a place of mind

ELEC 341: Systems and Control

Lecture 18

Nyquist stability criterion: Introduction

Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- Models for systems
  - ✓ Electrical
  - ✓ Electromechanical
  - ✓ Mechanical
- ✓ Linearization, delay

Analysis

- ✓ Stability
  - ✓ Routh-Hurwitz
  - ➤ Nyquist
- ✓ Time response
  - ✓ Transient
  - ✓ Steady state
- ✓ Frequency response
  - ✓ Bode plot

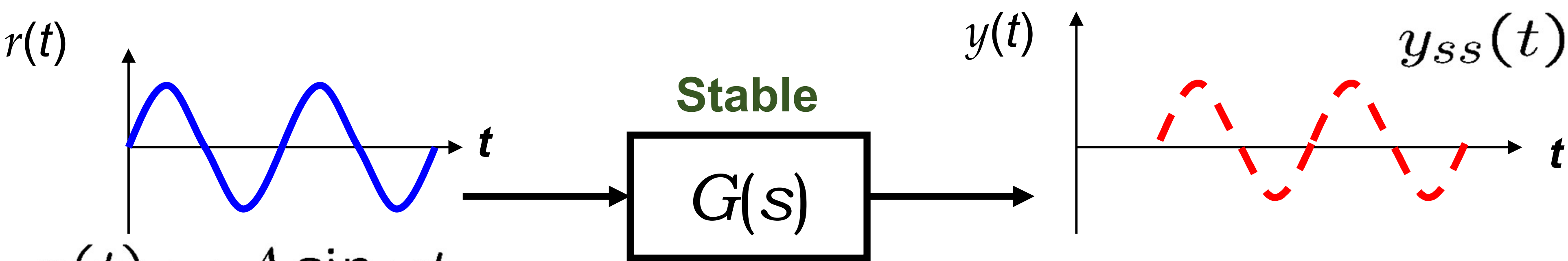
Design

- Design specs
- ✓ Root locus
- Frequency domain
- ✓ PID & Lead-lag
- Design examples

Matlab simulations

Frequency response (review)

- Steady state output $y_{ss}(t) = A\,|G(j\omega)|\,\sin(\omega t + \angle G(j\omega))$
  - Frequency is same as the input frequency ω
  - Amplitude is that of input (A) multiplied by $|G(j\omega)|$ ← Gain
  - Phase shift is $\angle G(j\omega)$

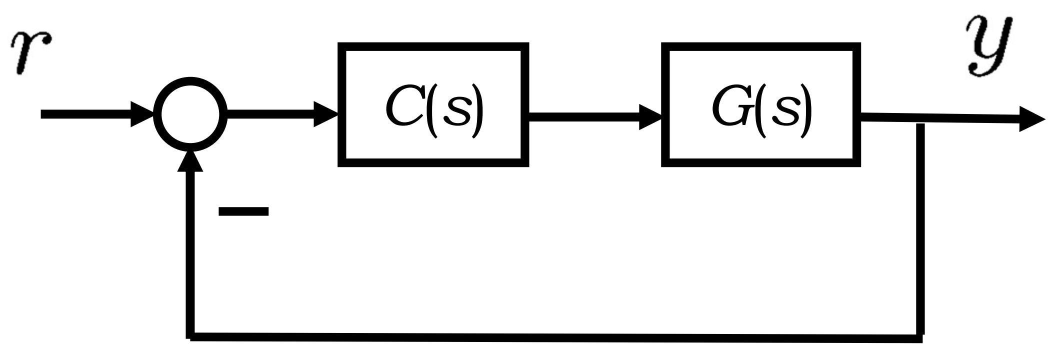


$r(t) = A \sin \omega t$

- Frequency response function (FRF): $G(jw)$
- Bode plot: Graphical representation of $G(jw)$

Stability of feedback system

- Consider the feedback system:



- Fundamental questions:
  - If $G(s)$ and $C(s)$ are stable, is the closed-loop system *always stable*?
  - If $G(s)$ and $C(s)$ are unstable, is the closed-loop system *always unstable*?

Closed-loop stability criterion

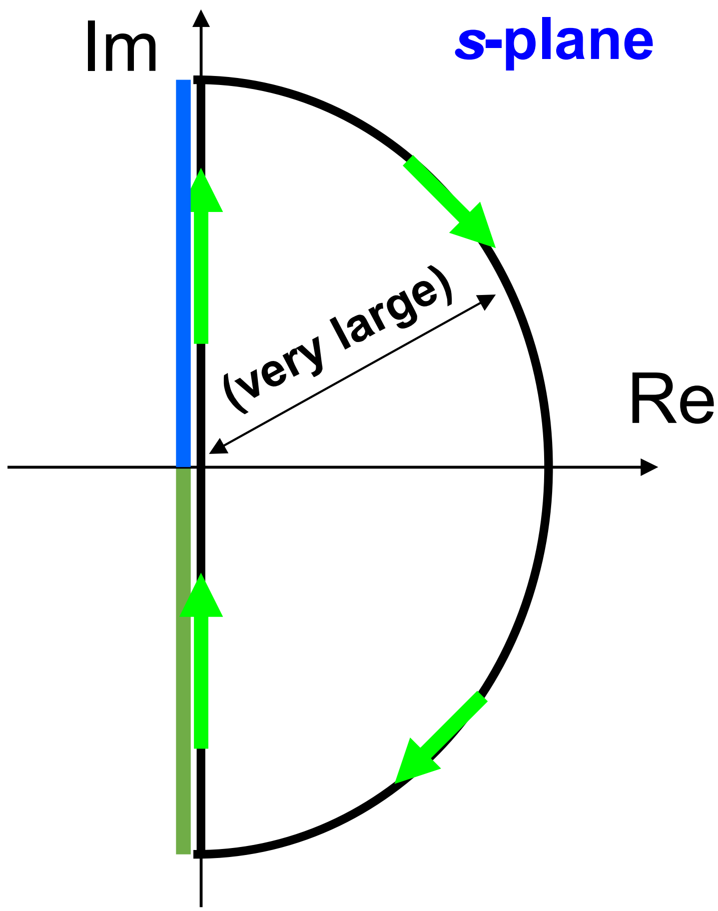
- Closed-loop stability can be determined by the roots of the **characteristic equation**:

$$1 + L(s) = 0,\ \ L(s)\ = G(s)C(s)$$

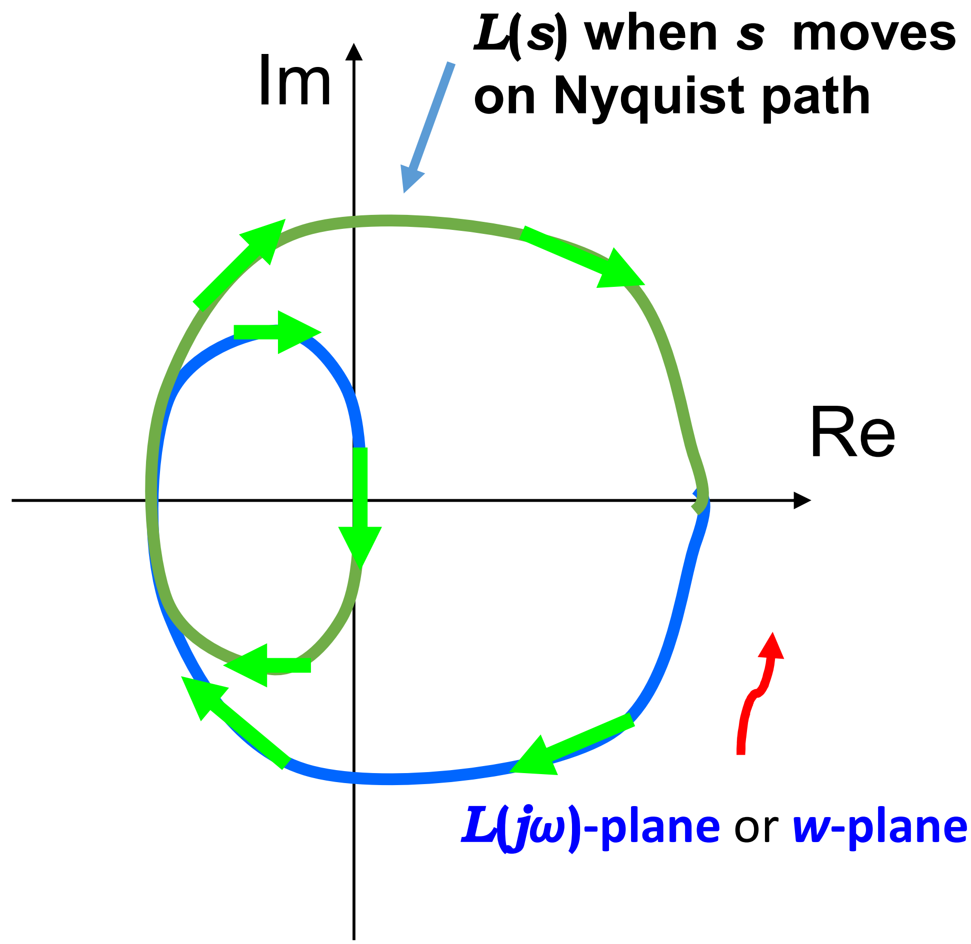
- Closed-loop system is stable if the Ch. Eq. has all roots in the open left half plane.
- How to check the closed-loop stability?
  - Computation of all the roots
  - Routh-Hurwitz stability criterion
  - *Nyquist stability criterion: Open-loop FRF $L(j\omega)$ contains information of closed-loop stability.*

Nyquist plot

- Nyquist path



- Nyquist plot



Note: We pick an s on the Nyquist path and then plug it into the OLTF, i.e., $L(s)$. This will give us a point which is located in another environment (w-plane). This new plot is called Nyquist plot. **Nyquist path** is in ***s*-plane** while **Nyquist plot** is in ***w*-plane**.

Nyquist Plot (or Polar Plot)

- Nyquist plots were invented by Nyquist - who worked at Bell Laboratories, the premiere technical organization in the U.S. at the time.
- Nyquist plots are a way of showing frequency responses of linear systems.
- There are several ways of displaying frequency response data, including Bode plots and Nyquist plots.
- **Bode plots** use frequency as the horizontal axis and use ***two separate plots*** to display gain and phase of the frequency response. This was covered earlier.
- **Nyquist plots** display both gain and phase angle on ***a single plot***, using ***frequency*** as a ***parameter*** in the plot. This will be covered in this lecture.
- Nyquist plots have properties that allow you to see whether a system is stable or unstable. They can also be used for designing various types of controllers.

Nyquist Plot

- Simply put, a **Nyquist plot** is basically a **polar plot** of the frequency response function of a linear system.

- That means a Nyquist plot is a plot of the transfer function, $G(s)$ with $s = j\omega$ (i.e., the **FRF**). Here, you will be plotting $G(j\omega)$.

- $G(j\omega)$ is a complex number for any angular frequency, ω. So, the plot is a plot of complex numbers.

- The complex number, $G(j\omega)$, depends upon frequency, so frequency will be a ***parameter*** if you plot the imaginary part of $G(j\omega)$ against the real part of $G(j\omega)$. Therefore, **Nyquist plot** is a graph of $\mathbf{Im}\{G(j\omega)\}$ vs. $\mathbf{Re}\{G(j\omega)\}$ in which $\boldsymbol{\omega}$ is the varying parameter.

How to sketch the Nyquist Plot of Frequency Response Function

- To sketch the Nyquist plot of $G(j\omega)$ for the entire range of frequency ω, i.e., from 0^+ to positive infinity, there are four key points that usually need to be known (**sometimes you will need more points!**):

  - **Key Point 1:** The start of plot where $\omega = 0$.
  - **Key Point 2:** The end of plot where $\omega = \infty$.
  - **Key Point 3:** Where the plot crosses the real axis, i.e., $\text{Im}(G(j\omega)) = 0$.
  - **Key Point 4:** Where the plot crosses the imaginary axis, i.e., $\text{Re}(G(j\omega)) = 0$.

- **Note for tests/exams/assignments:** In practice, using the above method sometimes does not provide you with enough number of key points to sketch more complicated FRF. For tests/exams/assignments (***unless you are specifically tasked with finding the above key points***), you can set up a table and plug in suitable number of arbitrary numerical values for ω and find the corresponding gain and phase values. Then, sketch the ***trend*** of the Nyquist plot.

Example 1: Nyquist Plot of First Order System

Consider a first order system $G(s) = \dfrac{1}{1+sT}$ where T is the time constant.

