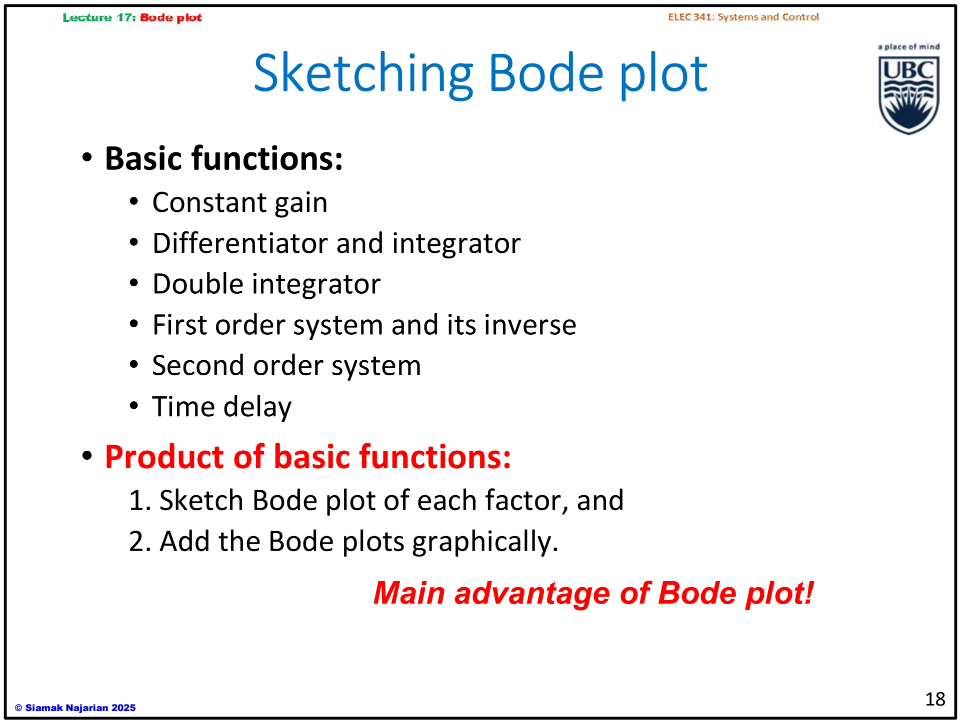
$$e^{-0.5T} \approx \left(\frac{(s-4+j4)(s-4-j4)}{(s+4+j4)(s+4-j4)}\right)$$

**Analysis and Design of Feedback Systems with Time Delays**

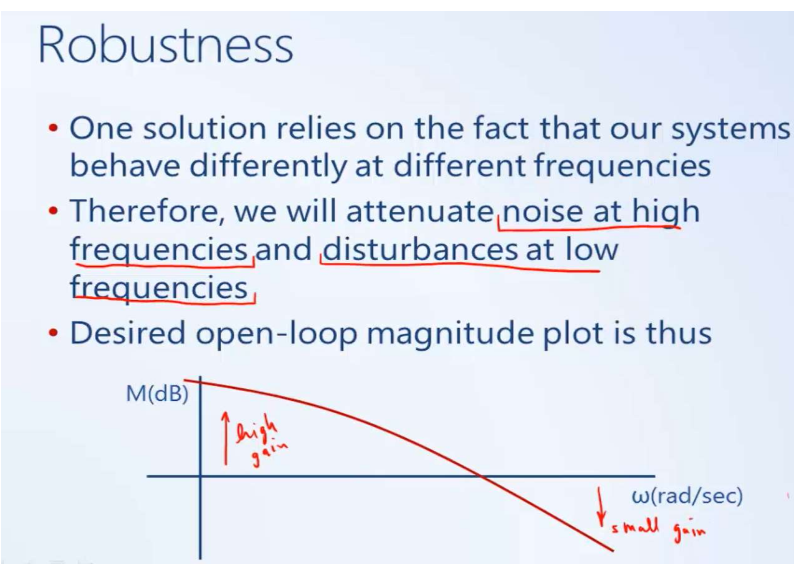
When working with time delay systems it is advantageous to work with analysis and design tools that directly support time delays so that performance and stability can be evaluated exactly. However, many control design techniques and algorithms cannot directly handle time delays. **A common workaround consists of** replacing delays by their **Pade approximations (all-pass filters).** Because this approximation is only valid at low frequencies, it is important to choose the right approximation order and check the approximation validity. Control System Designer provides a variety of design and analysis tools. Some of these tools support time delays exactly while others support time delays indirectly through approximations. Use these tools to design compensators for your control system and visualize the compromises made when using approximations.

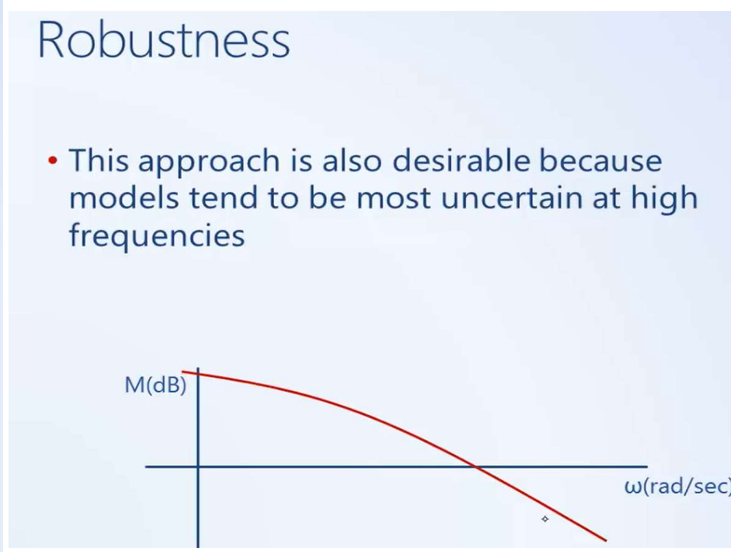
**Differences between root locus plot and Nyquist plot?**

Both methods assess stability but with different means. The root locus plot is most often used when you are dealing with one design-parameter (most time simple P-controller with gain K). It will show how the roots change when changing the design-parameter. Hence, it is a direct way to assess stability (negative real part) and also to see for which parameter range the system oscillates (has overshoot). The root locus plot cannot be used for systems with **dead time**. The Nyquist plot is an indirect way to assess stability. **We see from the Nyquist plot if the given open loop system is closed loop stable.** It also gives information about the stability margins like phase margin and gain margin. It can be used for systems with dead time.




