a place of mind

ELEC 341: Systems and Control

Lecture 19

Nyquist stability criterion: Examples on relative stability

Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- Models for systems
  - ✓ Electrical
  - ✓ Electromechanical
  - ✓ Mechanical
- ✓ Linearization, delay

Analysis

- ✓ Stability
  - ✓ Routh-Hurwitz
  - ➤ Nyquist
- ✓ Time response
  - ✓ Transient
  - ✓ Steady state
- ✓ Frequency response
  - ✓ Bode plot

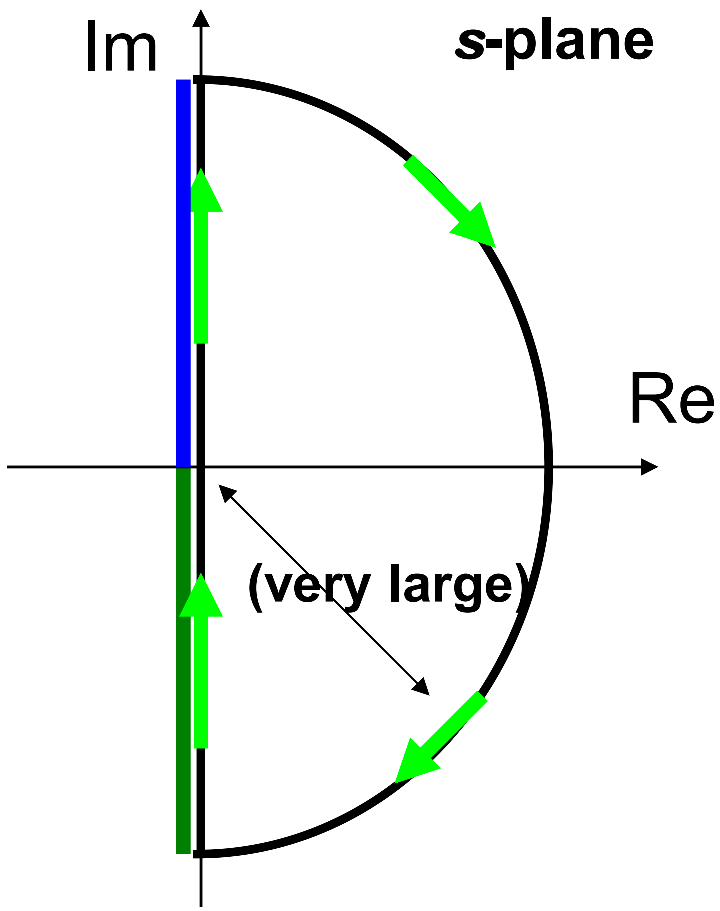
Design

- Design specs
- ✓ Root locus
- Frequency domain
- ✓ PID & Lead-lag
- Design examples

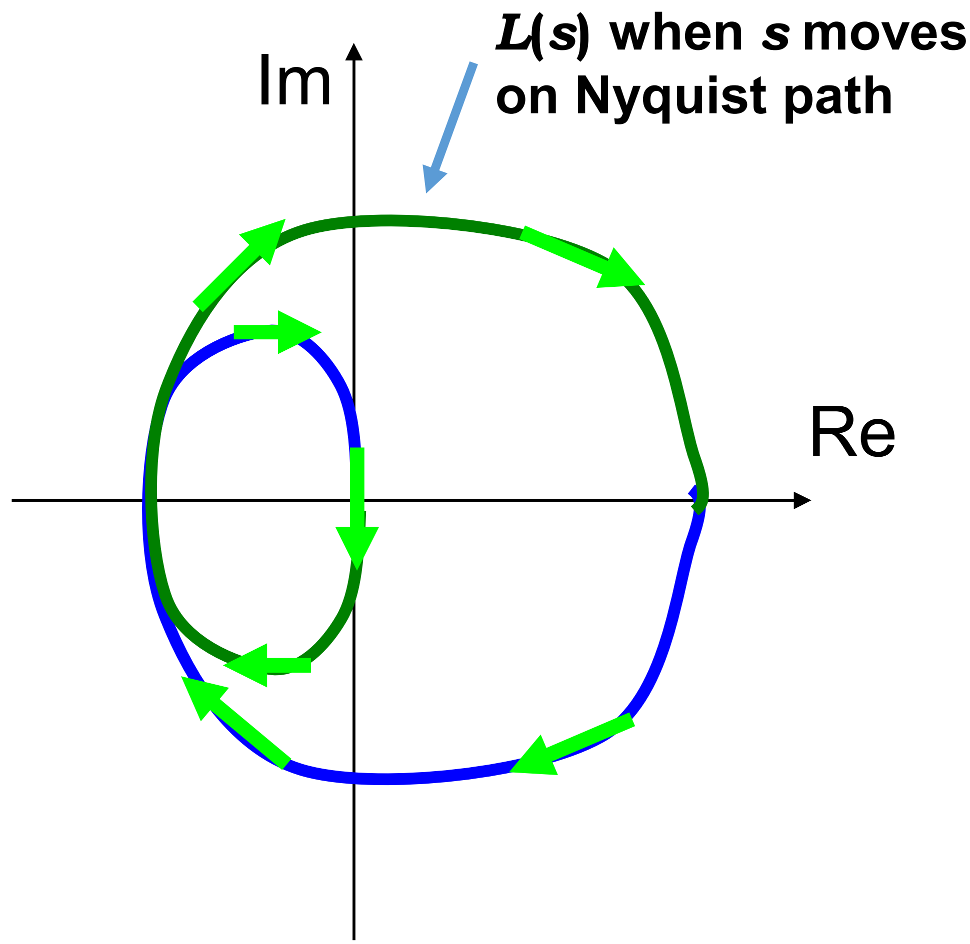
Matlab simulations

Nyquist plot (review)