By representing $G(s)$ in the form of **frequency response function** $G(j\omega)$ and by replacing $s = j\omega$:

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{1+\omega^2T^2} - j\,\frac{\omega T}{1+\omega^2T^2}$$

The **magnitude** of $G(j\omega)$, i.e., $|G(j\omega)|$, is obtained as:

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2T^2}}$$

The **phase** of $G(j\omega)$, denoted by, $\angle G(j\omega)$, is obtained as:

$$\angle G(j\omega) = \tan^{-1}\left(\frac{-\frac{\omega T}{1+\omega^2T^2}}{\frac{1}{1+\omega^2T^2}}\right) \Rightarrow \angle G(j\omega) = -\arctan(\omega T) = -\tan^{-1}(\omega T)$$

Example 1 (cont'd): Nyquist Plot of First Order System

The start of plot where $\omega = 0^+$

$$|G(j\omega)| = \frac{1}{\sqrt{1+0}} = 1,$$

For $\omega = 0^+$: $\angle G(j.0^+) = -\tan^{-1}(0^+.T) = 0°$

The end of plot where $\omega = +\infty$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\infty}} = 0,$$

For $\omega = +\infty$: $\angle G(j.(+\infty)) = -\tan^{-1}(+\infty.T) = -90°$

The mid part of plot where $\omega = 1/T$

$$|G(j\omega)| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}},$$

For $\omega = \frac{1}{T}$: $\angle G\left(j.\frac{1}{T}\right) = -\tan^{-1}(\frac{1}{T}.T) = -\tan^{-1}(1) = -45°$

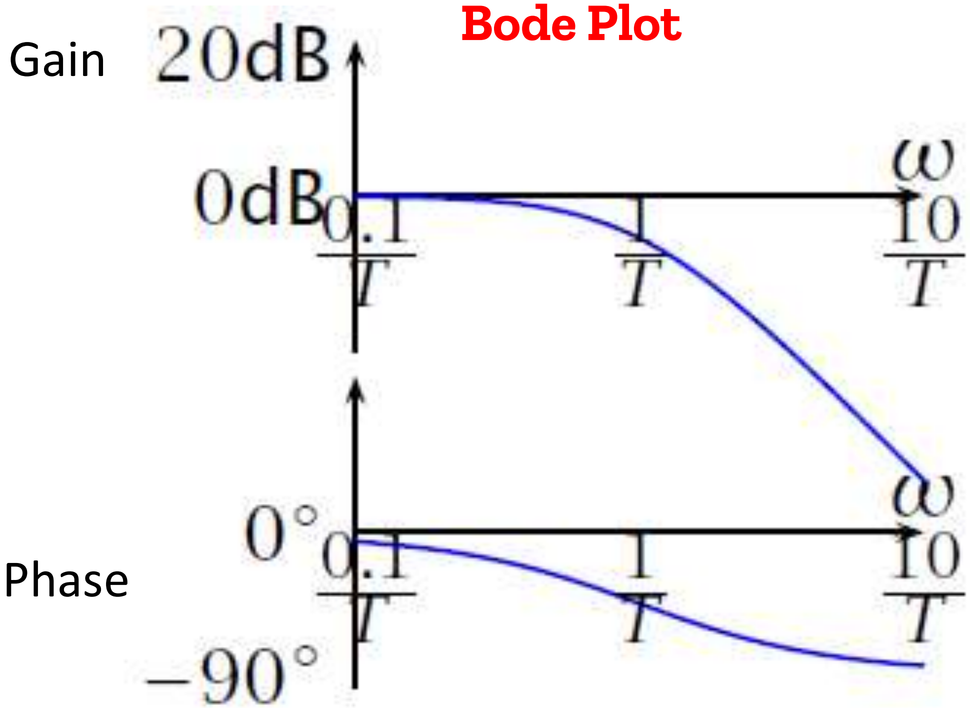
Example 1 (cont'd): Nyquist Plot of First Order System

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
$\omega = 0^+$	1	$0°$
$\omega = \frac{1}{T}$	$\frac{1}{\sqrt{2}}$	$-45°$
$\omega = +\infty$	0	$-90°$

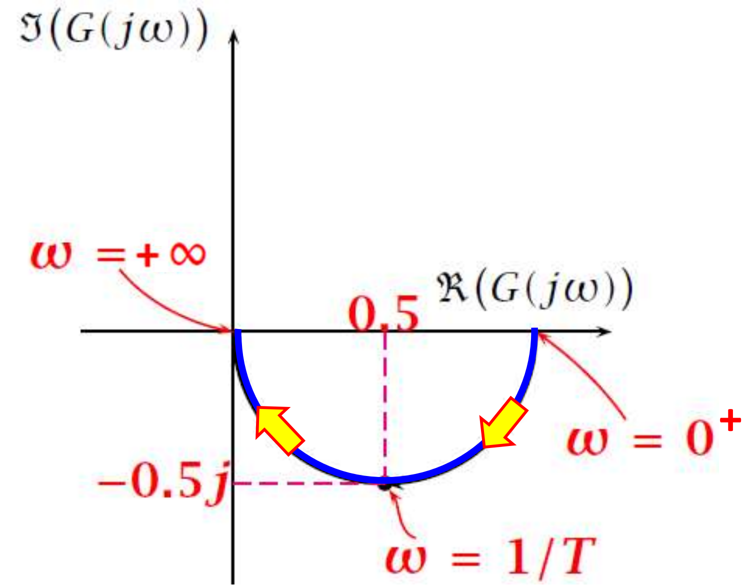
ω	Re + Im.j
0.0100	0.9999 - 0.0100 j
0.1000	0.9901 - 0.0990 j
1.0000	0.5000 - 0.5000 j
2.0000	0.2000 - 0.4000 j
3.0000	0.1000 - 0.3000 j
5.0000	0.0385 - 0.1923 j
10.0000	0.0099 - 0.0990 j
100.0000	0.0001 - 0.0100 j

For $T = 1$

Bode Plot



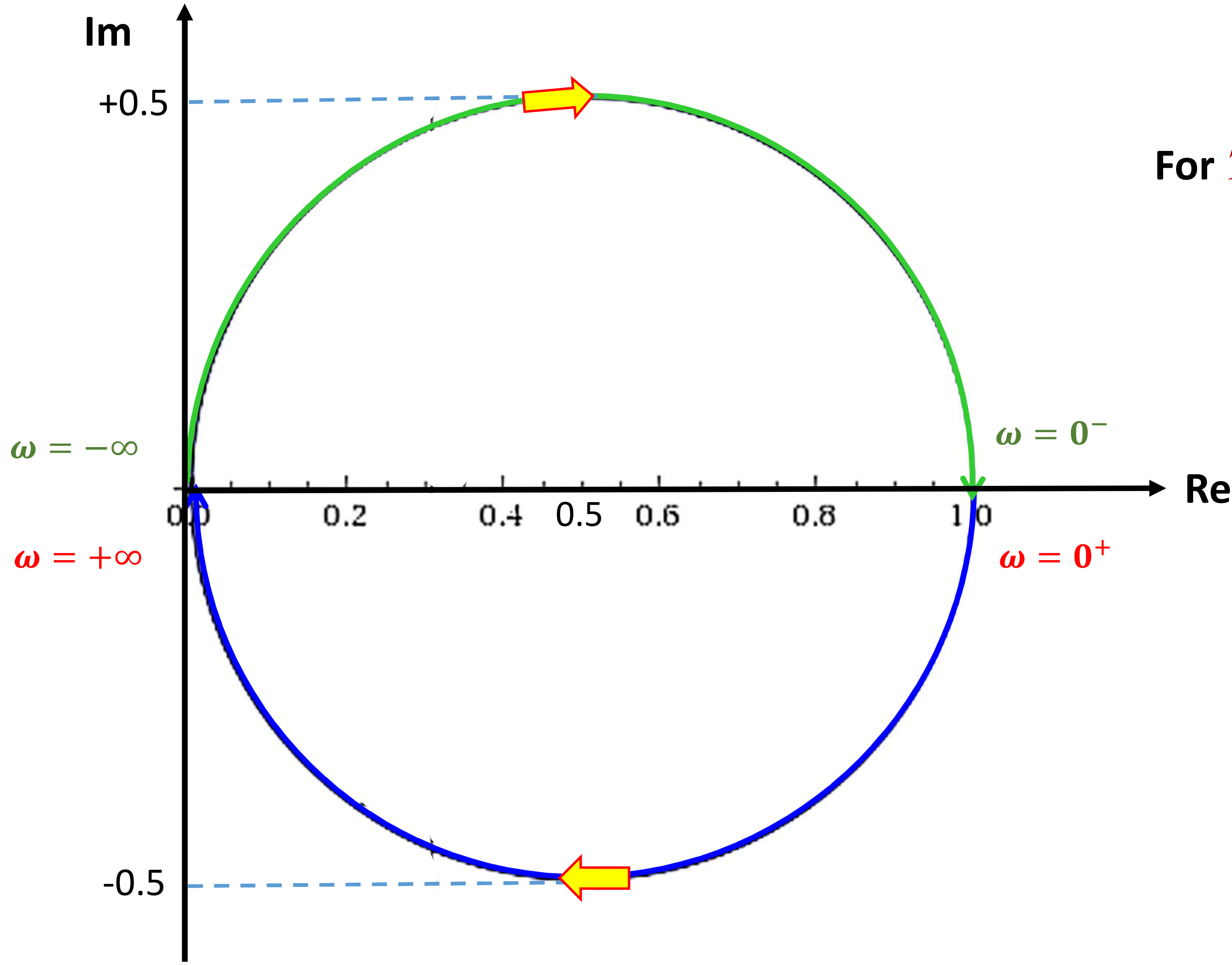
Nyquist Plot



Note: You can use either i or j.

Example 1 (cont'd): Nyquist Plot of First Order System

Nyquist Plot:



Example 2: Nyquist Plot of Integrator

Consider an integrator system, $G(s) = \frac{1}{s}$

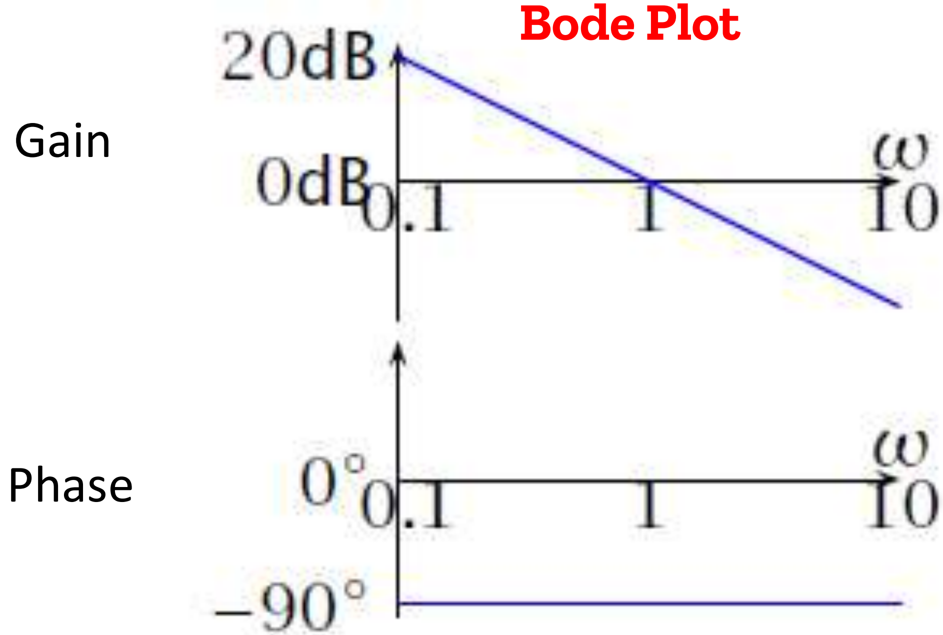
By representing $G(s)$ in the form of **frequency response function** $G(j\omega)$ and by replacing $s = j\omega$:

$$G(j\omega) = \frac{1}{j\omega} = \frac{-j}{\omega}$$

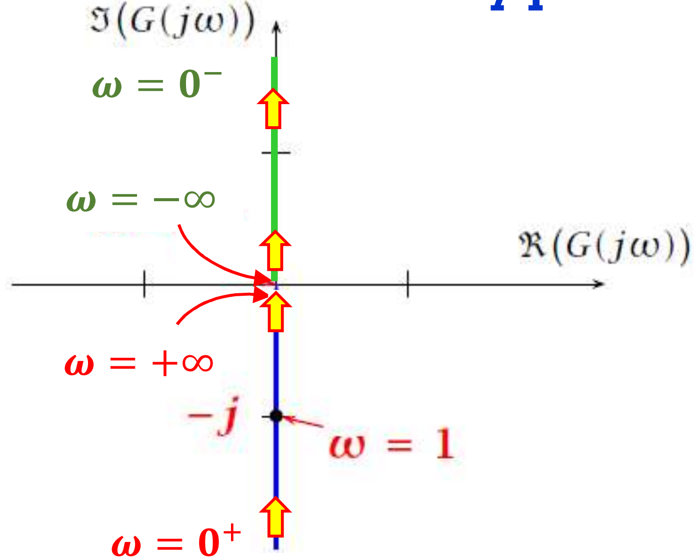
The **magnitude** of $G(j\omega)$, i.e., $|G(j\omega)|$, is obtained as: $|G(j\omega)| = 1/\omega$

The **phase** of $G(j\omega)$, denoted by, $\angle G(j\omega)$, is obtained as: $\angle G(j\omega) = -90°$