See below:
https://www.bing.com/videos/search?q=different+shapes+of+bode+plot&&view=detail&mid=717C4D3C935D4CBC9B98717C4D3C935D4CBC9B98&&FORM=VRDGAR



## An advantage of Bode plot

- Bode plot of a series connection $G_1(s)G_2(s)$ is the addition of each Bode plot of $G_1(s)$ and $G_2(s)$.
  - Gain:

$$20\log_{10}|G_1(j\omega)G_2(j\omega)| = 20\log_{10}|G_1(j\omega)| + 20\log_{10}|G_2(j\omega)|$$

  - Phase:

$$\angle G_1(j\omega)G_2(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$$




I deleted "Later, we use this property to design C(s) so that G(s)C(s) has a "desired" shape of Bode plot."

**bode** /bohd/verb 1. to be an omen of (good or ill, esp of ill)

However, Bode plot is pronounced [**boh**-dee]. Some people pronounce it just [bohd].

See



## Short proofs

- Use polar representation

$$G_1(j\omega) = |G_1(j\omega)|e^{j\angle G_1(j\omega)} \qquad G_2(j\omega) = |G_2(j\omega)|e^{j\angle G_2(j\omega)}$$

Then,
$$\begin{aligned} G_1(j\omega)G_2(j\omega) &= |G_1(j\omega)||G_2(j\omega)|e^{j\angle G_1(j\omega)}e^{j\angle G_2(j\omega)} \\ &= |G_1(j\omega)||G_2(j\omega)|e^{j\{\angle G_1(j\omega)+\angle G_2(j\omega)\}} \end{aligned}$$

Therefore,

$$20\log_{10}|G_1(j\omega)G_2(j\omega)| = 20\log_{10}|G_1(j\omega)|\cdot|G_2(j\omega)| = 20\log_{10}|G_1(j\omega)| + 20\log_{10}|G_2(j\omega)|$$

$$\angle G_1(j\omega)G_2(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$$

# Example 1 $G(s) = \frac{10}{s}$

- Sketch the Bode plot of the following transfer function:

$$G(s) = \frac{10}{s}$$

**Step 1:** Decompose $G(s)$ into a product form:

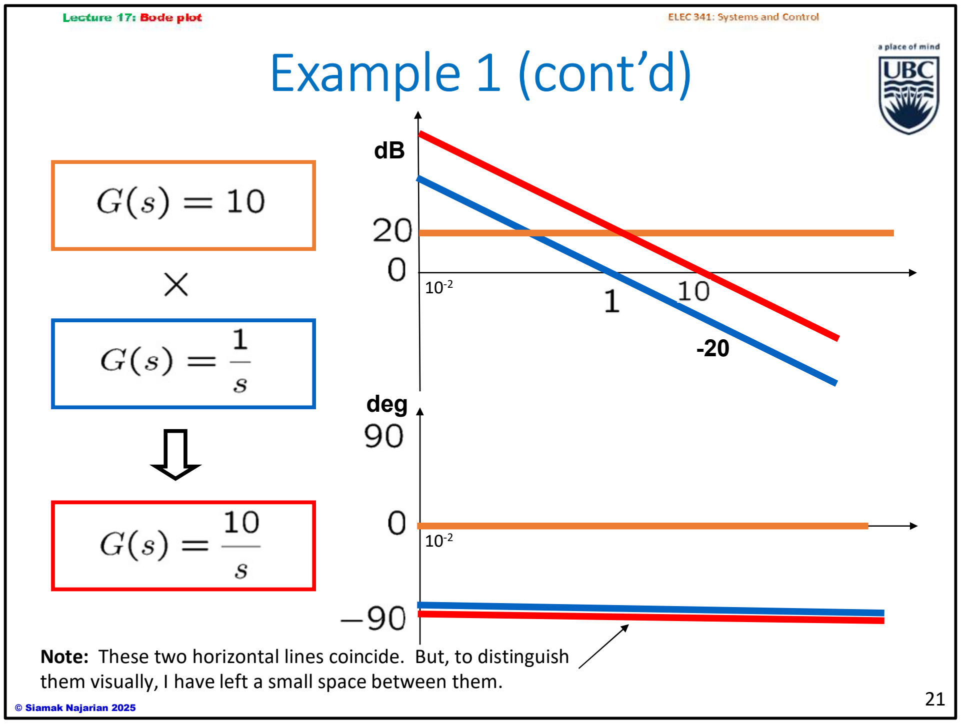
$$G(s) = 10 \cdot \frac{1}{s}$$

**Step 2:** Sketch a Bode plot for each component on the same graph.

**Step 3:** Add them all on both gain and phase plots.



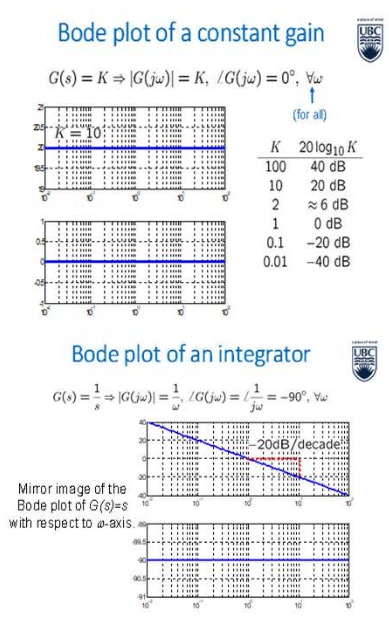

**decompose** [dee-kem-**pohz**]

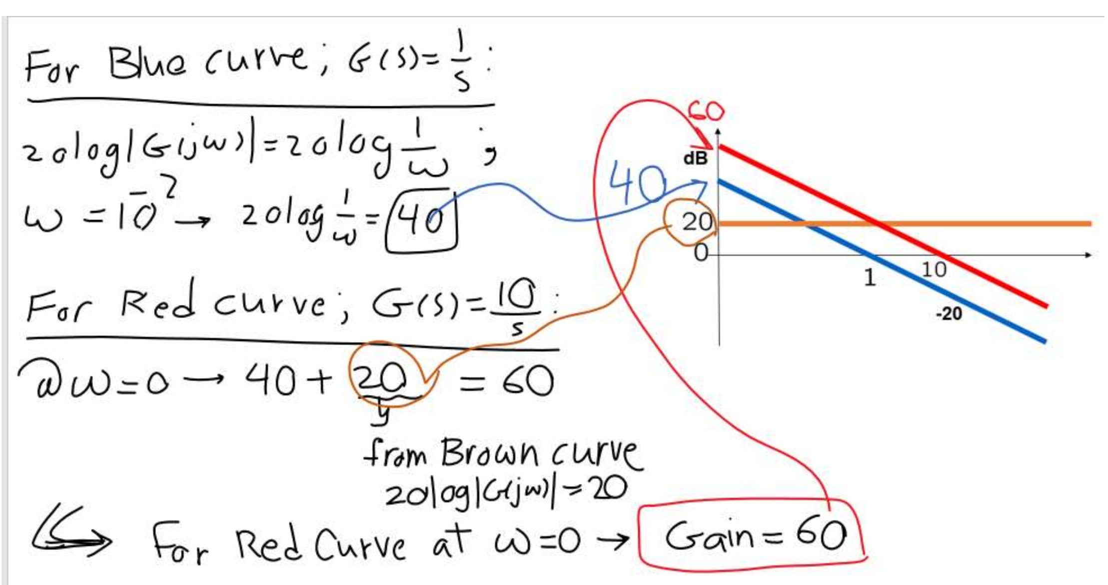



See below for proof: In the top graph, the red curve is the final gain plot and in the bottom graph, the purple (magenta) is the final phase plot.

**Important Note:** The numbers in bold face (such as -20) are angles per 1 decade and are the slope of the lines (and not the angle of the lines). They are the amount of gain (or phase shift) per one decade.

**How to sketch Bode of a multiple function (such as in the above slide):**
**Step 1:** Start with the far-left hand side point (i.e., ω = 0) and add the values at ω = 0. This should give us the starting point on the final gain or phase plot.
**Step 2:** Start moving on the final gain/phase plot and add the values of the slopes. If one of them is a straight line it will have no effect on the outcome. For the ones that are sloped lines (let us say we have only one sloped line), the final gain/phase plot will follow its slope until there is a change in the sloped line. If there is a change in the slope of the other sloped line, the final gain/phase plot will become steeper (with the new slope being increased by the sum of the slopes of the other sloped lines). **The slope of each point on the final plot is equal to the sum of the slopes of the other lines.**



# Example 1 (cont'd)

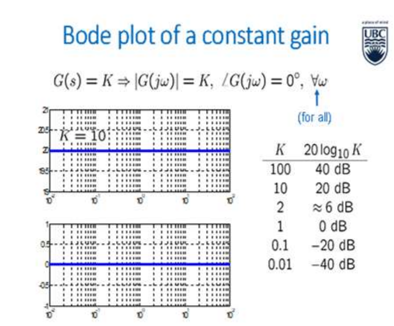
**How to the sketch Bode of product of basic functions (Slope Method):**

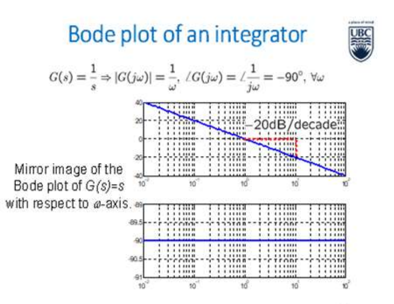
- **Step 1:** Start with the far-left hand side point (i.e., $\omega = 10^{-2}$) and add the values at $\omega = 10^{-2}$. This should give us the starting point on the final gain or phase plot.

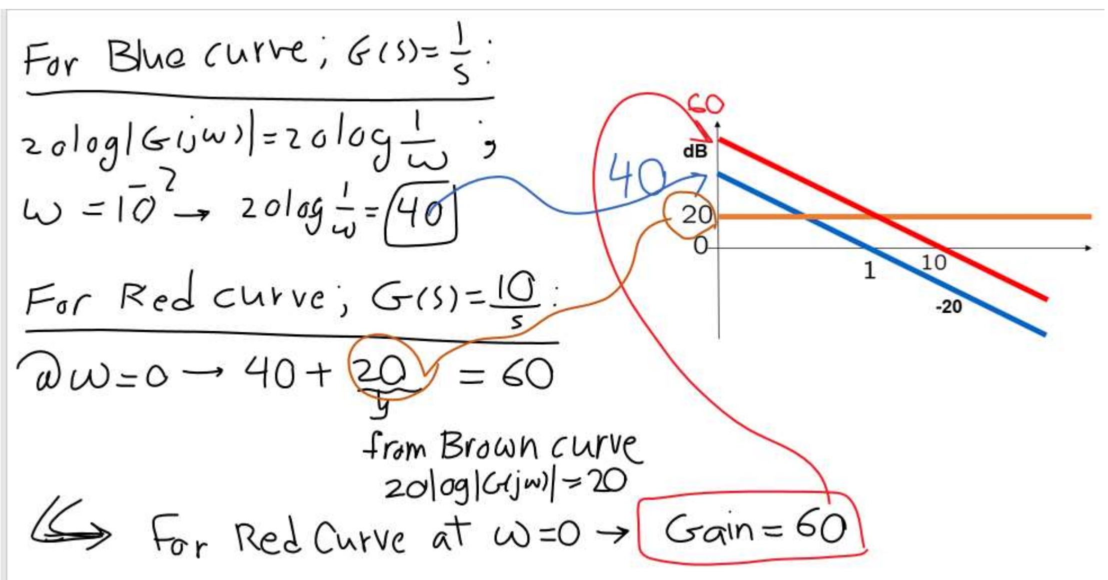
- **Step 2:** Start moving on the final gain/phase plot and add the values of the slopes. If one of them is a straight line it will have no effect on the outcome. For the ones that are sloped lines (let us say we have only one sloped line), the final gain/phase plot will follow its slope until there is a change in the sloped line. If there is a change in the slope of the other sloped line, the final gain/phase plot will become steeper (with the new slope being increased by the sum of the slopes of the other sloped lines). **The slope of each point on the final plot is equal to the sum of the slopes of the other lines.**