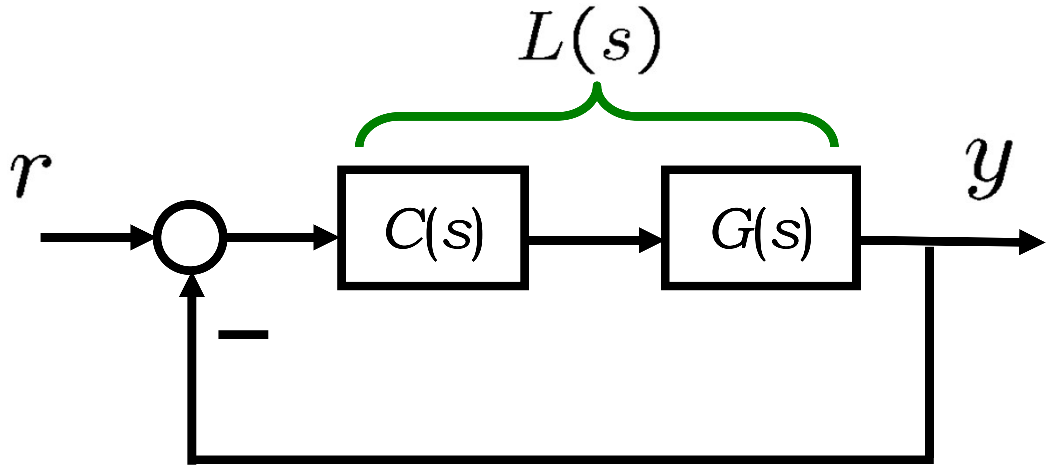
- Nyquist path



- Nyquist plot



Nyquist stability criterion (review)



CL system is stable $\Leftrightarrow Z = P + N = 0$

Z: # of CL poles in open RHP

P: # of OL poles in open RHP (given)

N: # of clockwise/counterclockwise encirclement of **-1** by Nyquist plot of OL transfer function $L(s)$ (counted by using Nyquist plot of $L(s)$)

Advantages of Nyquist stability criterion

- The Nyquist stability criterion can deal with closed-loop systems with a **time-delay** subsystem in the open-loop, while the Routh-Hurwitz criterion cannot.
- Compared to the Routh-Hurwitz criterion, the **Nyquist stability criterion** does not require an explicit transfer function. ***Experimental frequency response data of the open-loop system are sufficient to assess closed-loop stability***. In contrast, the Routh-Hurwitz criterion requires the characteristic equation of the system, typically derived from the transfer function.
- The Nyquist criterion introduces the concept of **stability margins**, such as **gain margin** and **phase margin**, which provide quantitative measures of how close the system is to instability. The Routh-Hurwitz criterion, while primarily used to determine whether a system is stable or not, ***can also offer some insight into how close the system is to instability***. For example, by examining how the coefficients in the first column of the Routh array depend on system parameters (such as gain), one can determine the range of acceptable values that preserve stability — indirectly offering a sense of how "close" the system is to becoming unstable.

Nyquist criterion: A special case

CL system is stable $\Leftrightarrow Z = P + N = 0$

- If $P = 0$ (i.e., if $L(s)$ has no pole in open RHP)

CL system is stable $\Leftrightarrow N = 0$

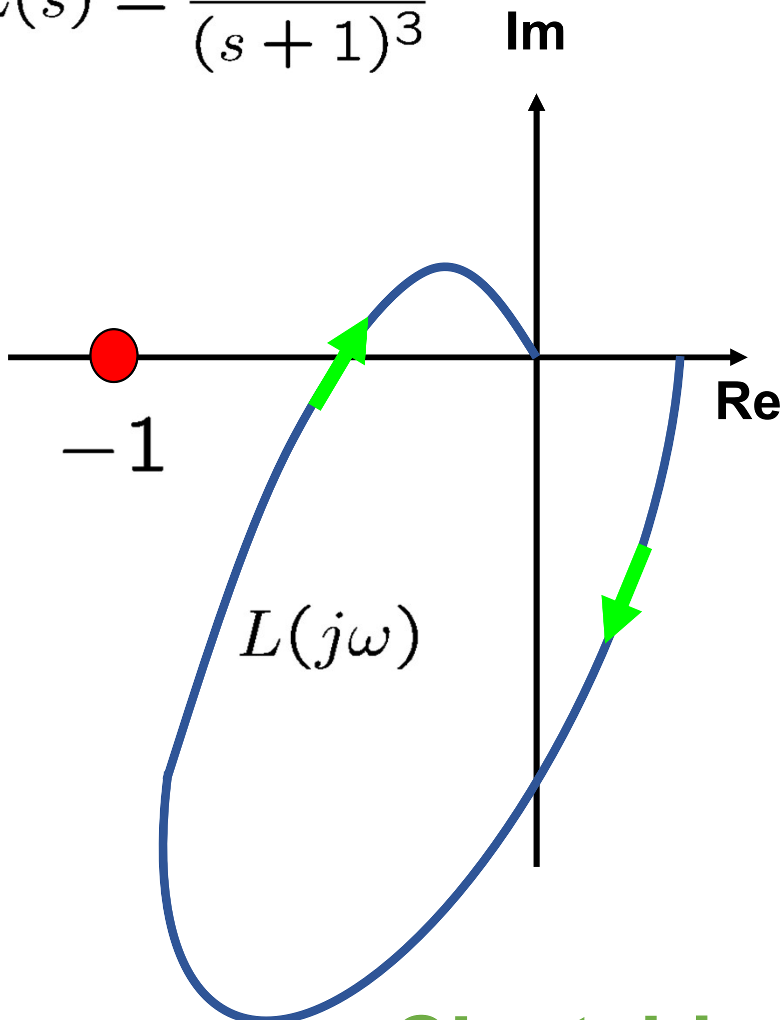
↑

We assume $P = 0$ from now on!