Bode Plot



Nyquist Plot

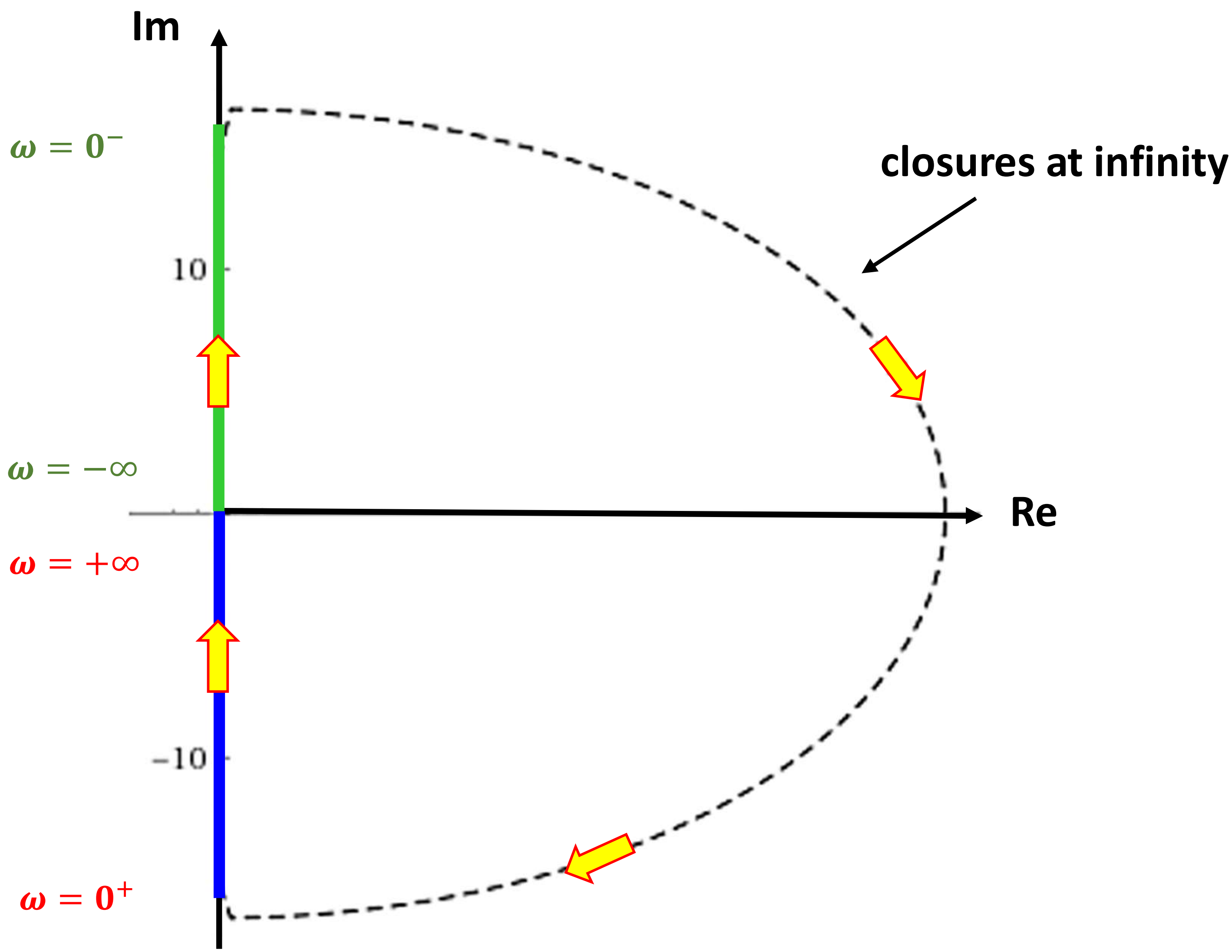


Example 2 (cont'd): Nyquist Plot of Integrator

$$G(s) = \frac{1}{s}$$

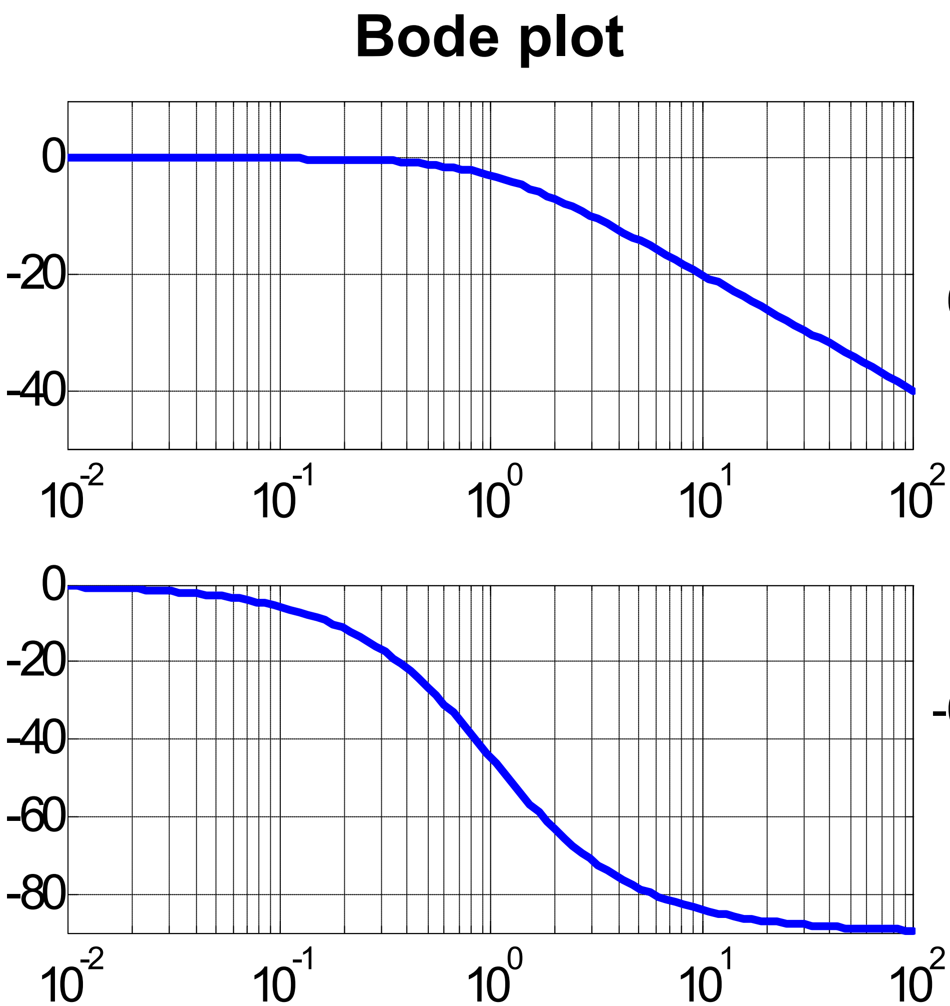
Nyquist Plot:

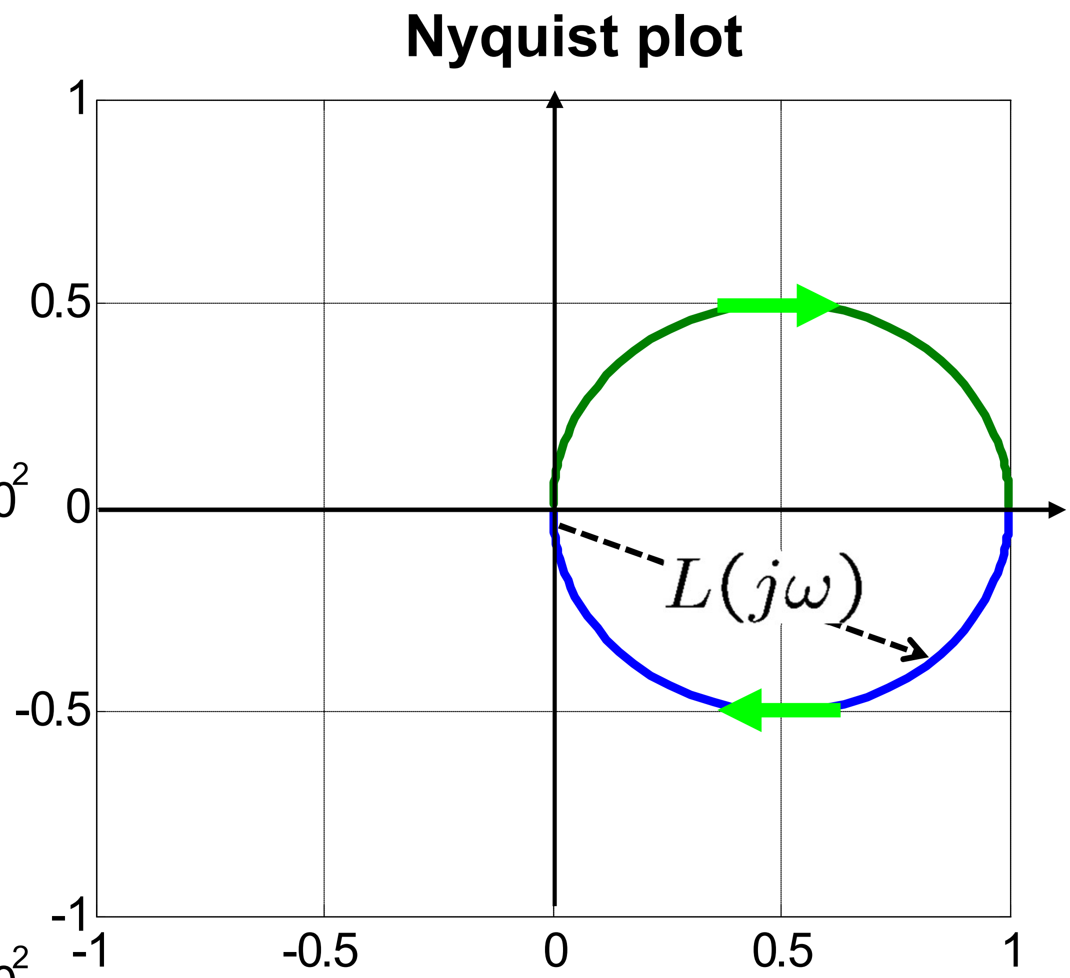
If the points of $\omega = 0^-$ and $\omega = 0^+$ are not at the same point, then connect them with a curve with an arbitrary large radius that lies in the 1st and 4th quadrant. That is, in order to obtain a complete Nyquist plot, we use **closures at infinity**. It means you connect the aforementioned points with an ***infinite radius***. By so doing, we generate a ***closed Nyquist plot***.



Example 3: (Bode & Nyquist plots)

- First order system $L(s) = \dfrac{1}{s+1}$



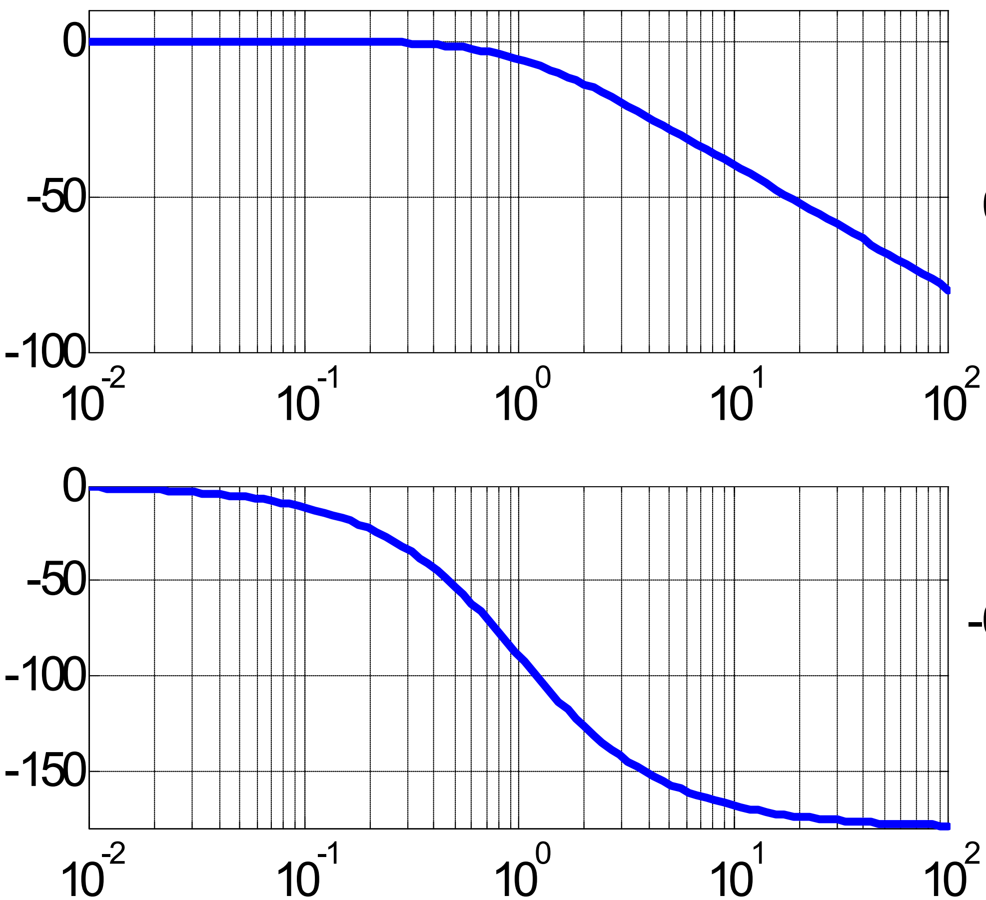


$$L(j\omega) = \frac{1}{1+\omega^2} - j\,\frac{\omega}{1+\omega^2}$$

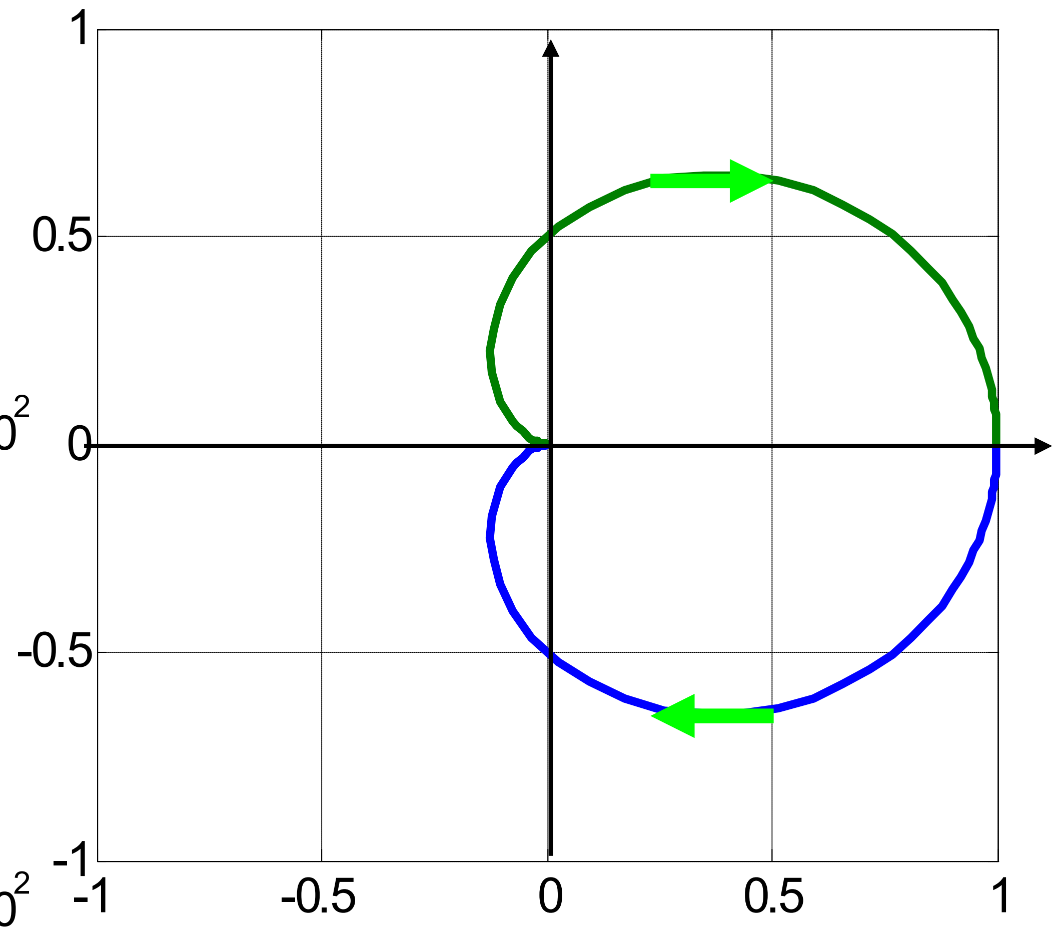
Example 4: (Bode & Nyquist plots)