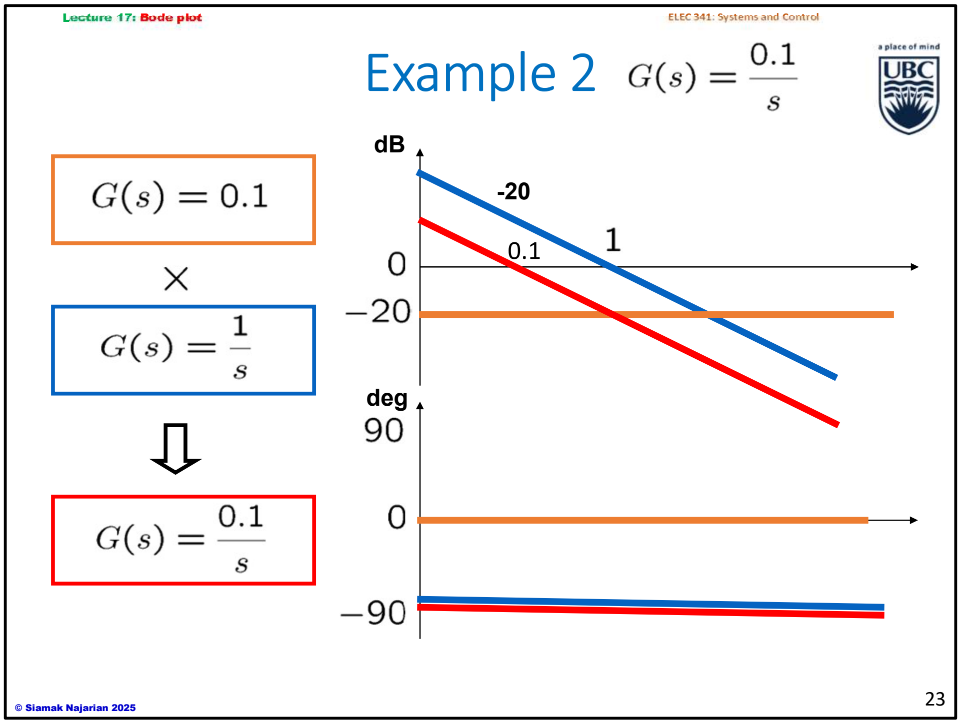


**See below:**




Lecture 17: Bode plot
ELEC 341: Systems and Control
Example 2 $G(s) = \frac{0.1}{s}$
a place of mind
UBC
$G(s) = 0.1$
×
$G(s) = \frac{1}{s}$
$G(s) = \frac{0.1}{s}$
dB
-20
1
0.1
0
−20
deg
90
0
−90
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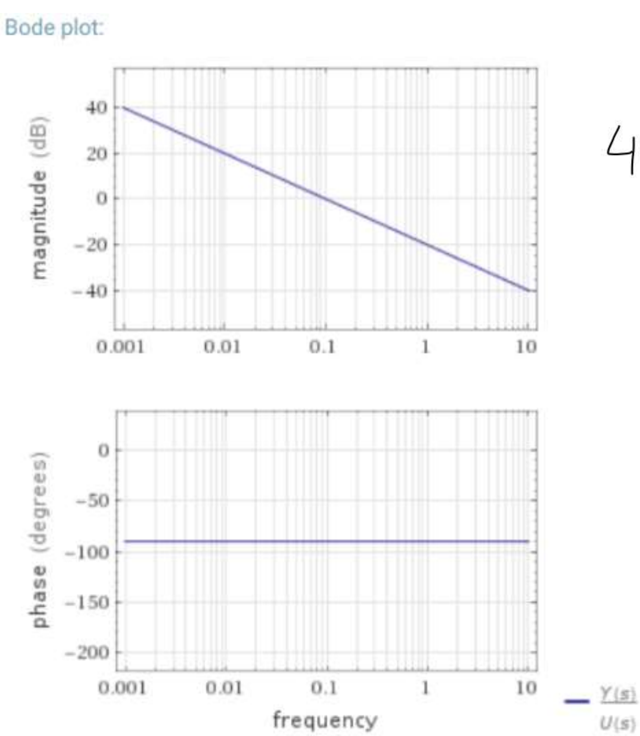
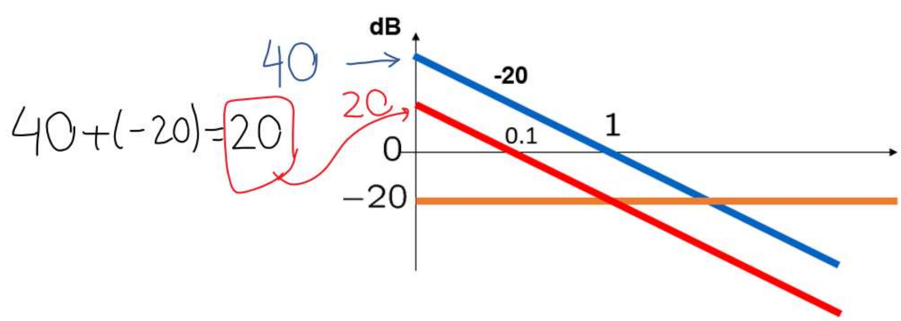
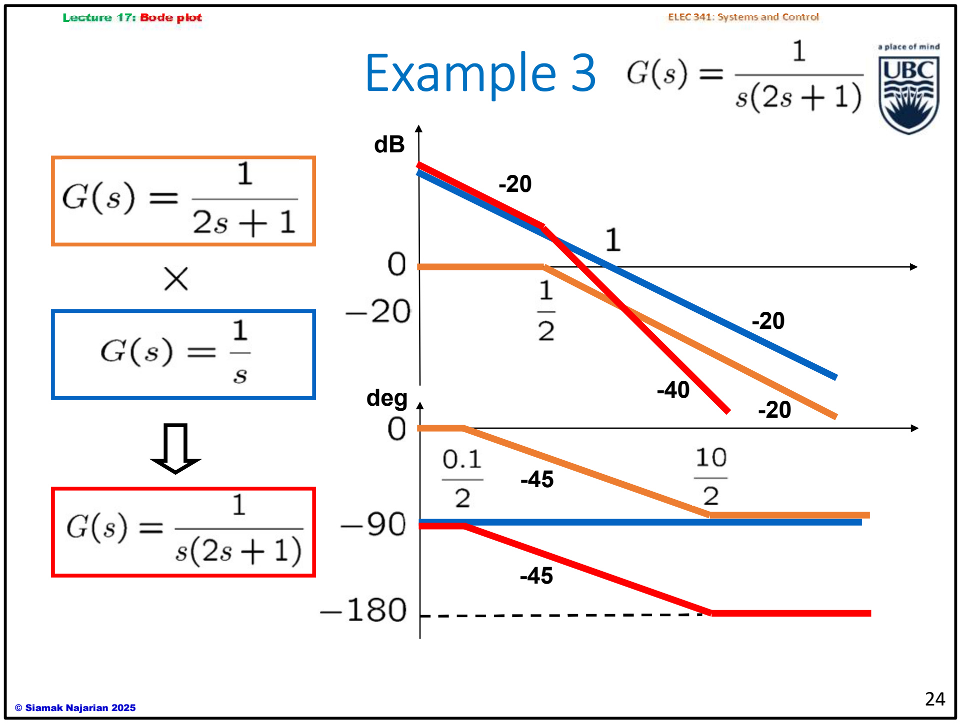
23