Examples when $P = 0$

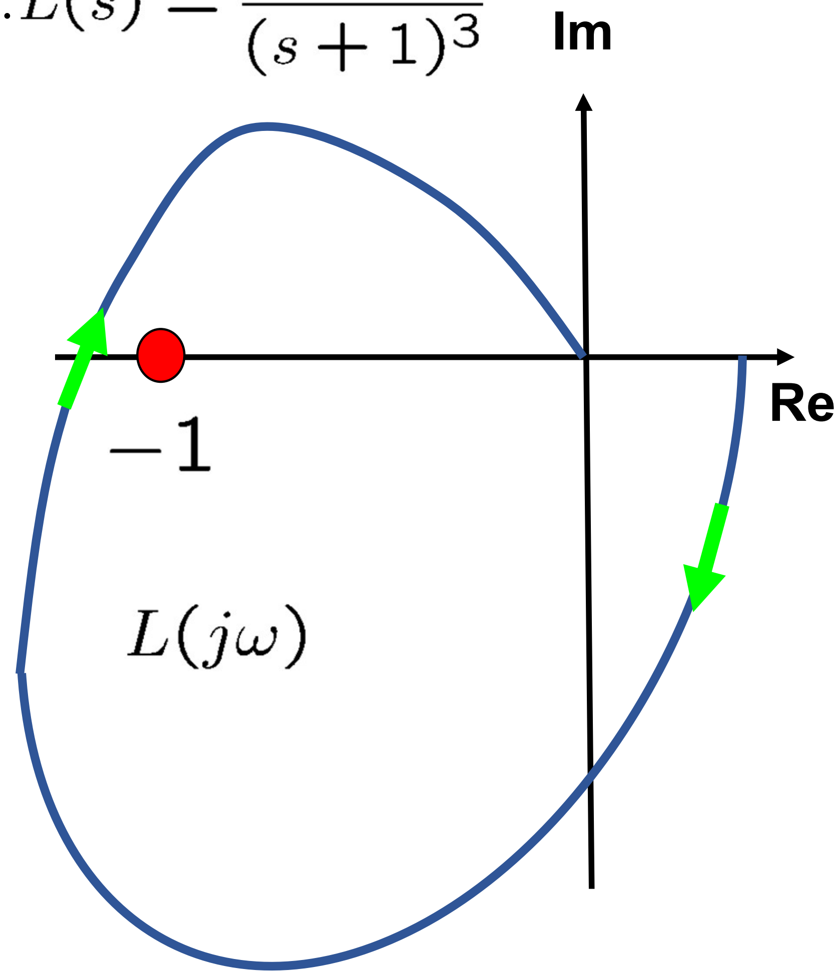
$$L(s) = \frac{1}{(s+1)^3}$$

$$K\,.L(s) = \frac{1}{(s+1)^3}$$



CL stable

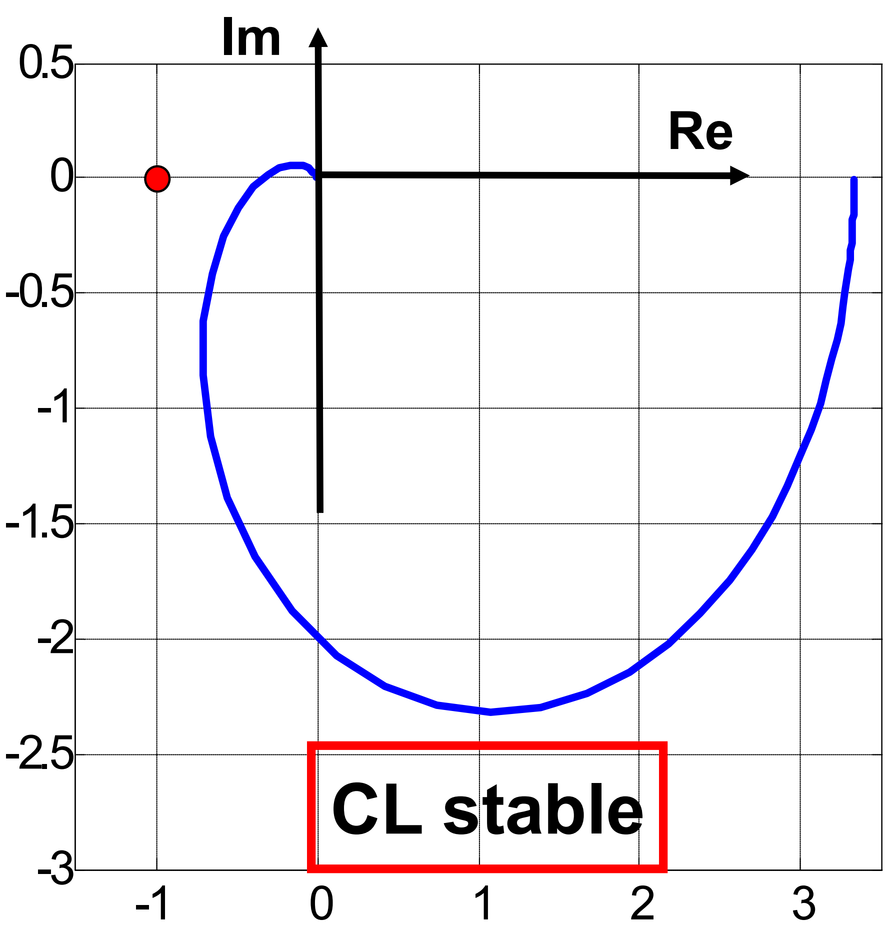
$$K\,.L(s) = \frac{15}{(s+1)^3}$$



CL unstable

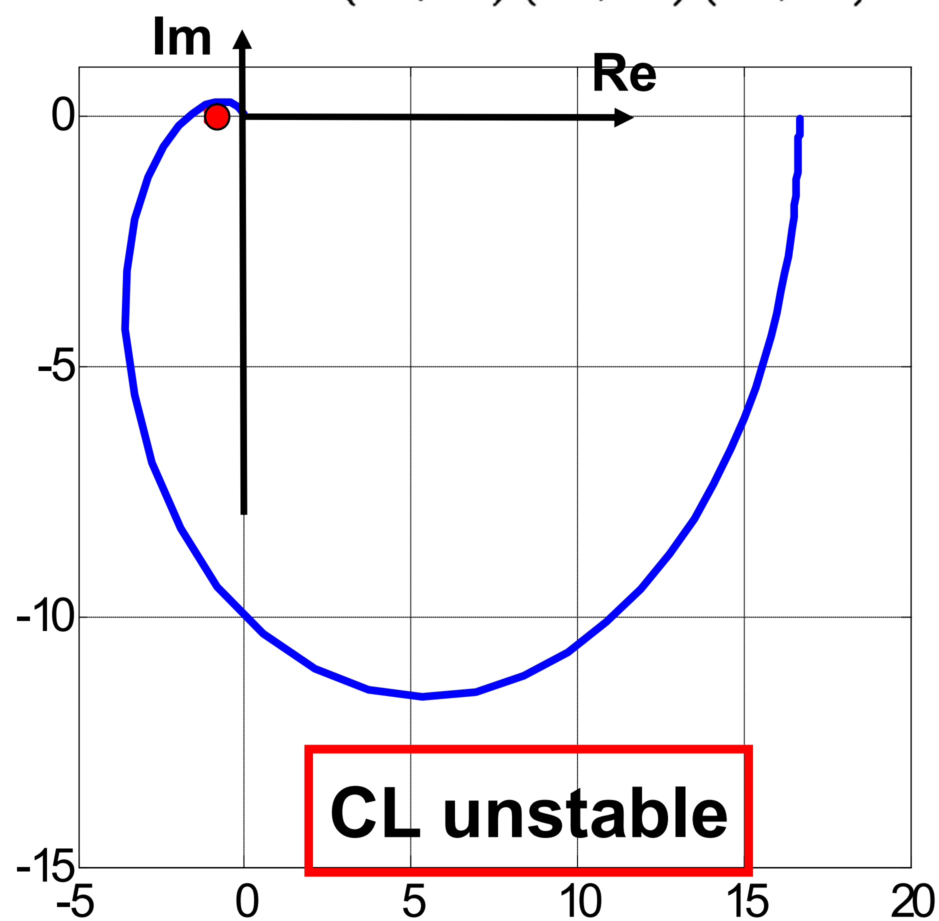
Example 1: Stable $L(s)$

$$K.L(s) = \frac{20}{(s+1)(s+2)(s+3)}$$

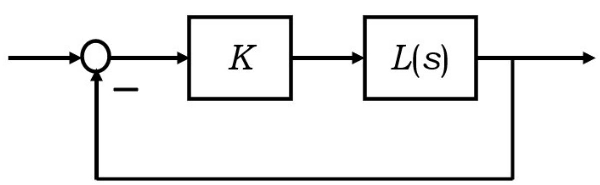


$$L(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

$$K.L(s) = \frac{100}{(s+1)(s+2)(s+3)}$$



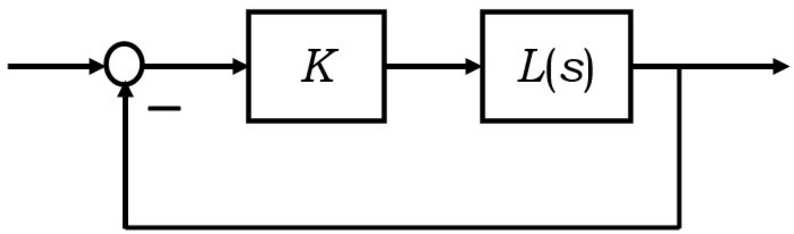
$$L(s) = \frac{1}{(s+1)(s+2)(s+3)}$$



Example 2: $L(s)$ with an integrator

In this example, we will see how to sketch Nyquist plot for transfer functions that have a single integrator in the denominator.