- Second order system $L(s) = \dfrac{1}{(s+1)^2}$

Bode plot



Nyquist plot

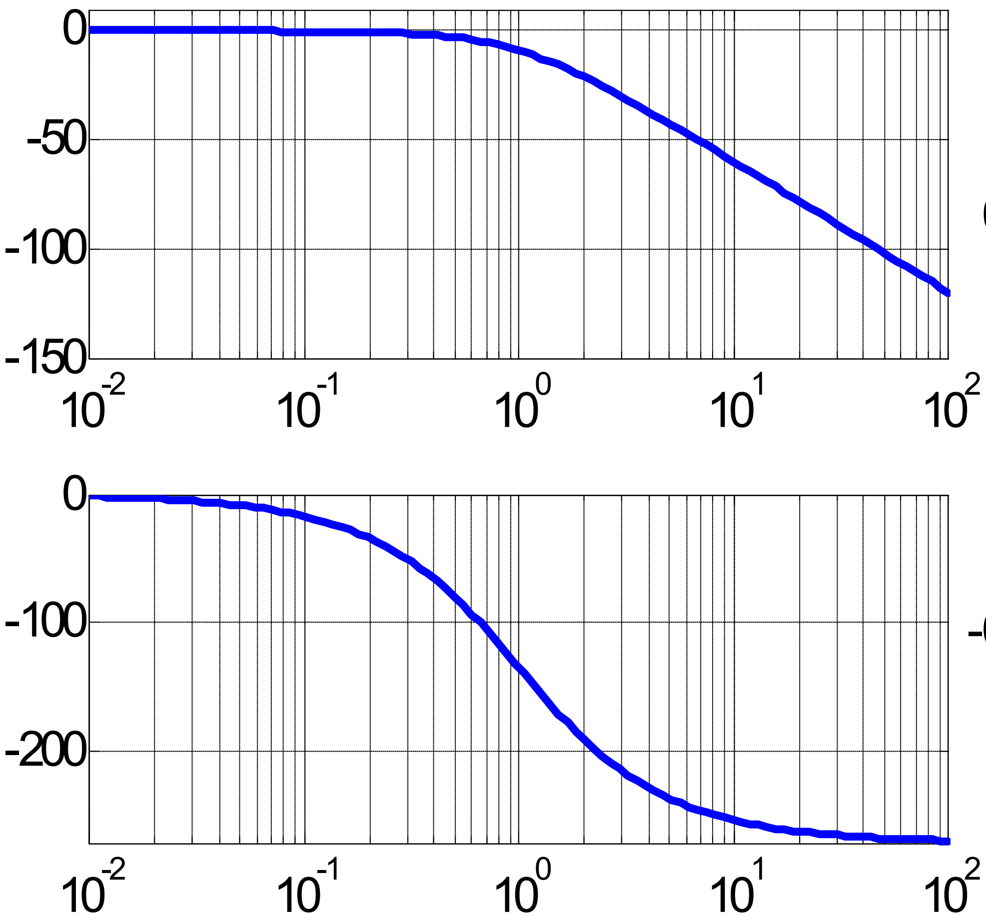


$$L(j\omega) = \frac{1-\omega^2}{1+\omega^4+2\omega^2} - j\frac{2\omega}{1+\omega^4+2\omega^2}$$

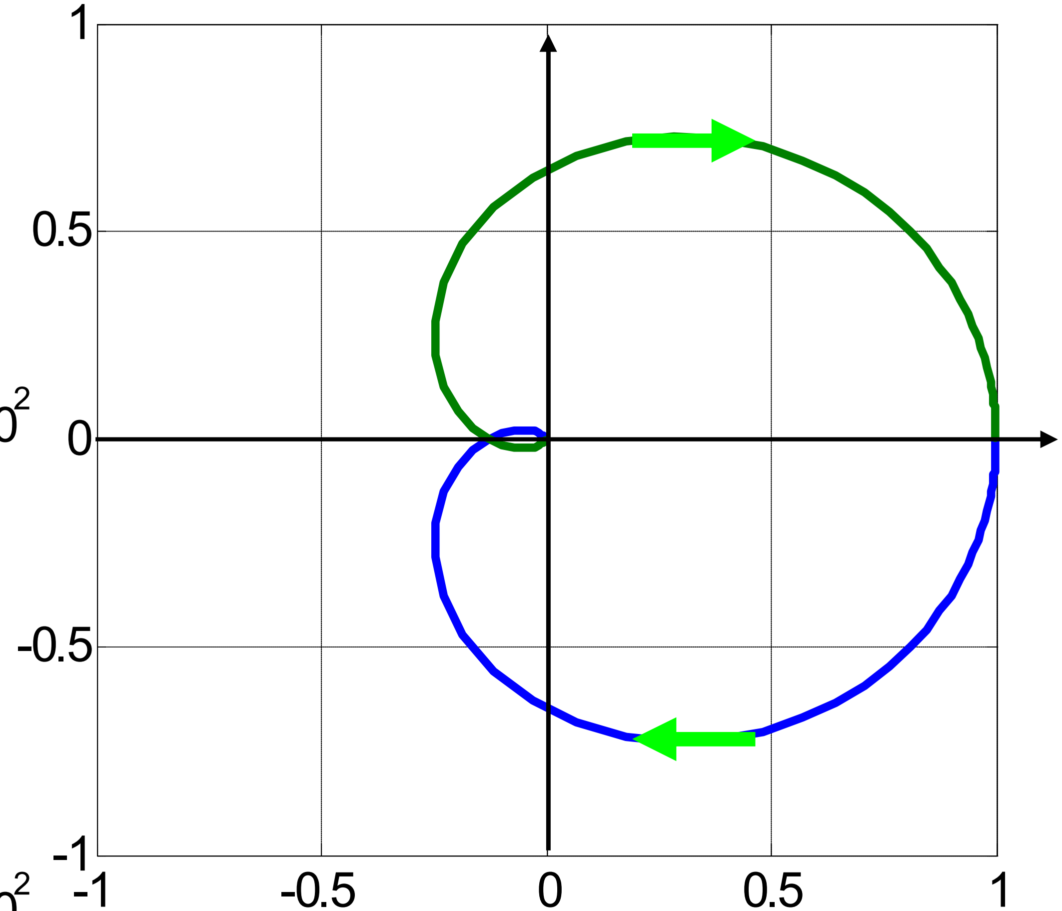
Example 5: (Bode & Nyquist plots)

- Third order system $L(s) = \dfrac{1}{(s+1)^3}$

Bode plot



Nyquist plot



$$L(j\omega) = \frac{1-3\omega^2}{1+\omega^6+3\omega^2+3\omega^4} + j\,\frac{\omega^3-3\omega}{1+\omega^6+3\omega^2+3\omega^4}$$

Nyquist stability criterion

$$\text{CL system is stable} \Leftrightarrow Z = P + N = 0$$

Z: # of CL poles in open RHP

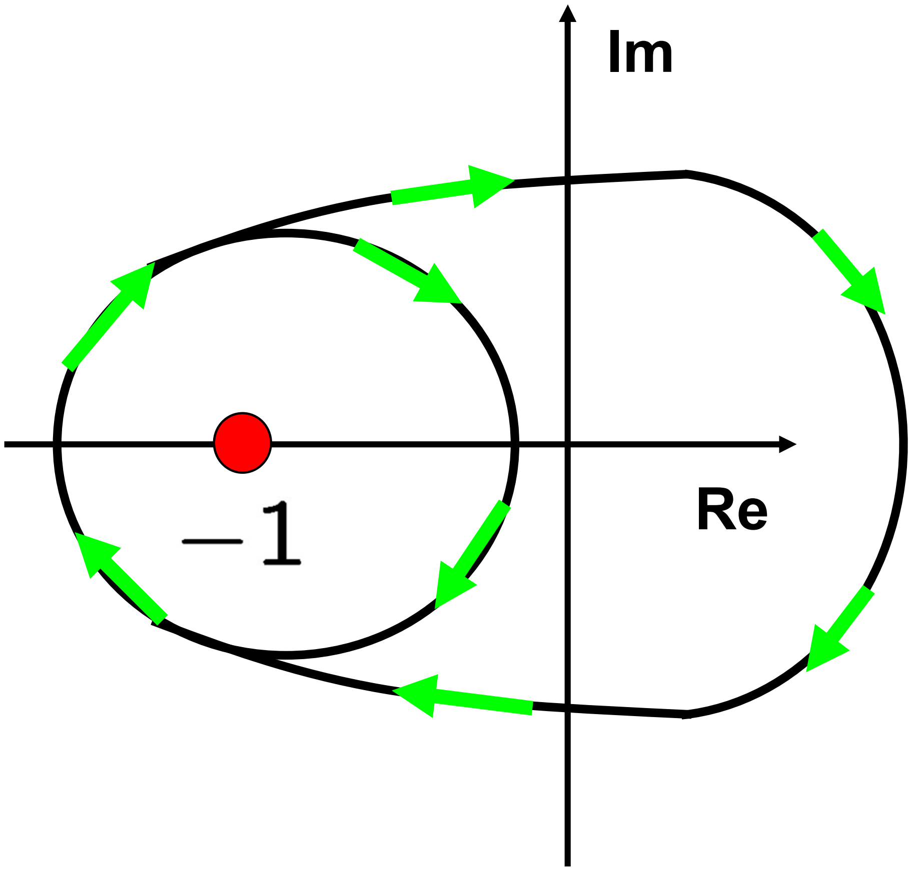
P: # of OL poles in open RHP (given)

N: # of clockwise/counterclockwise encirclement of -1 by Nyquist plot of OL transfer function $L(s)$ (counted by using Nyquist plot of $L(s)$)

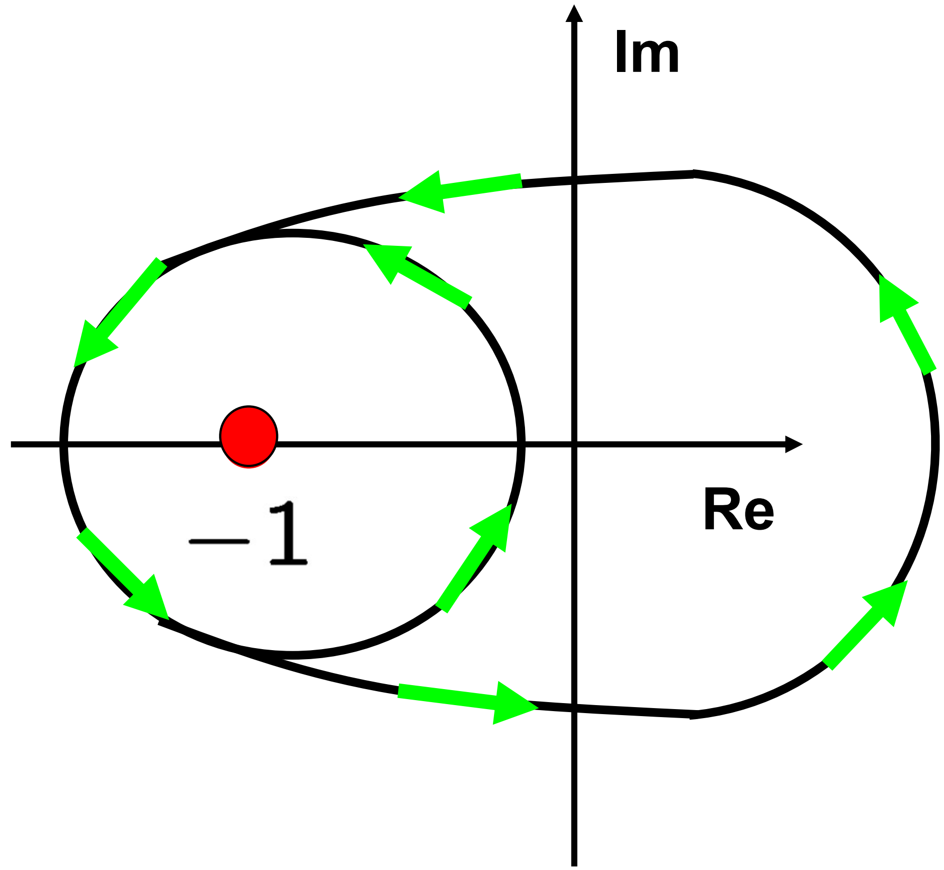
Remark: A ***negative value*** for N means a ***counterclockwise*** encirclement. For example, N = -2 means we encircle point -1 twice and in counterclockwise direction.