See below:

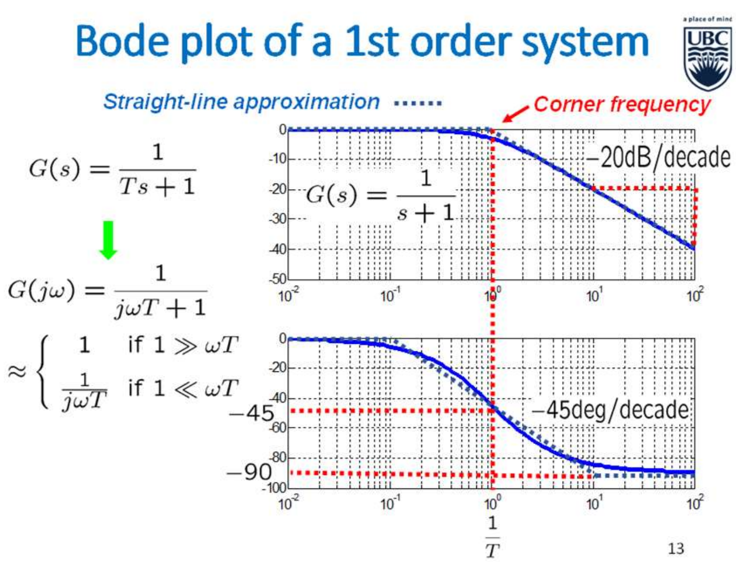

Bode plot:
magnitude (dB)
40
20
0
−20
−40
0.001
0.01
0.1
1
10
phase (degrees)
0
−50
−100
−150
−200
frequency
Y(s) / U(s)

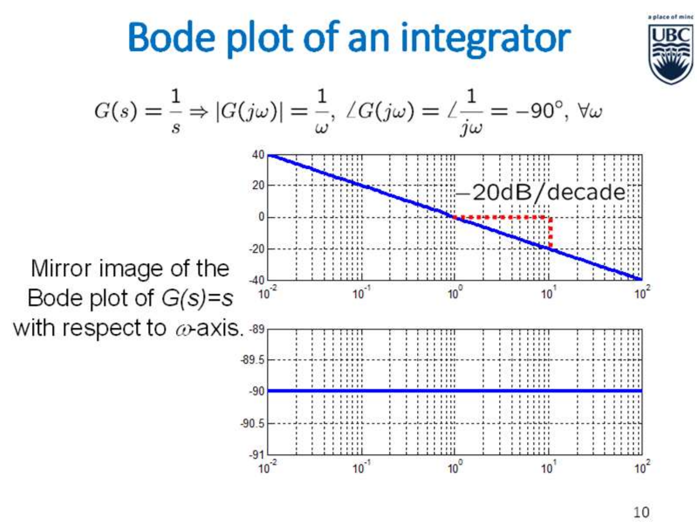

dB
40
-20
20
40+(-20)=20
1
0.1
0
−20

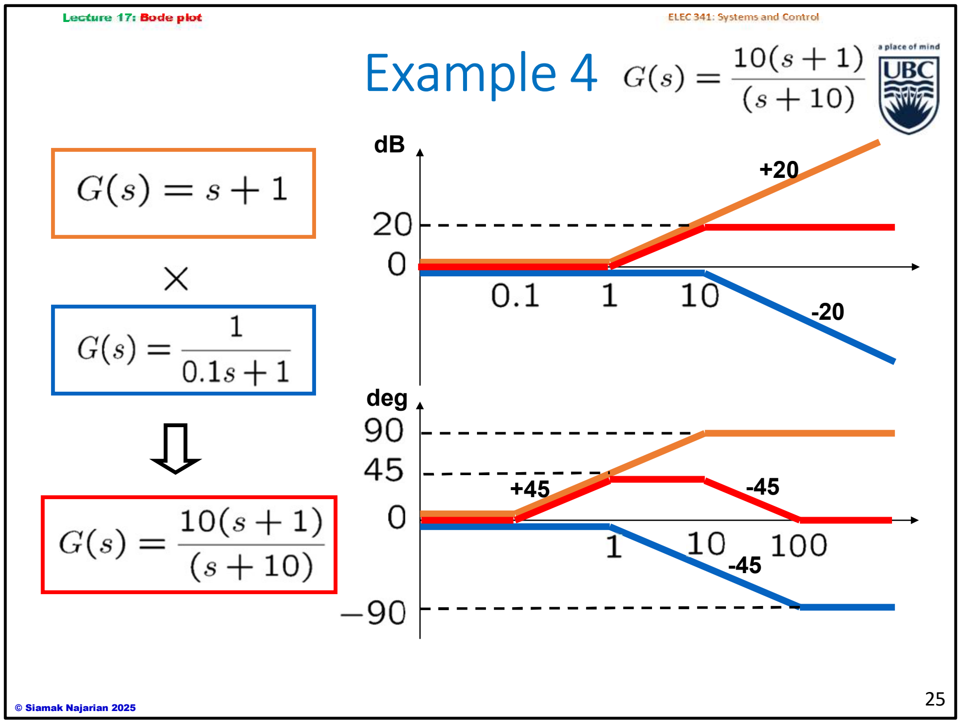



See below for proof:

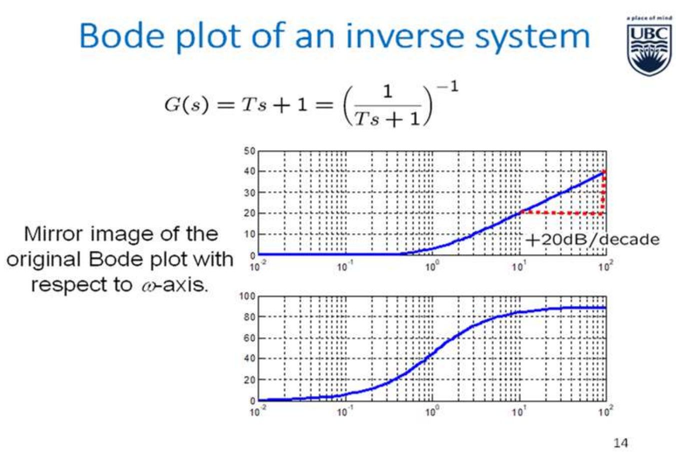
**Note that the beginning and the end of the 45-degree line are always at $0.1\frac{1}{T}$ and $10\frac{1}{T}$.**

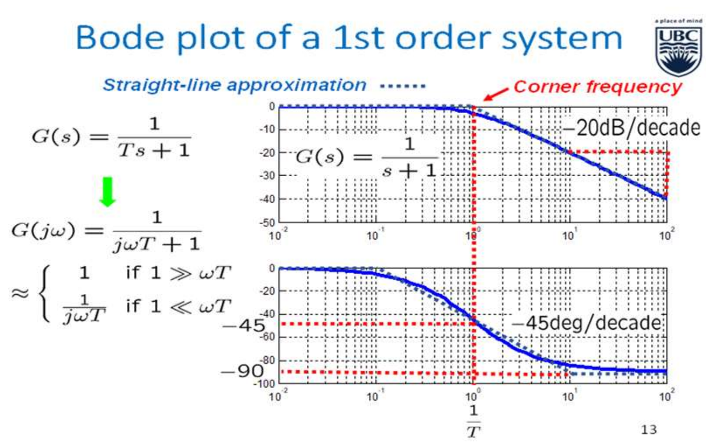





See below for proof:




$$G(S) = \frac{10(S+1)}{(S+10)} = \frac{(S+1)}{0.1(S+10)} \rightarrow$$

$$G(S) = \frac{(S+1)}{(0.1S+1)}$$

Corner frequency range for $G(S) = \frac{1}{0.1S+1}$ :

$T = 0.1$ : $0.1\left(\frac{1}{T}\right) \rightarrow 10\left(\frac{1}{T}\right)$

$0.1 \times \frac{1}{0.1} \rightarrow 10 \times \frac{1}{0.1}$

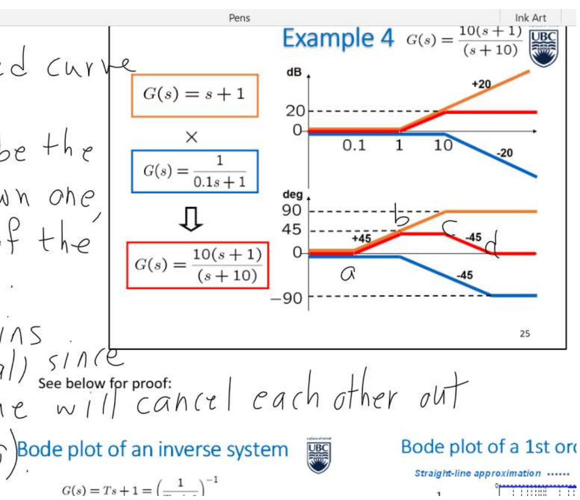
⇓

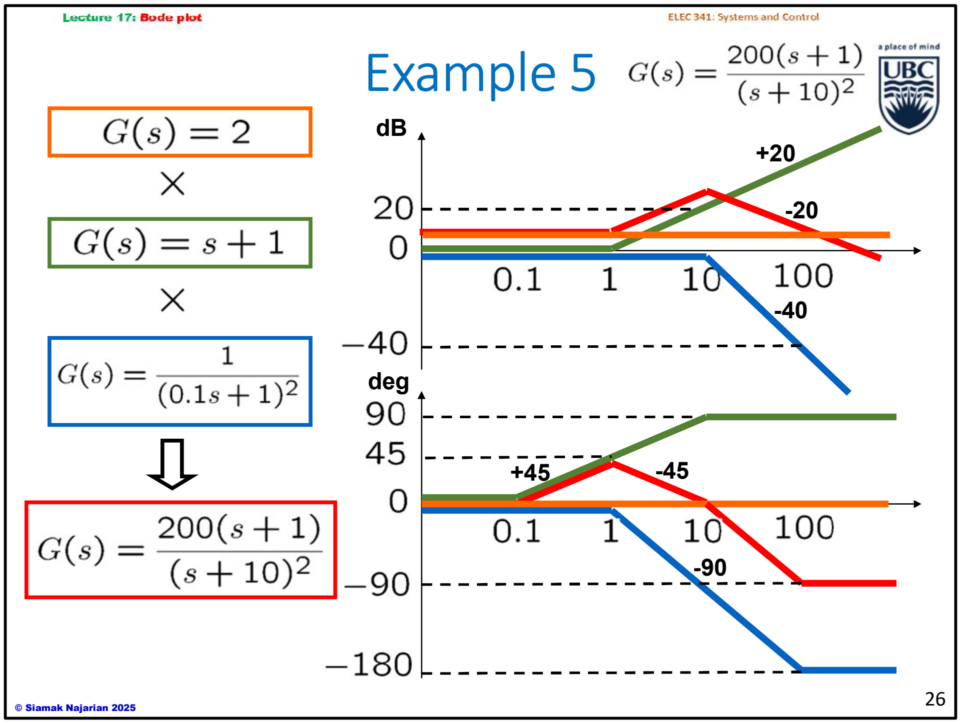
$1 \rightarrow 100$

Graphing the red curve (from a→d):