$$L(s) = \frac{1}{s(s+1)}$$

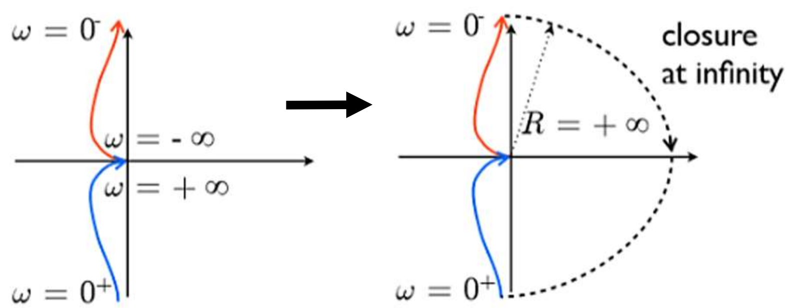


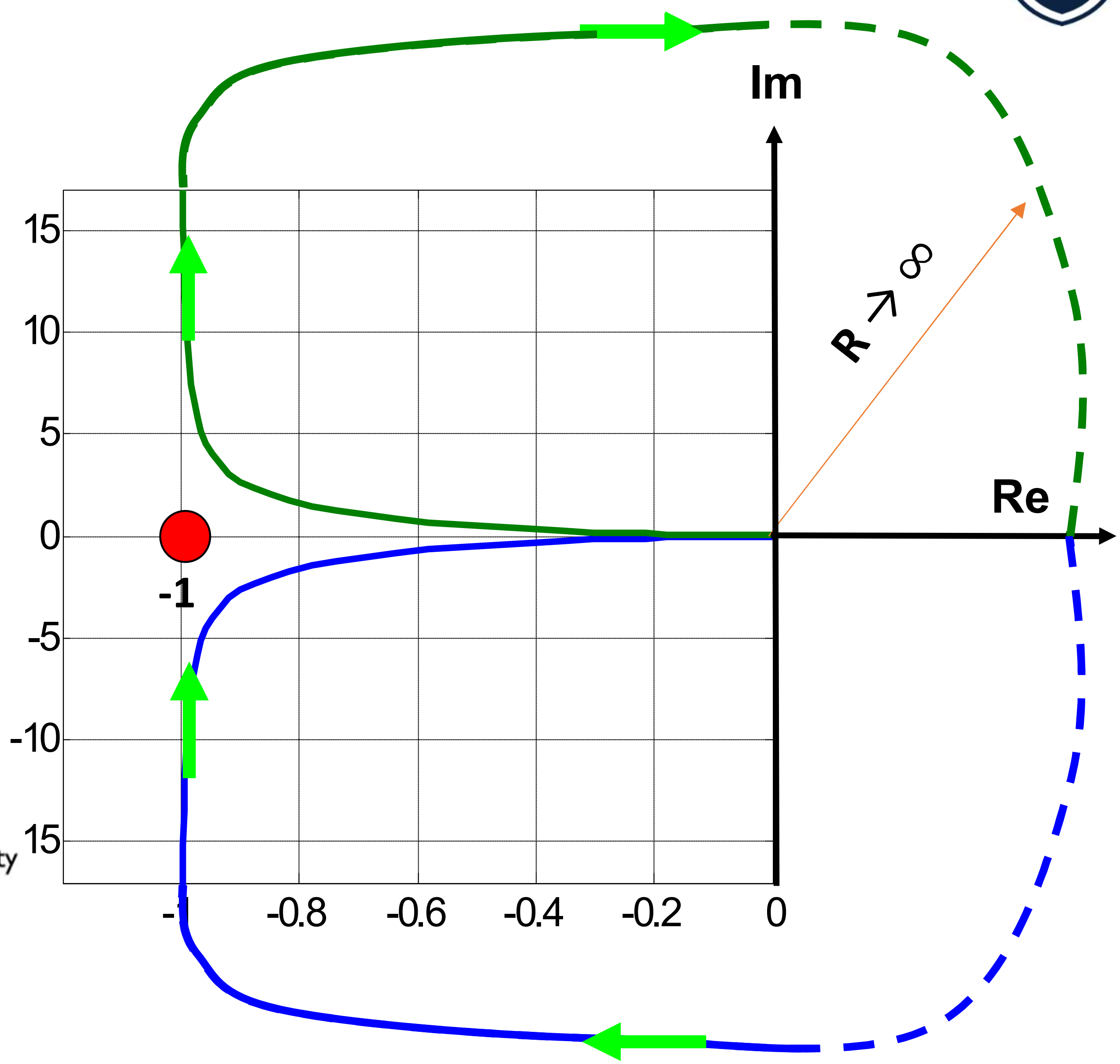
Example 2 (cont'd): $L(s)$ with an integrator

$$L(s) = \frac{1}{s(s+1)}$$

How to use the Nyquist stability criterion?

$P = 0,\ N = 0 \Rightarrow$ CL stable





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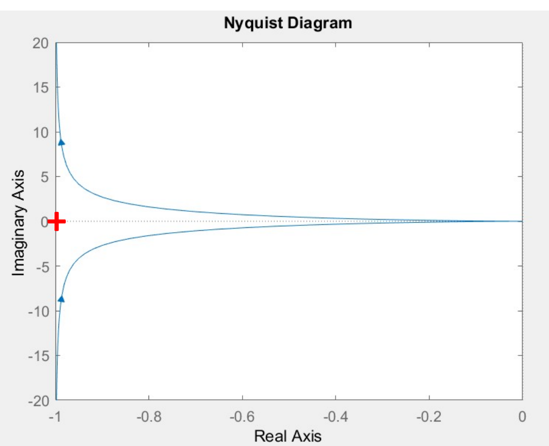
Example 2 (cont'd): $L(s)$ with an integrator

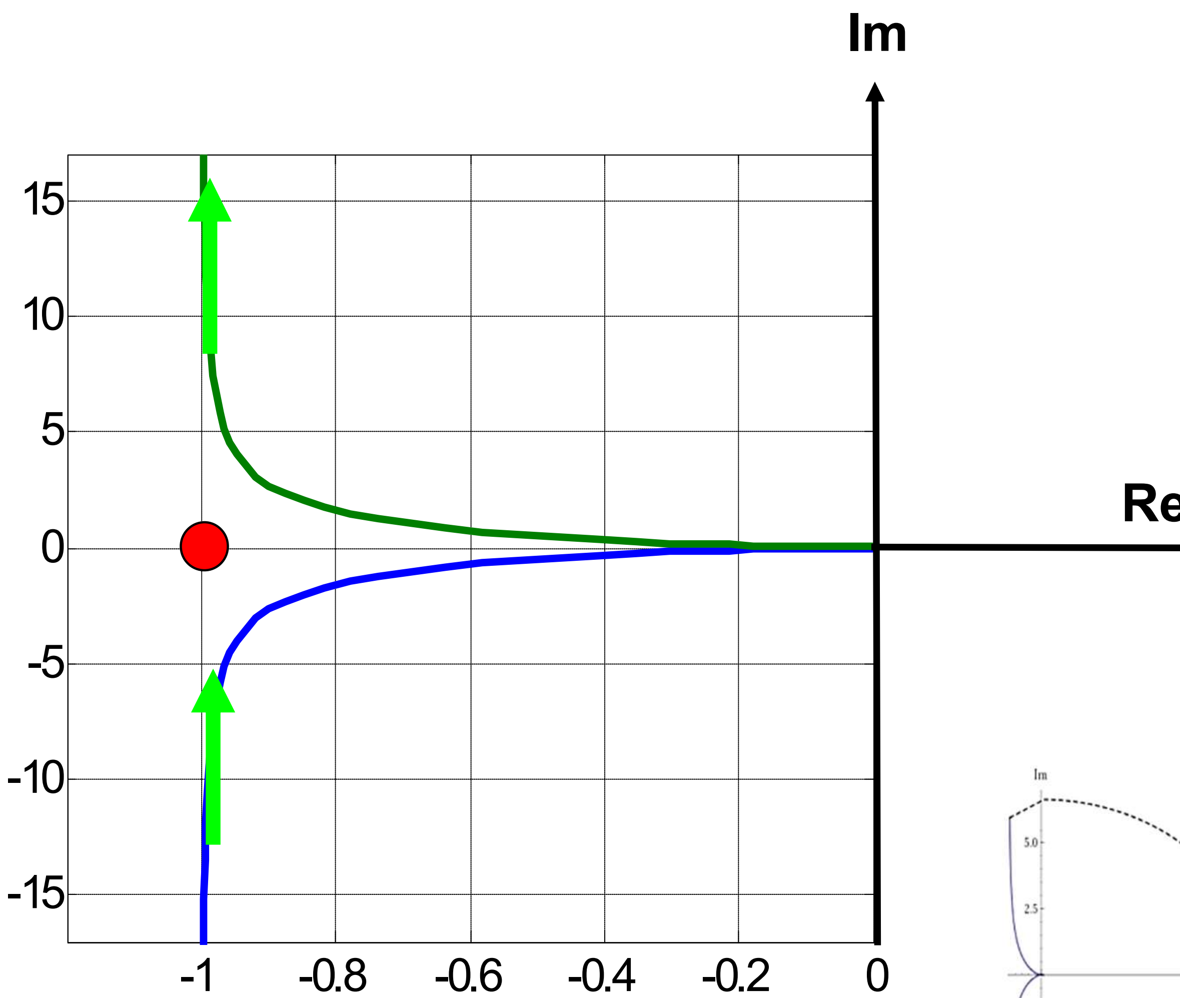
Nyquist in MATLAB:

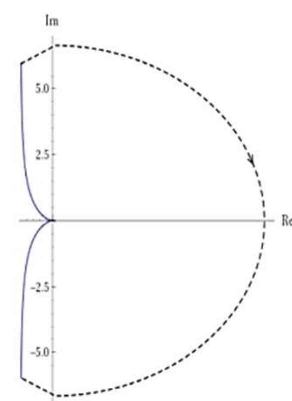
$$L(s) = \frac{1}{s(s+1)}$$

```
sys = tf(1,[1 1 0]);
nyquist(sys)
```

Note: In MATLAB, the critical point of **-1** is shown by "**+**" and not by a red dot.

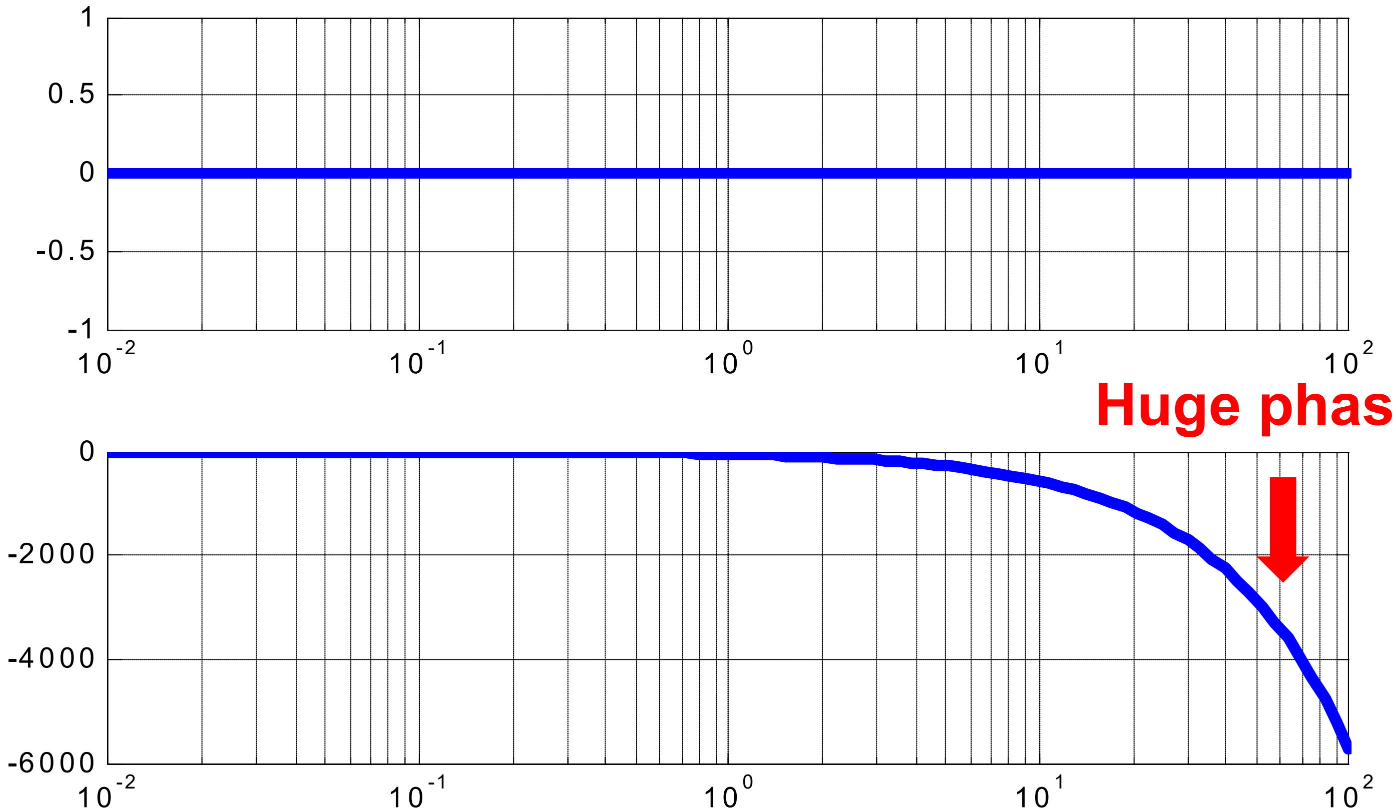






Bode plot of a time delay (review)

$$G(s) = e^{-Ts} \Rightarrow |G(j\omega)| = 1, \forall\omega, \ \angle G(j\omega) = -\omega T (\text{rad})$$