Encirclements in Nyquist plot

- Clockwise

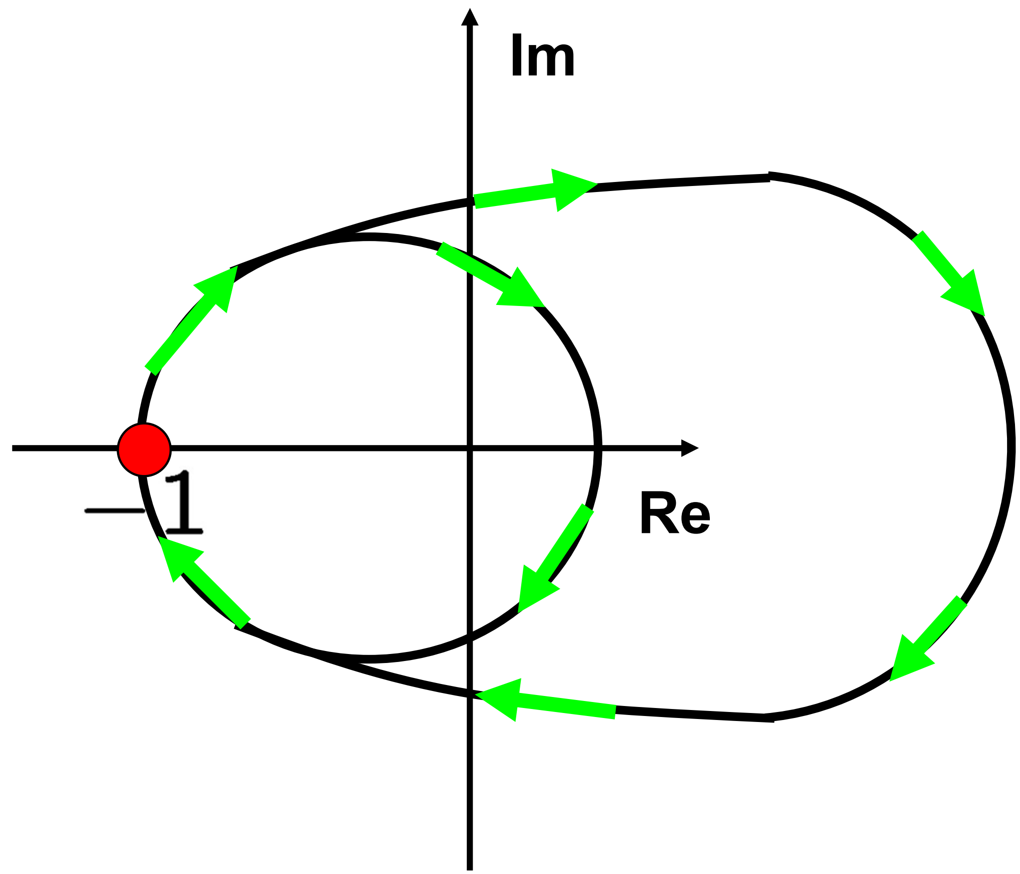


- Counter-clockwise

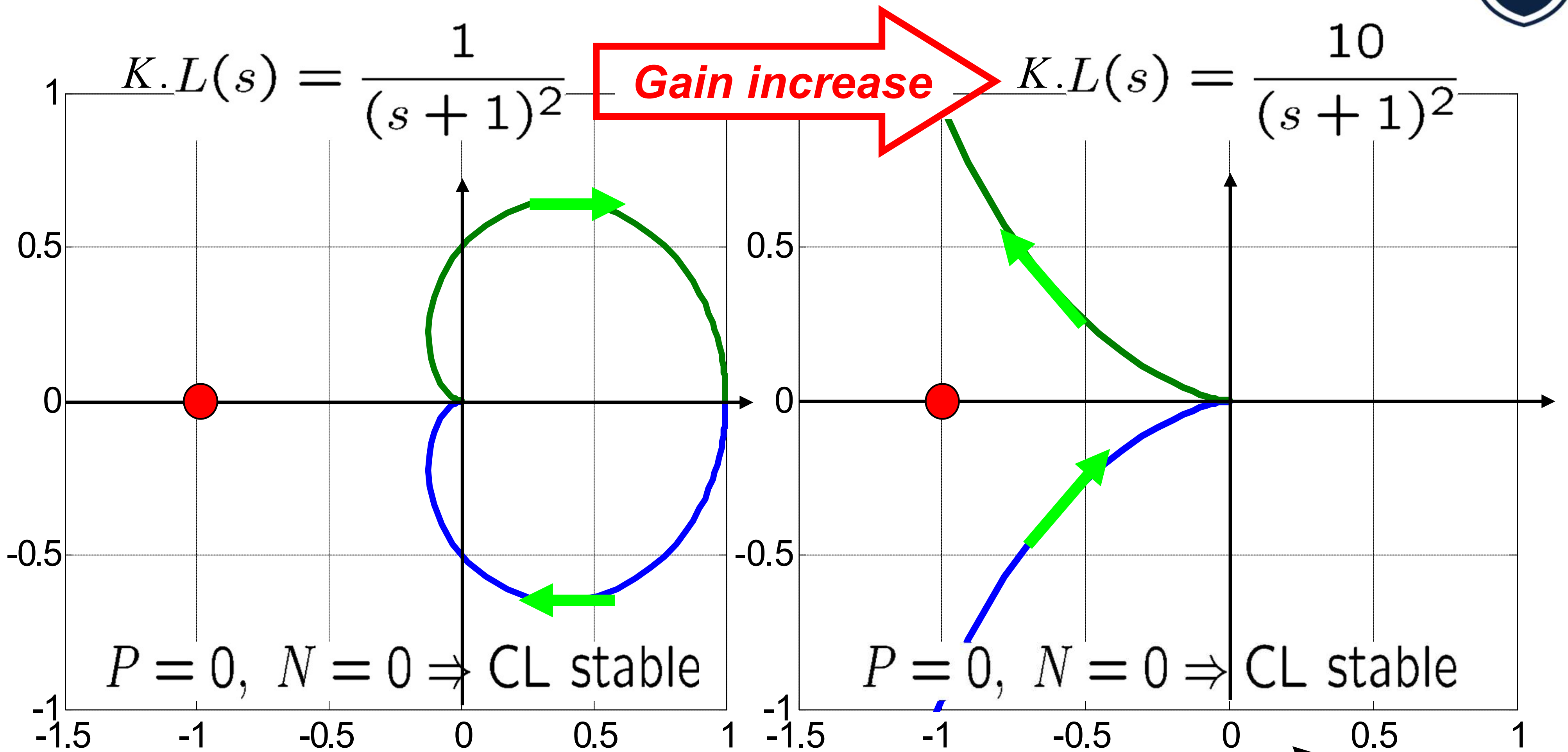


Remark

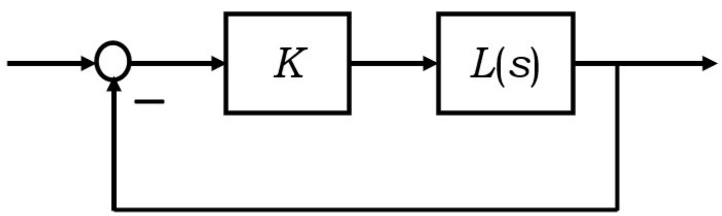
- If Nyquist plot passes the point **-1**, it means that the closed-loop system has a pole on the imaginary axis (and thus, marginally stable).



Example 6: (for 2nd order $L(s)$)

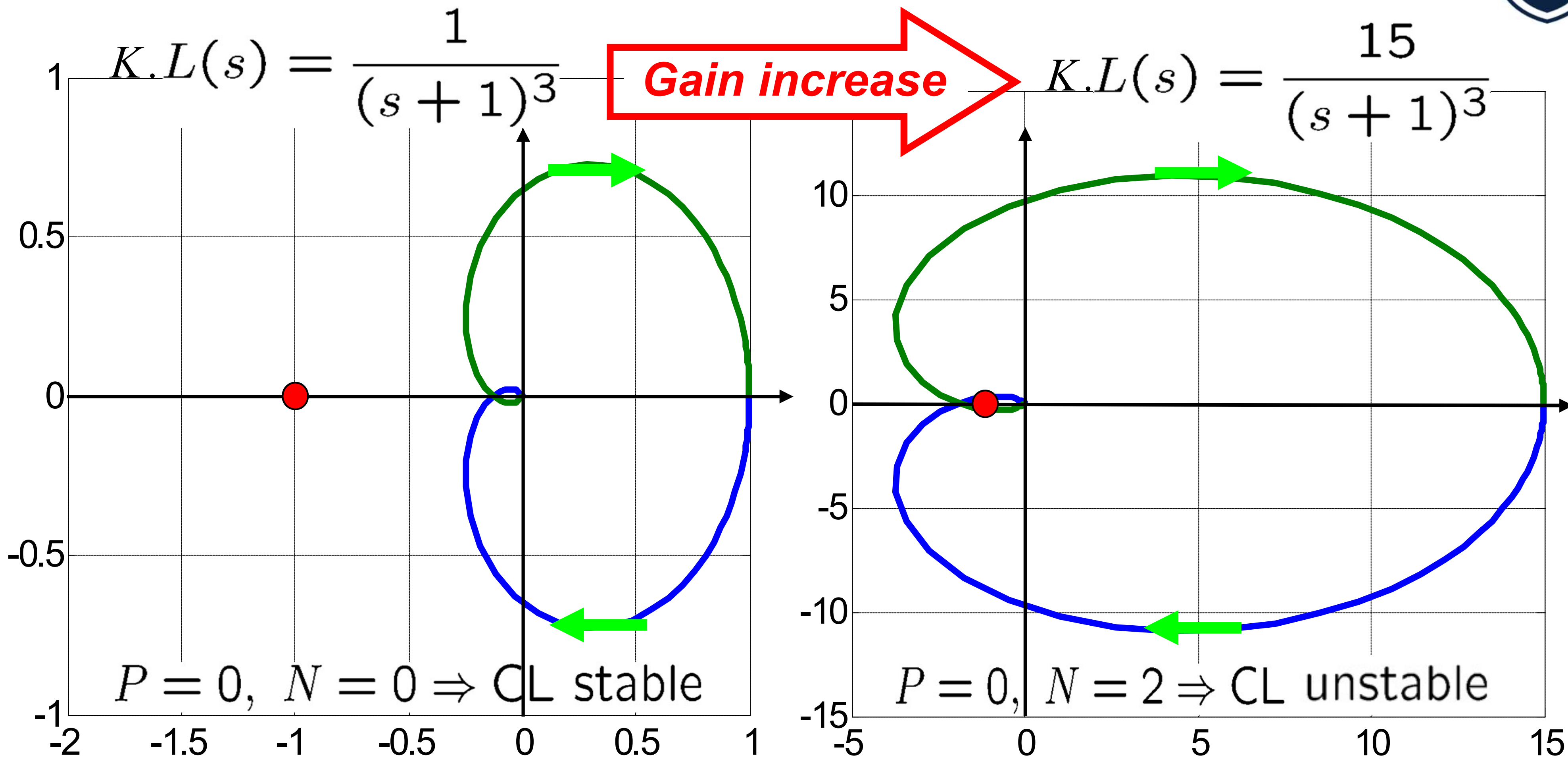


$$\begin{cases} L(s) = \frac{1}{(s+1)^2} \\ K = 1 \end{cases}$$

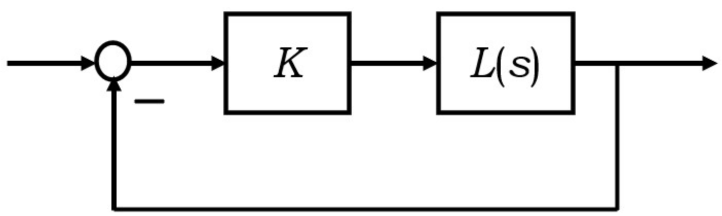


$$\begin{cases} L(s) = \frac{1}{(s+1)^2} \\ K = 10 \end{cases}$$

Example 7: (for 3rd order $L(s)$)



$$L(s) = \frac{1}{(s+1)^3}$$

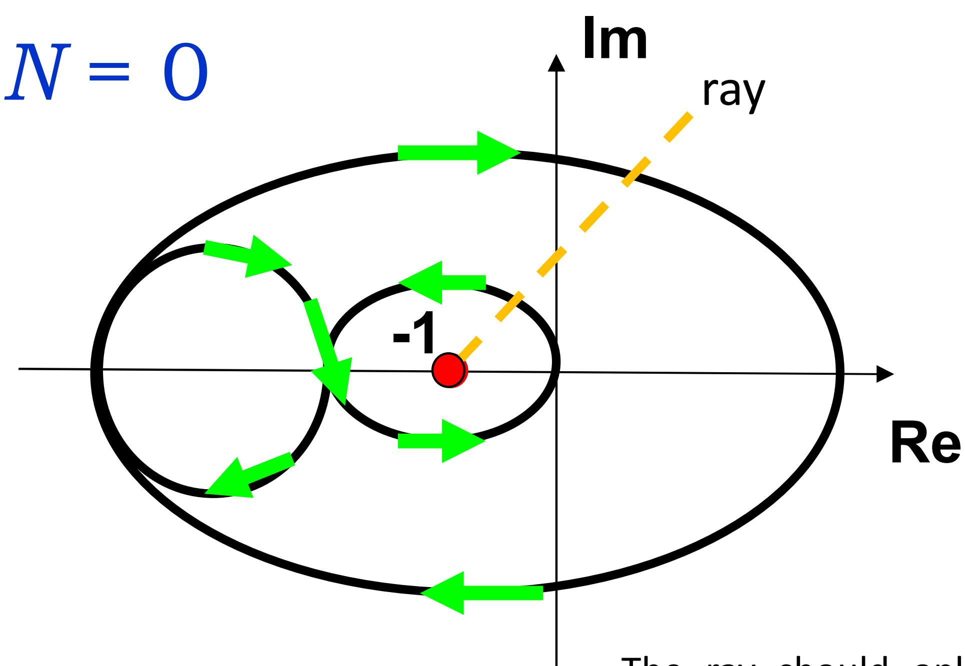


How to count # of encirclement (N)

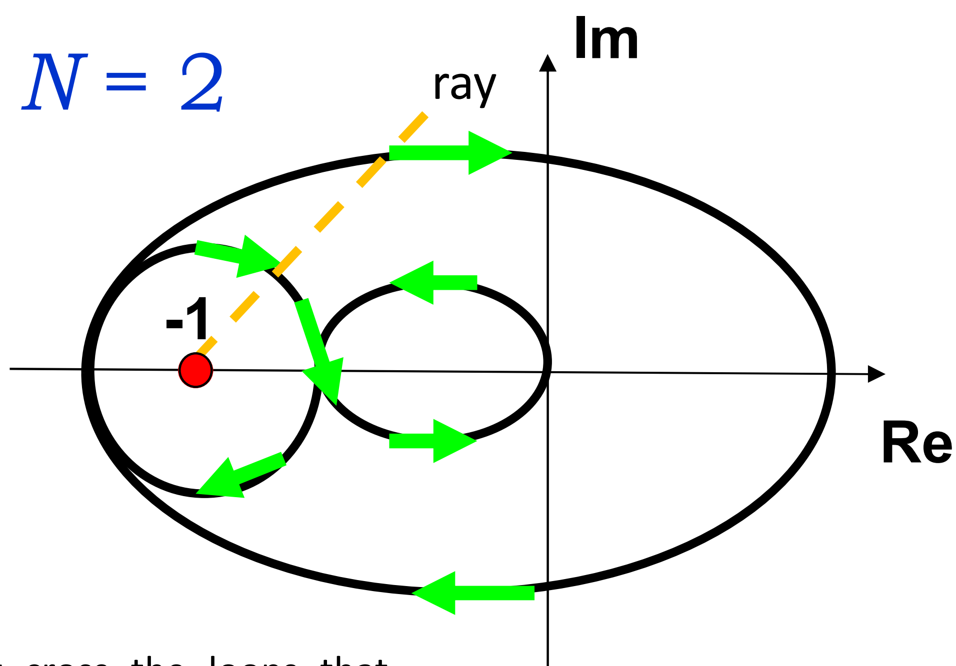
- A ray is drawn from **-1** point in any convenient direction. Then,

$$N = (\text{\# of crossing of ray by Nyquist plot in clockwise direction}) - (\text{\# of crossing of ray by Nyquist plot in counterclockwise direction})$$

$N = 0$



$N = 2$



The ray should only cross the loops that somehow encircle the "**-1**" point and not the irrelevant loops (i.e., non-encircling ones).