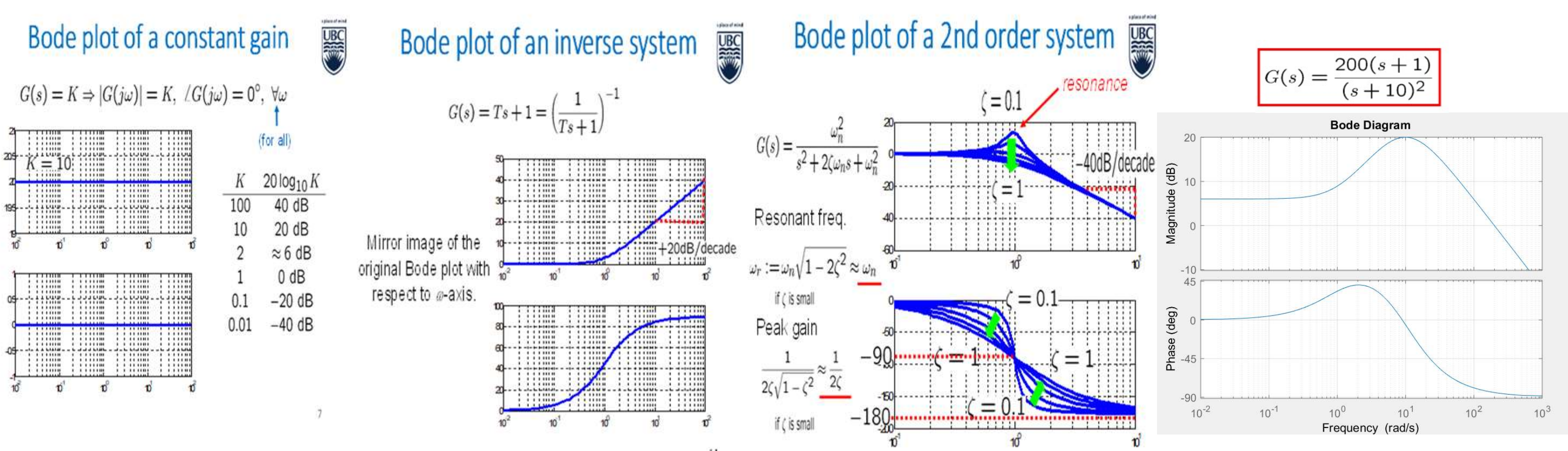
a→b: It will be the same as the brown one, since the slope of the blue is zero

b→c: It remains constant (horizonal) since the brown and blue will cancel each other out $(+45° + (-45°) = 0)$.

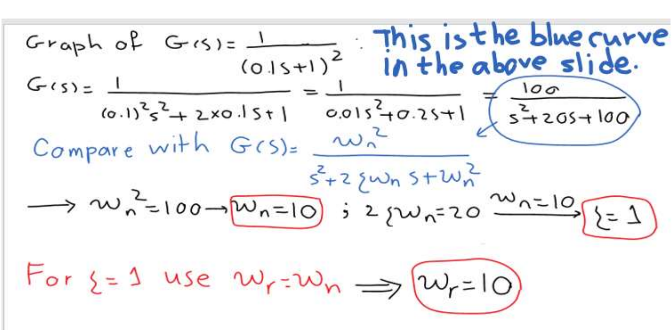


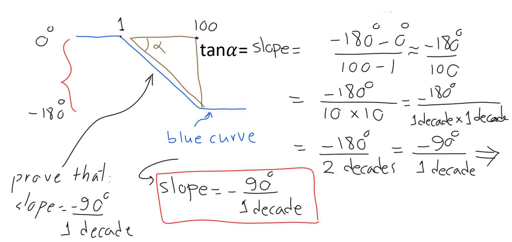


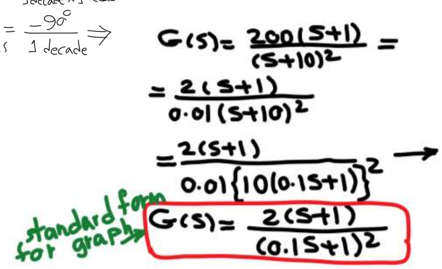

Remember that in all standard forms of 1st order systems, we have Ts+1. That is, T is multiplied by s and there is always a "+1". See below:




| K | $20\log_{10}K$ |
|---|---|
| 100 | 40 dB |
| 10 | 20 dB |
| 2 | ≈ 6 dB |
| 1 | 0 dB |
| 0.1 | −20 dB |
| 0.01 | −40 dB |




**Important Note:** The boldface numbers in the graphs are degrees per decade and the slope of the curves. Slope of a curve is a number and not an angle. That is, slope = (-90°)/(1 decade) = $\tan\alpha$. By solving this equation we will actually find the angle of alpha, which will be the angle of the line. For example, in the bottom graph (phase plot), we have -90 degrees per one decade.

# Remarks

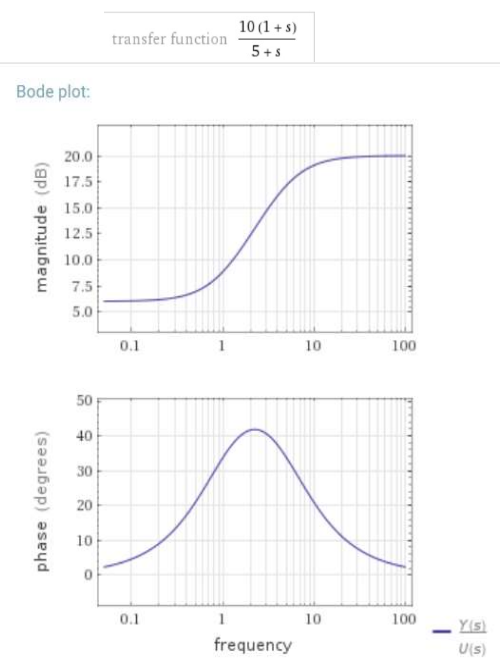
- ALWAYS check the correctness of the following:
  - Low frequency gain (DC gain): $G(0)$
  - High frequency gain: $G(\infty)$
- Example

$$G(s) = \frac{10(s+1)}{s+5}$$






You can use MATLAB command “bode.m” to obtain precise shape. See below: The above G(s) is actually a lead compensator.