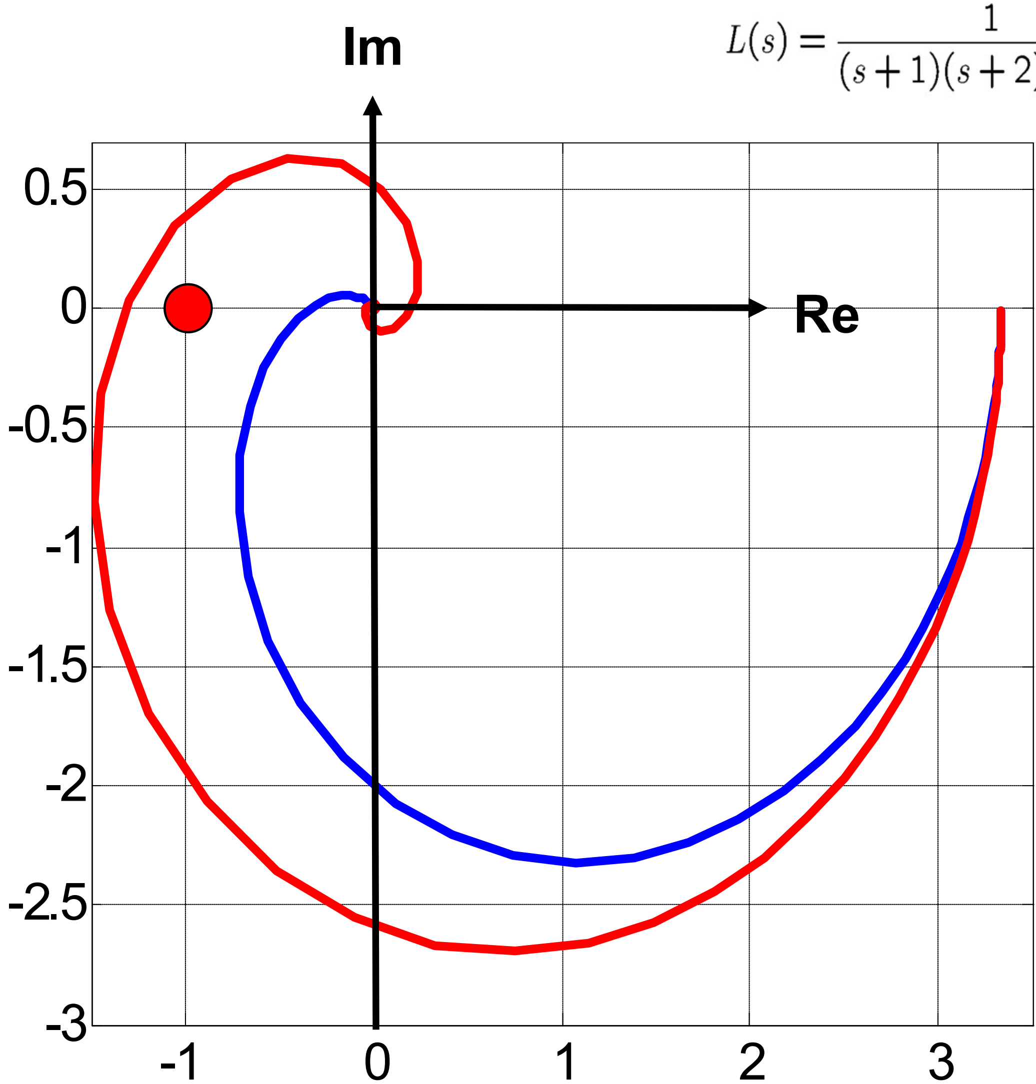


The phase lag can cause instability of the closed-loop system, and thus, the difficulty in control.