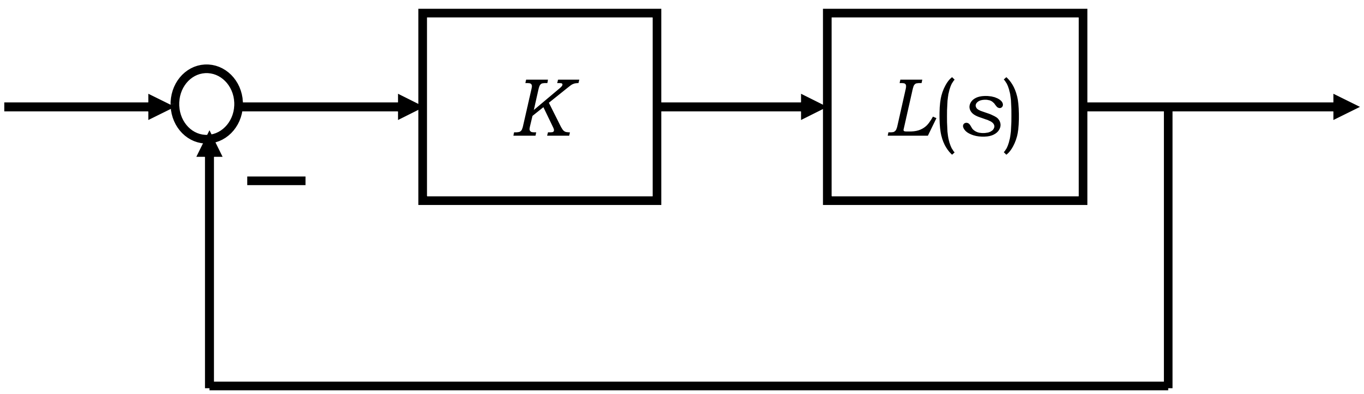
Notes on Nyquist stability criterion

- Nyquist stability criterion allows us to determine the stability of CL system from knowledge of the OL system.
- It can deal with time delay (next lecture), which Routh-Hurwitz criterion cannot.
- You can draw only half of Nyquist plot and then draw the other half using the notion that it will be the mirror image with respect to the real axis.
  - ➢ **Important Note:** For determining CL system stability, you should always draw ***the whole*** **Nyquist plot**.
- Nyquist plot in MATLAB: nyquist(sys).

Example 8: (for unstable $L(s)$)

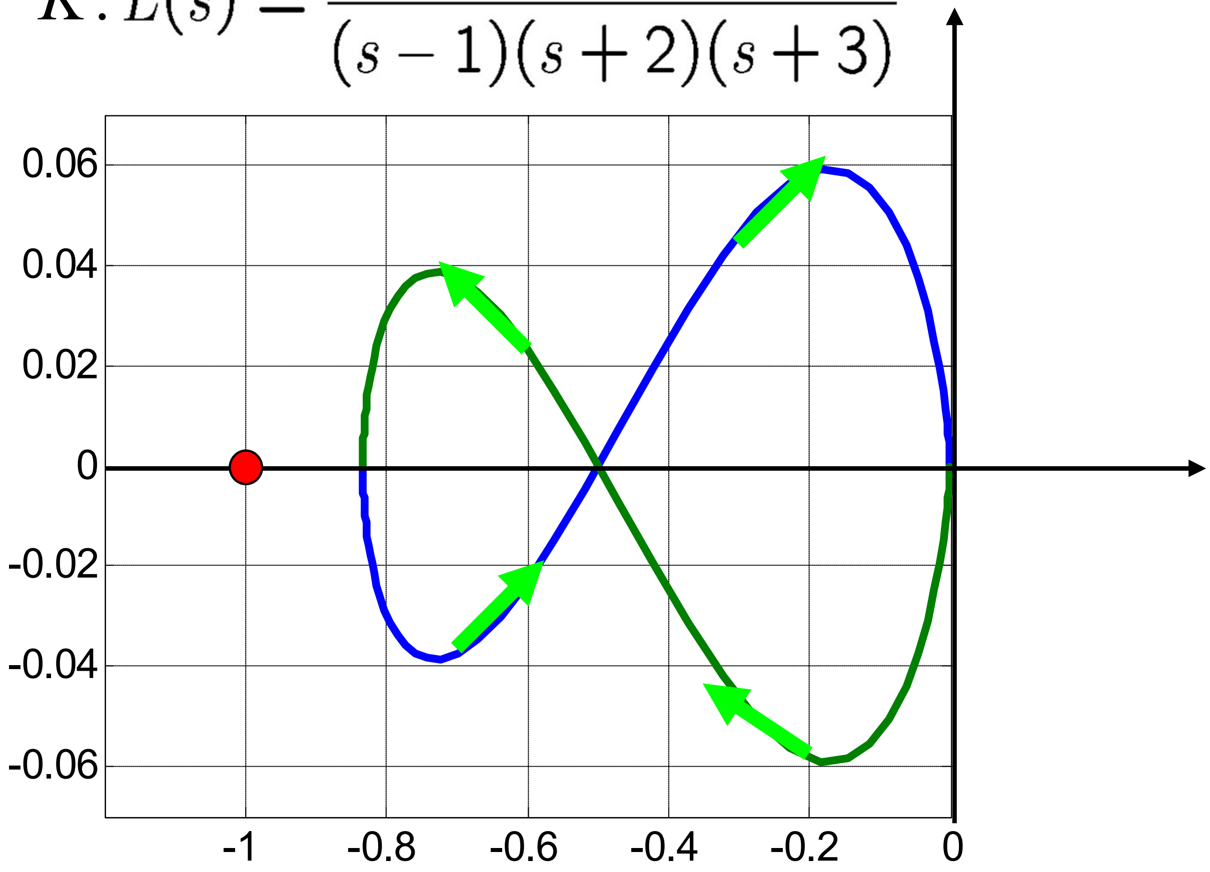
Using the numerical values of 5, 8, and 11 for the gain of a proportional controller (K), investigate the stability of the following closed-loop system for the given OLTF. Use Nyquist criterion.

$$L(s) = \frac{1}{(s-1)(s+2)(s+3)}$$



Example 8: (for unstable $L(s)$, cont'd)

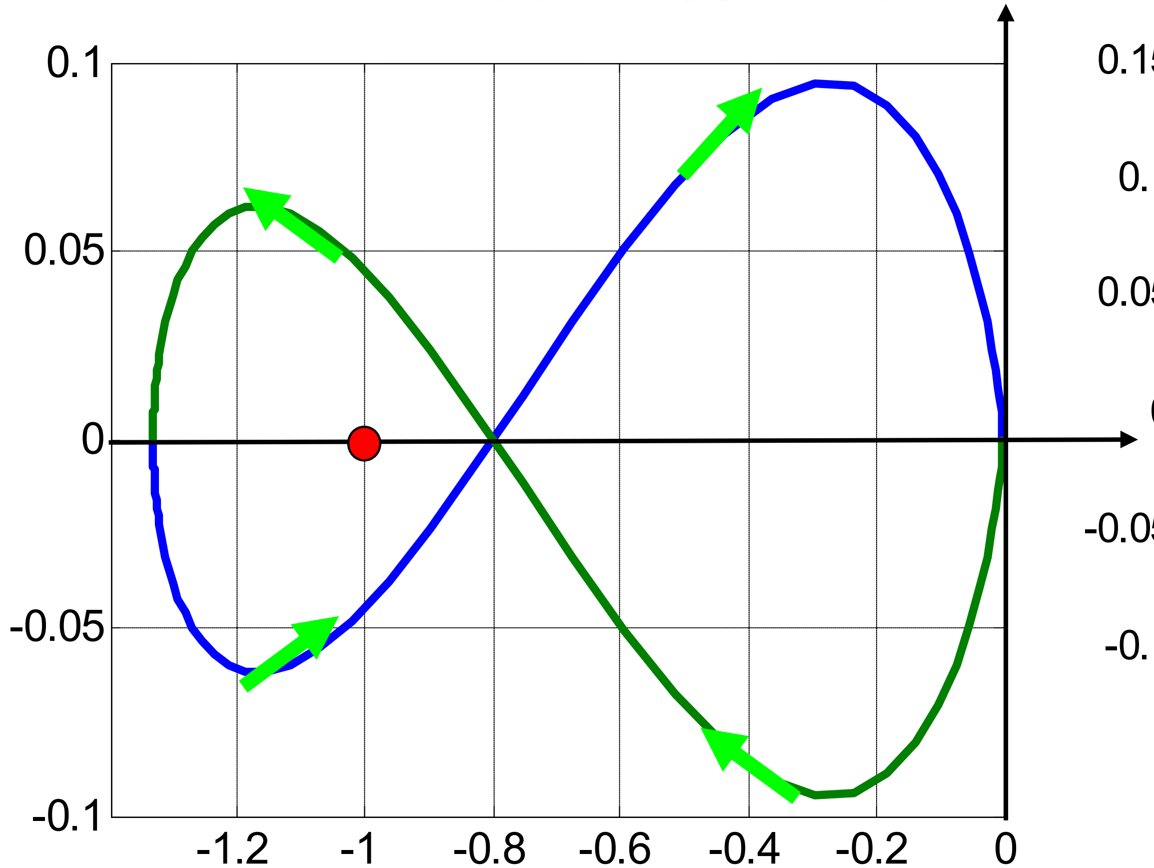
$$K.L(s) = \frac{5}{(s-1)(s+2)(s+3)}$$



$P = 1,\ N = 0 \Rightarrow$ CL unstable

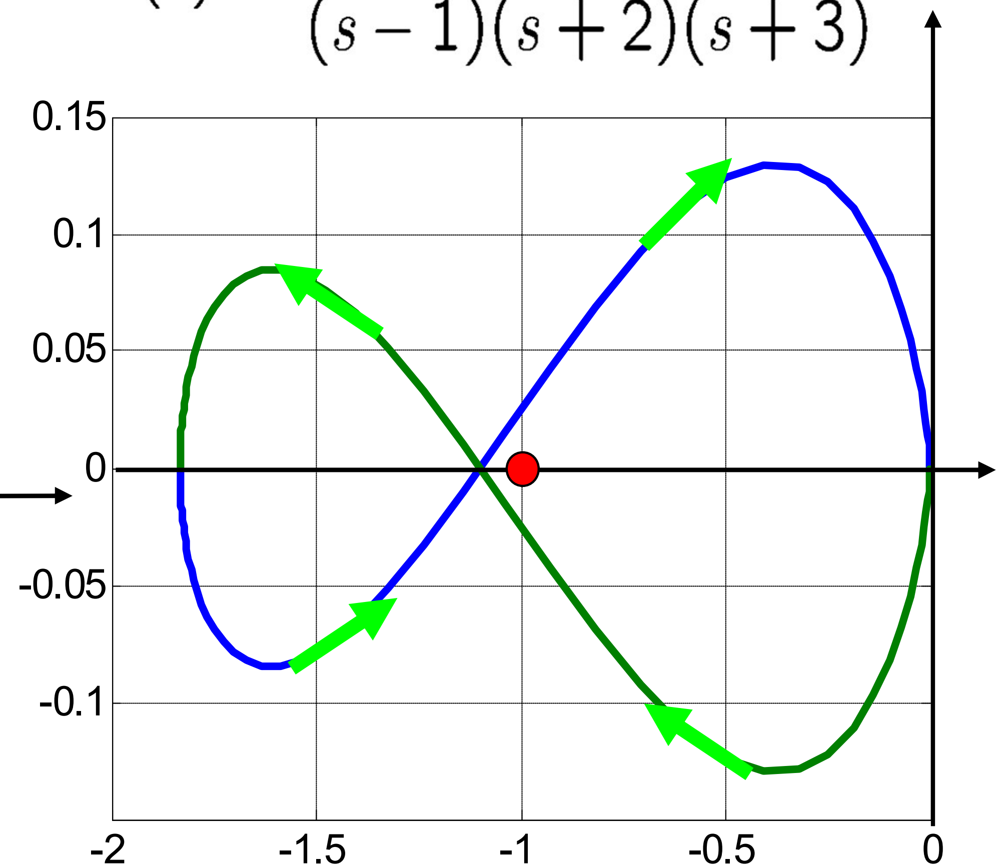
Example 8: (for unstable $L(s)$, cont'd)