$G(s) = \frac{10(s+1)}{s+5}$ ; $G(0) = \frac{10(0+1)}{0+5} = 2 \rightarrow G(0) = 2$

$G(\infty) = \frac{10(\infty+1)}{\infty+5} = 10 \rightarrow G(\infty) = 10$

$\text{gain (dB)} = 20\log_{10}|G(j\omega)| = 20\log_{10}|G(s)|$

For $s = 0 \rightarrow \text{gain (dB)}\big|_{s=0} = 20\log_{10}|G(0)| = 20\log_{10}|2| = 6$

For $s = \infty \rightarrow \text{gain (dB)}\big|_{s=\infty} = 20\log_{10}|G(\infty)| = 20\log_{10}|10| = 20$

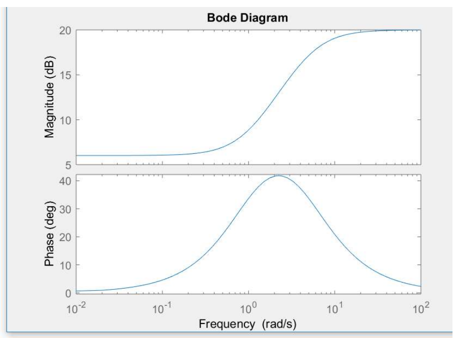
⇒ Check these numbers against those you obtained and put on your Bode plot. Hope they match ✓

# Remarks (cont'd)

- In Matlab, use "bode" command:

$$G(s) = \frac{10(s+1)}{s+5}$$

```
>> s=tf('s')

s =

  s

Continuous-time transfer function.

>> g=10*(s+1)/(s+5)

g =

  10 s + 10
  ---------
    s + 5

Continuous-time transfer function.

>> bode(g)
>>
```



# Summary

- Sketches of Bode plot
  - Basic functions
  - Products of basic functions
- Sketching Bode plot is useful for the following reasons:
  - To get a rough idea of the characteristics of a system.
  - To interpret the result obtained from computer.
- Next
  - *Nyquist stability criterion*




See below for more on resonance frequency:

Peaks in the frequency response can only exist in systems with conjugate complex poles. For an underdamped ($\zeta < 1$ or $Q > 0.5$) second-order system, the peak appears specifically for $\zeta < 1/\sqrt{2} = 0.707$.

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

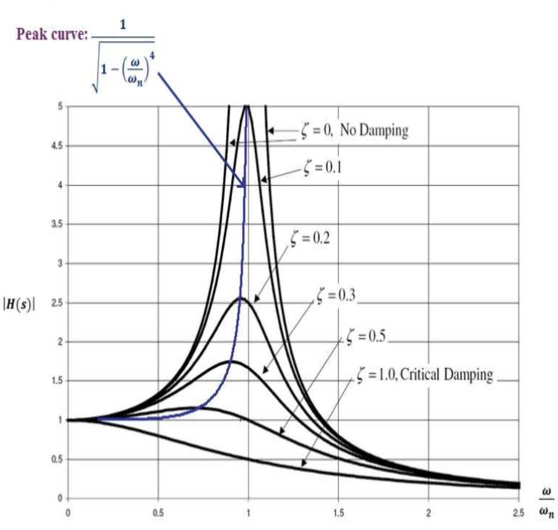
where $\omega_n$ is the natural frequency (also called corner frequency when considering assymptotes), the peak

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

occurs at resonant frequency

$$\omega_p = \omega_n\sqrt{1-2\zeta^2}$$

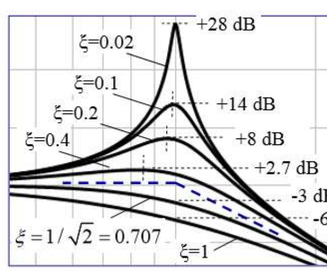
Note on figure below: When varying the damping ratio $\zeta$, the peak follows a specific curve. In filter theory, that special value for $\zeta = 0.707$ corresponds to a Butterworth response. The magnitude curve is sais to be *maximally flat* (no peak). The meaning of $w_n$ for the Butterworth response is the same as for the first-order case, that is, $w_n$ represents the -3 dB frequency, also called cuttoff frequency. **Only in this case.** Also, $w_n = w_p$, causes an infinite response (undamped system - oscillator).




For small $\xi$ the curves are peaked sharply near the corner frequency. Exactly at the corner frequency the curve must pass through the point

$$|H(j\omega_n)| = \frac{1}{2\xi} \quad \Rightarrow \quad -20\log 2\xi \ \text{[dB]} \tag{1.35}$$

Note that this correction may be above the asymptote (positive) or below (negative) depending on the value of the damping factor $\xi$.




| $\xi$ | $\lvert H(j\omega_n)\rvert$ | $\omega_p / \omega_n$ | $\lvert H(j\omega_p)\rvert$ |
|---|---|---|---|
| 0.02 | +28 dB | ~1 | +28 dB |
| 0.05 | +20 dB | 0.997 | +20 dB |
| 0.1 | +14 dB | 0.990 | +14dB |
| 0.2 | +8 dB | 0.959 | +8.1 dB |
| 0.4 | +1.9 dB | 0.825 | +2.7dB |
| 0.5 | 0 dB | 0.707 | +1.3 dB |
| 0.707 | -3 dB | 0 | 0 dB |
| 1 | -6 dB | — | — |

Figure 1-9 – Behavior near the corner frequency for various values of the damping factor $\xi$.

Also note that the peak value is not necessarily centered exactly at the corner frequency; to find the peak location we set the first derivative equal to zero, giving

$$\frac{\partial}{\partial\omega}|H(j\omega)| = 0 \quad \Rightarrow \quad \omega = \omega_p = \omega_n\sqrt{1-2\xi^2} \quad \text{(low-pass)} \tag{1.36}$$

This result tells us that there is a peak or maximum in the response only when $1-2\xi^2 > 0$, or equivalently for $0 \le \xi \le 1/\sqrt{2}$. In this range the peak amplitude is given by

$$|H(j\omega_p)| = \frac{1}{2\xi\sqrt{1-\xi^2}} \quad \Rightarrow \quad -20\log\left(2\xi\sqrt{1-\xi^2}\right) \ \text{[dB]} \tag{1.37}$$