$$K.L(s) = \frac{8}{(s-1)(s+2)(s+3)}$$



$P = 1,\ N = -1 \Rightarrow$ CL stable

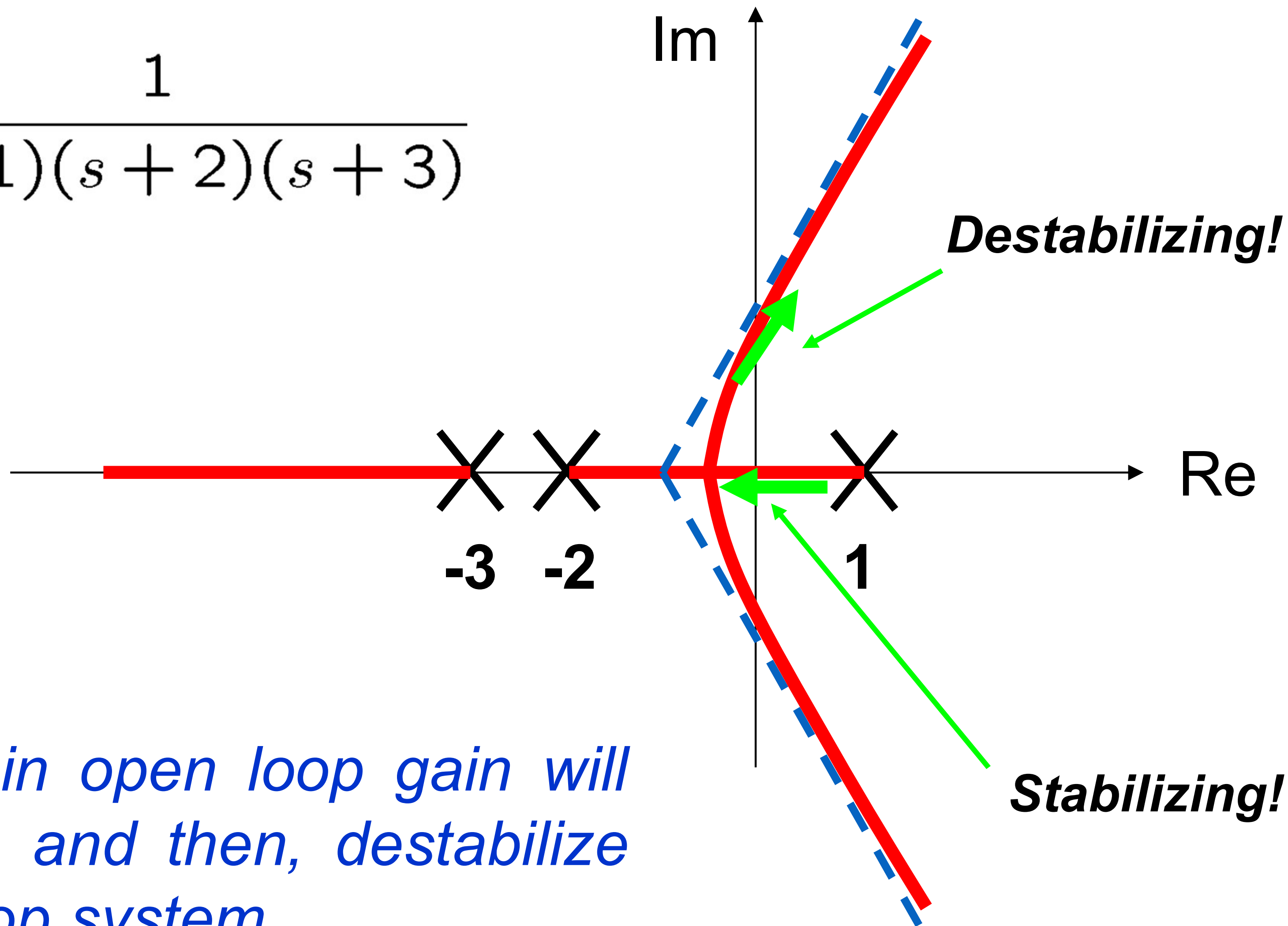
$$K.L(s) = \frac{11}{(s-1)(s+2)(s+3)}$$



$P = 1,\ N = 1 \Rightarrow$ CL unstable

Example 8: (for unstable $L(s)$, cont'd), Interpretation by root locus

$$L(s) = \frac{1}{(s-1)(s+2)(s+3)}$$



An increase in open loop gain will first stabilize, and then, destabilize the closed-loop system.

Nyquist criterion: A special case

$$\text{CL system is stable} \Leftrightarrow Z = P + N = 0$$

- If $P = 0$ (i.e., if $L(s)$ has no pole in open RHP)

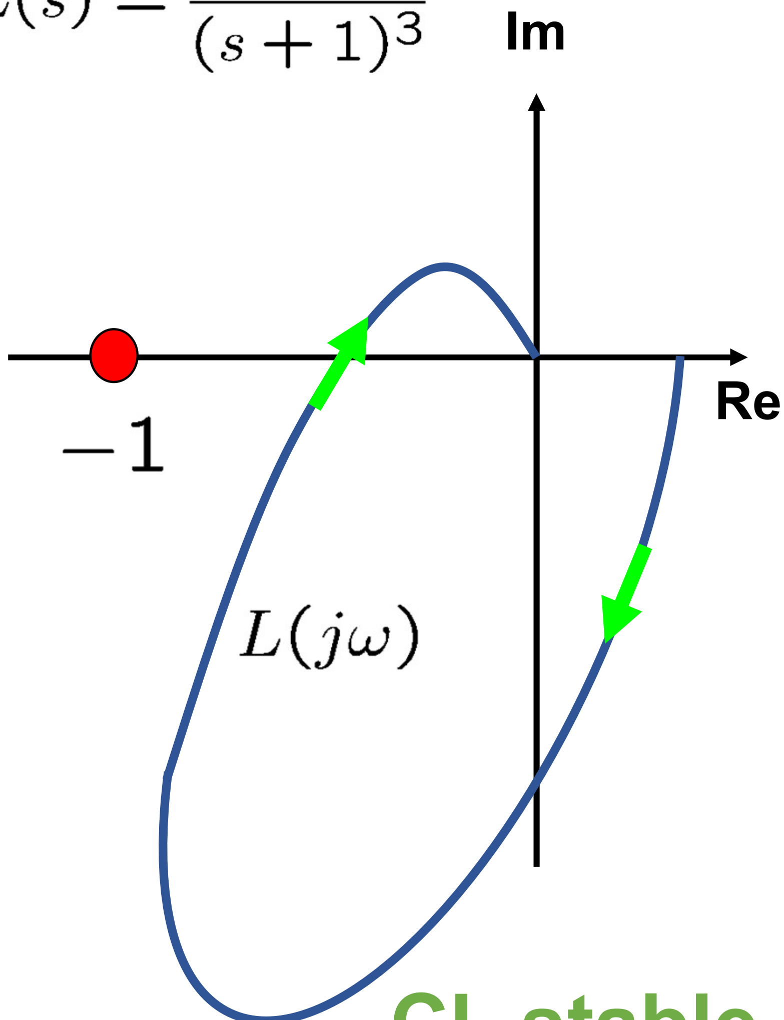
$$\text{CL system is stable} \Leftrightarrow N = 0$$

↑

This fact is important since open-loop systems in real-life engineering problems usually have no pole in open RHP!

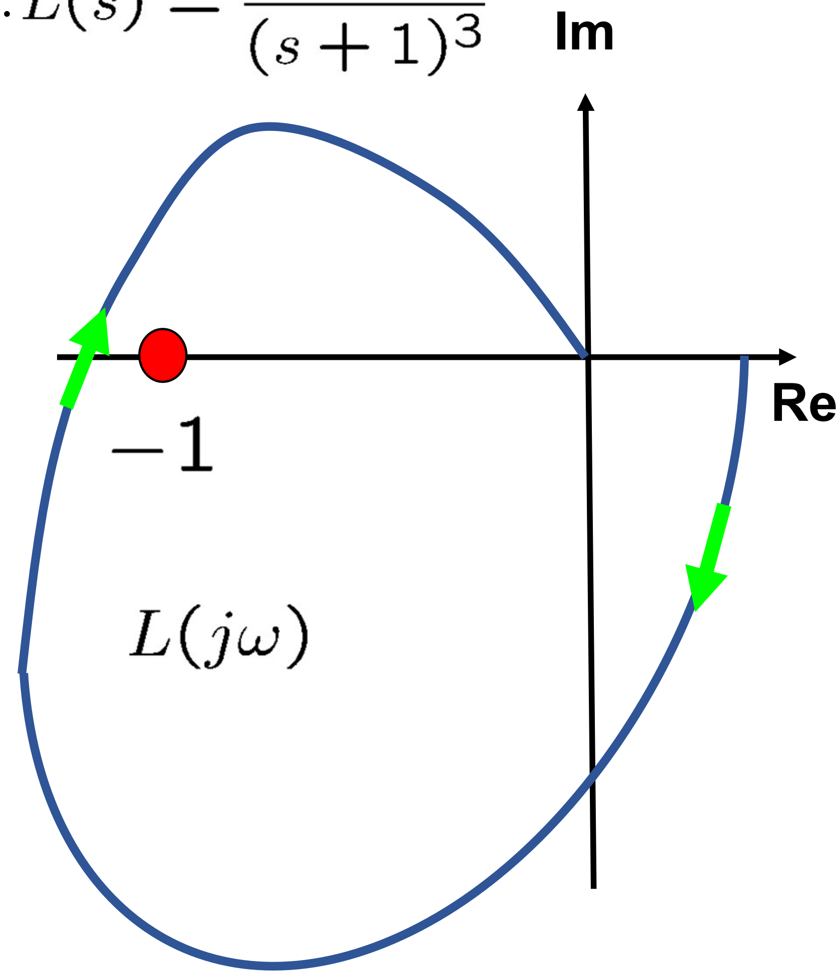
Example 9: (when $P = 0$)

$$K\,.\,L(s) = \frac{1}{(s+1)^3}$$



CL stable

$$K\,.\,L(s) = \frac{15}{(s+1)^3}$$

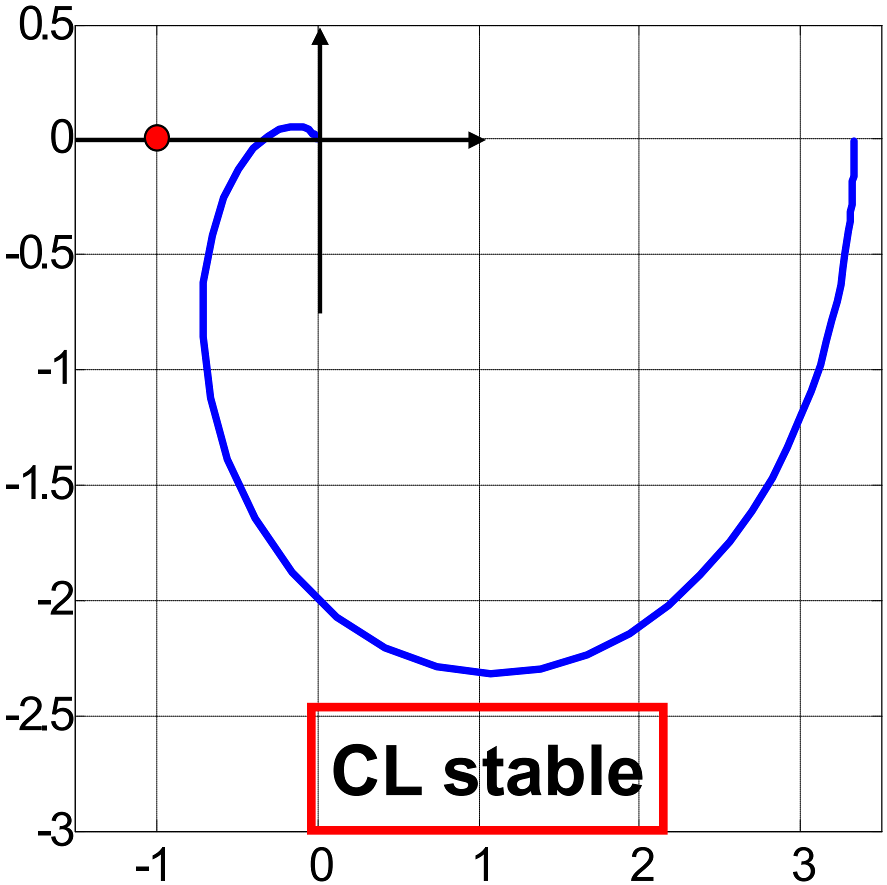


CL unstable

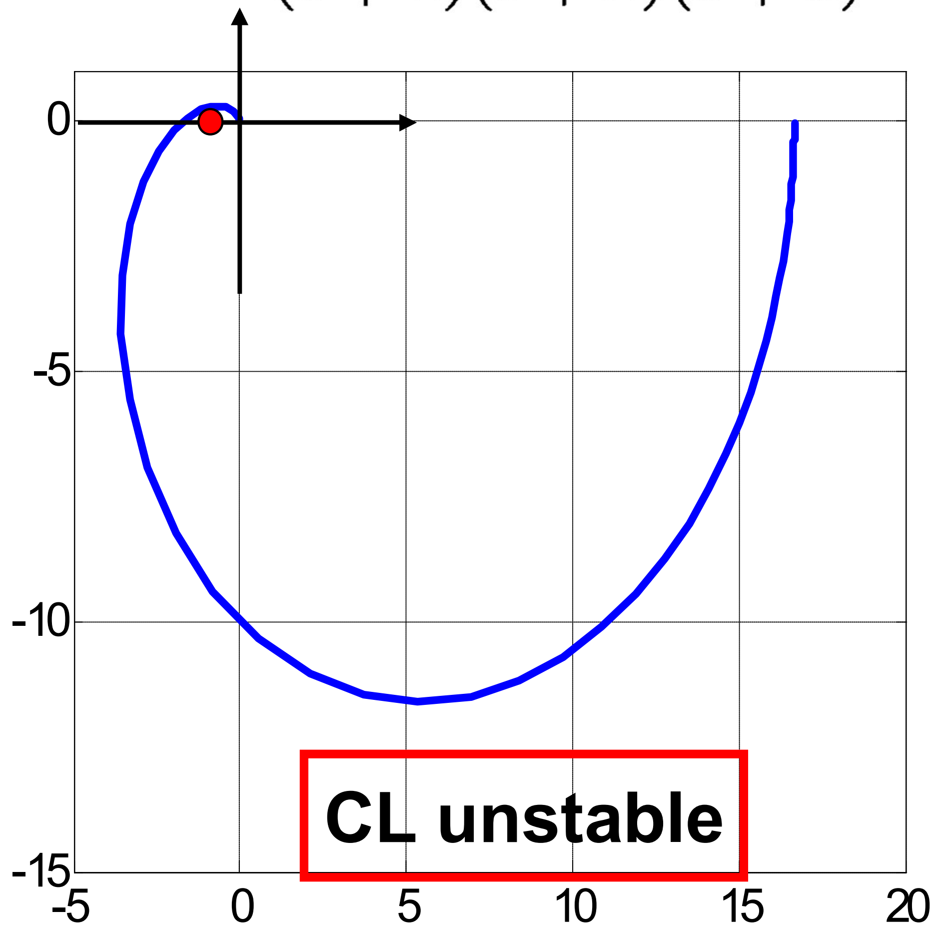
Note: Sometimes, engineers draw only half of the Nyquist plot (for simplicity or to save space). For investigating, stability, you should always draw ***the whole Nyquist plot***.

Example 10: (for stable $L(s)$, cont'd)

$$K.L(s) = \frac{20}{(s+1)(s+2)(s+3)}$$

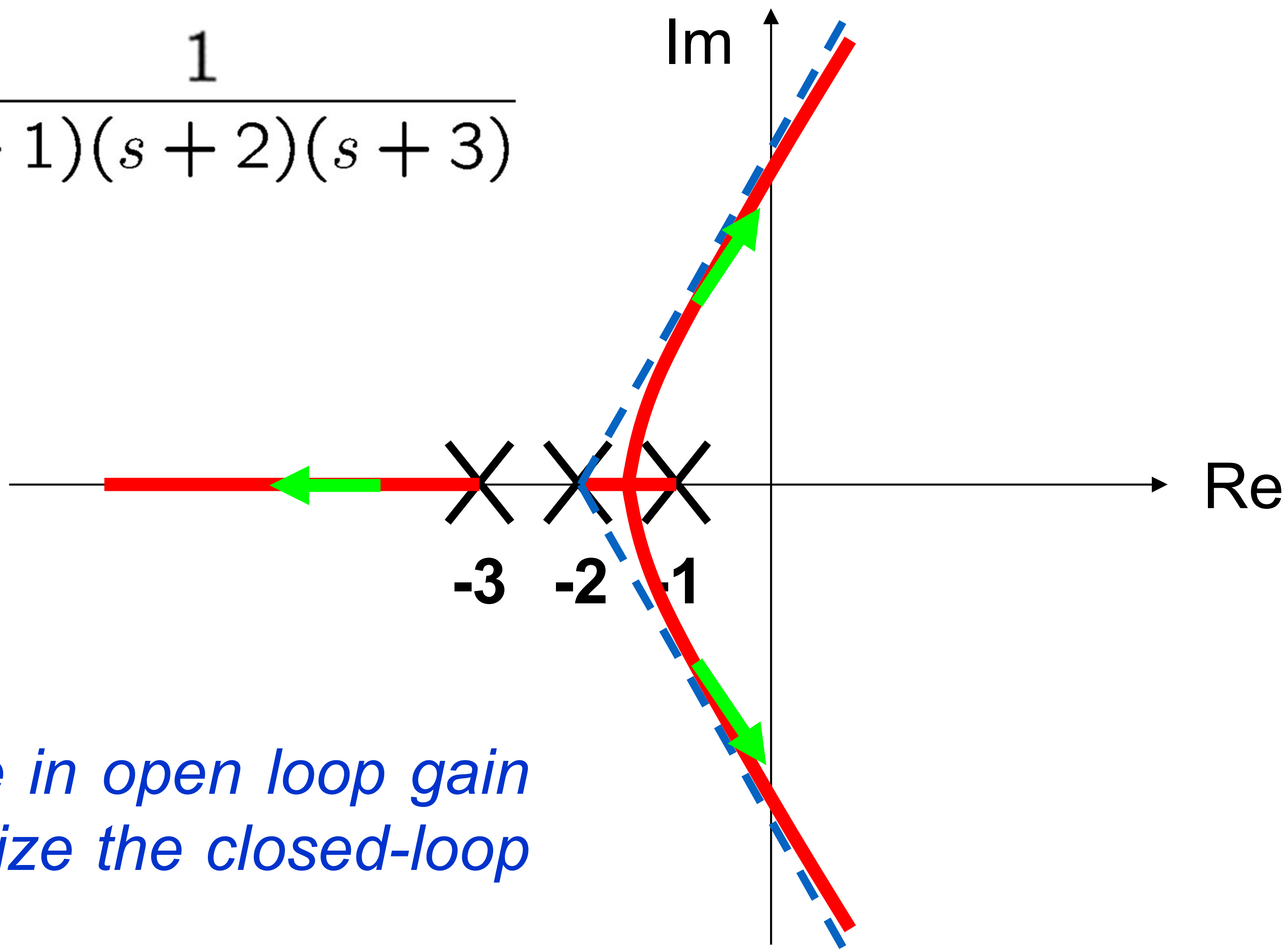


$$K.L(s) = \frac{100}{(s+1)(s+2)(s+3)}$$



Example 10: (Interpretation by root locus, cont'd)

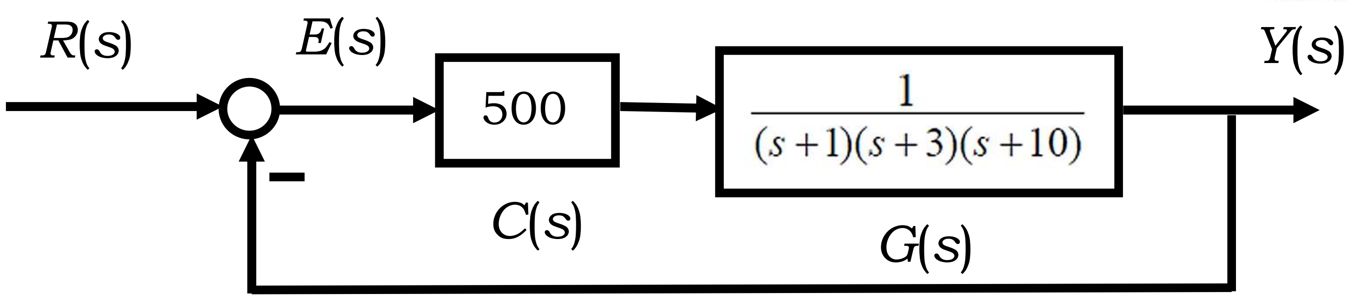
$$L(s) = \frac{1}{(s+1)(s+2)(s+3)}$$



An increase in open loop gain will destabilize the closed-loop system.

Example 11: Stability using the Nyquist criterion

The following block diagram is given. **(a)** Find the 4 key points for sketching the Nyquist plot. **(b)** Find FRF in the form of a complex number for various values of frequency, i.e., in the form of $a(\omega) \pm b(\omega)j$. **(c)** Sketch the Nyquist plot, and then determine the system stability using the Nyquist criterion.



(a)

The open-loop transfer function: $C(s)G(s) = \dfrac{500}{(s+1)(s+3)(s+10)}$.

Replacing s with $j\omega$ yields the frequency response of $C(s)G(s)$, i.e.,

$$C(j\omega)G(j\omega) = \frac{500}{(j\omega+1)(j\omega+3)(j\omega+10)} = \frac{500}{(-14\omega^2+30)+j(43\omega-\omega^3)} \rightarrow$$

$$\rightarrow C(j\omega)G(j\omega) = \frac{500(-14\omega^2+30)}{(-14\omega^2+30)^2+(43\omega-\omega^3)^2} + j\left[\frac{-500(43\omega-\omega^3)}{(-14\omega^2+30)^2+(43\omega-\omega^3)^2}\right] \quad \textbf{(1)}$$

Example 11 (cont'd)

Magnitude response:

$$|C(j\omega)G(j\omega)| = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = \frac{500}{\sqrt{(-14\omega^2+30)^2 + (43\omega - \omega^3)^2}}$$

Phase response:

$$\angle C(j\omega)G(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = \tan^{-1}\left(\frac{-\left(43\omega - \omega^3\right)}{-14\omega^2 + 30}\right)$$

Now that when we have expressions for the magnitude and phase of the frequency response, we can sketch the polar plot using the **4 key points**.

Key Point 1: The start of plot where $\omega = 0$

$$|C(0)G(0)| = \frac{500}{\sqrt{(30)^2}} = 16.67$$

$$\angle C(0)G(0) = \tan^{-1}\frac{0}{30} = 0^\circ$$

Example 11 (cont'd)

Key Point 2: The end of plot where $\omega = \infty$

$$|C(\infty)G(\infty)| = \frac{500}{\sqrt{\infty}} = 0$$
$$\angle C(\infty)G(\infty) = \tan^{-1}\infty = -270°$$

Key Point 3: Where the plot crosses the real axis, i.e., $\mathrm{Im}\{C(j\omega)G(j\omega)\} = 0$

Take the imaginary part of equation **(1)**, and set it equal to zero, to get the value of frequency ω at the interception of real axis:

$$\frac{-(43\omega-\omega^3)}{(-14\omega^2+30)^2+(43\omega-\omega^3)^2} = 0 \Rightarrow \begin{cases} 43\omega-\omega^3 = 0 \quad \Rightarrow \quad \omega = 0 \quad \text{and} \quad \omega = 6.56\ \text{rad/s} \\ \omega = \infty \end{cases}$$

Example 11 (cont'd)

Key Point 4: Where the plot crosses the imaginary axis, $\mathrm{Re}\{C(j\omega)G(j\omega)\} = 0$

Take the real part of equation **(1)**, and set it equal to zero, to get the value of frequency ω at the interception of imaginary axis:

$$\frac{-14\omega^2+30}{(-14\omega^2+30)^2+(43\omega-\omega^3)^2}=0 \Rightarrow \begin{cases} -14\omega^2+30=0 & \Rightarrow \quad \omega=1.46\,\text{rad/s} \\ \omega=\infty \end{cases}$$

(b)

ω	Re + Im.j
0	16.6667 + 0.0000 j
1.0000	3.9604 -10.3960 j
1.4600	0.0221 - 8.3797 j
2.0000	-1.9231 - 5.7692 j
6.5600	-0.8734 + 0.0003 j
20.0000	-0.0340 + 0.0435 j
73.0000	-0.0002 + 0.0012 j

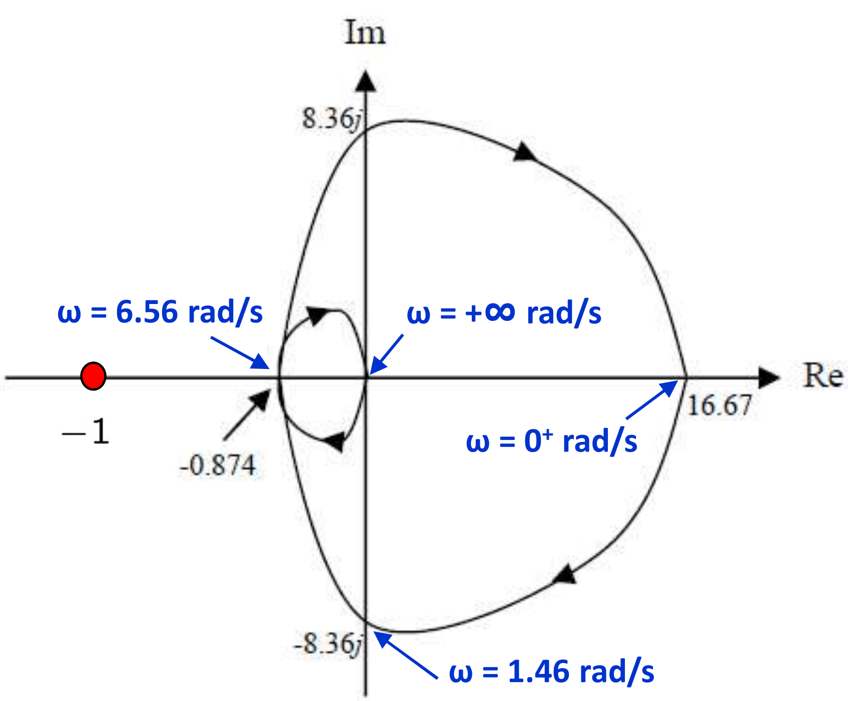
Example 11 (cont'd)

(c)

Key Points of the polar plot:

	$\lvert C(j\omega)G(j\omega)\rvert$	$\angle C(j\omega)G(j\omega)$
$\omega = 0$ **Start Point**	16.67	0°
$\omega = \infty$ **End Point**	0	-270°
Re Crossing:		
$\omega = 0$	16.67	0°
$\omega = \infty$	0	-270°
$\omega = 6.56$ rad/s	0.874	-180°
Im Crossing:		
$\omega = \infty$	0	-270°
$\omega = 1.46$ rad/s	8.36	-90°

Nyquist Plot



$P = 0,\ N = 0$ (no encirclements of -1). So based on $Z = P + N = 0$ (**CL** is **stable**).

Summary

- Nyquist stability criterion for feedback stability
- Examples for Nyquist stability criterion
- Next
  - Relative stability