$$\angle G(j\omega) = -\omega T \times \frac{180^\circ}{\pi} \text{(degrees)}$$

Example 3: $L(s)$ with a time-delay

$$L(s) = \frac{1}{(s+1)(s+2)(s+3)}$$



$$L(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

Blue: $K.L(s) = \frac{20}{(s+1)(s+2)(s+3)}$

$P = 0,\ N = 0 \Rightarrow$ CL stable

Red: $K.L(s) = \frac{20e^{-0.7s}}{(s+1)(s+2)(s+3)}$

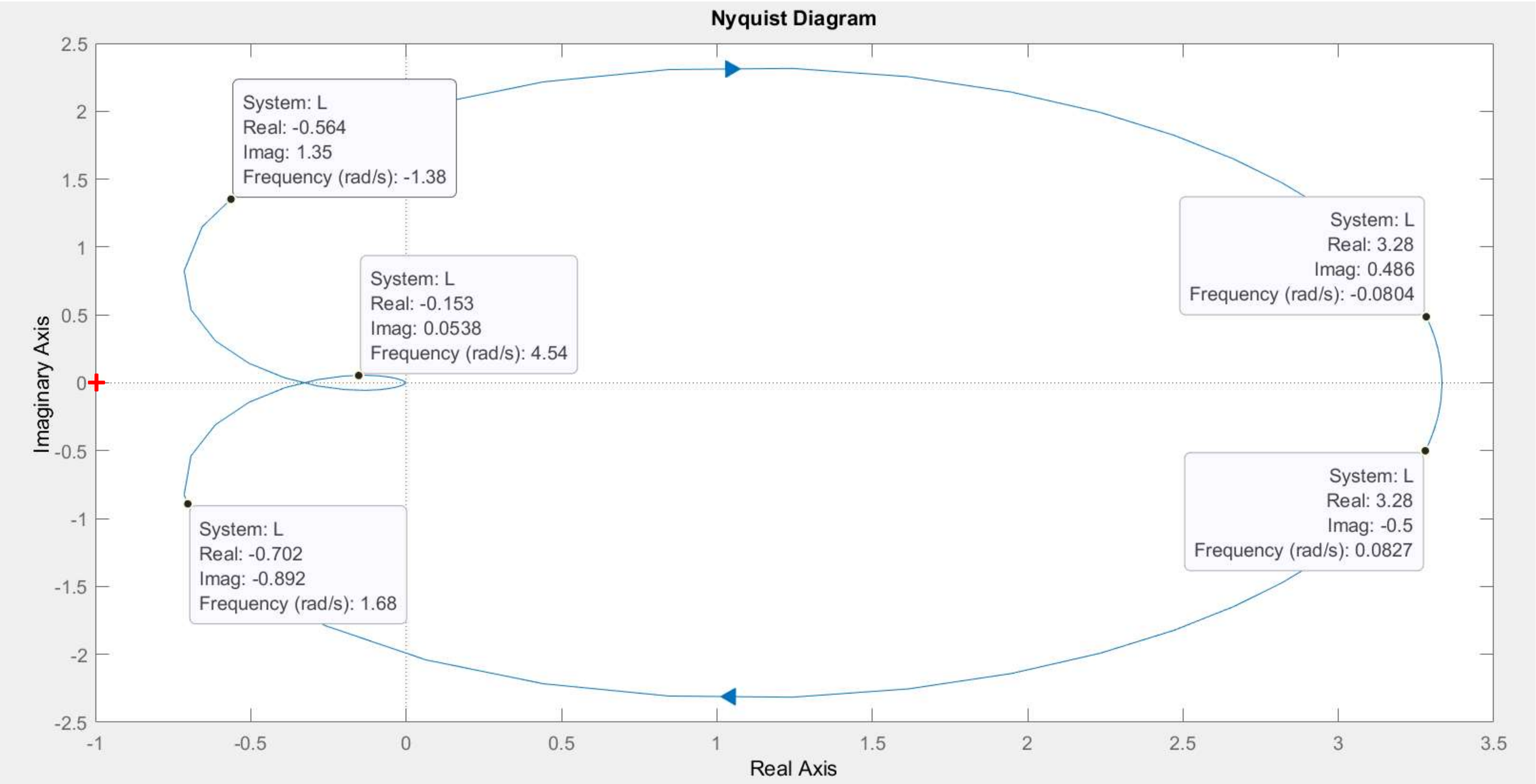
$P = 0,\ N = 2 \Rightarrow$ CL unstable

Routh-Hurwitz is NOT applicable!

Example 3: $L(s)$ with a time-delay

Without a time delay:

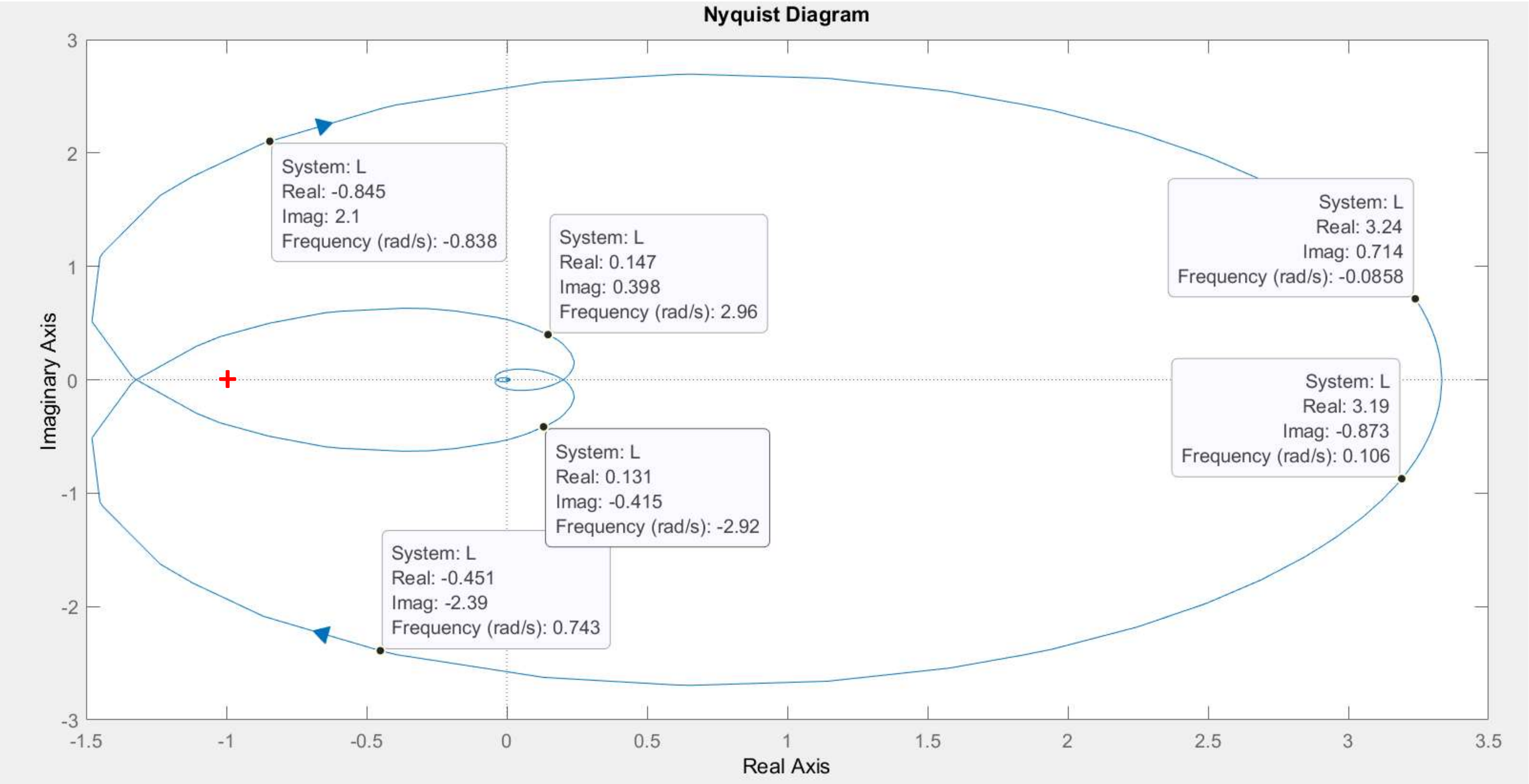
$$K.L(s) = \frac{20}{(s+1)(s+2)(s+3)}$$



Example 3: $L(s)$ with a time-delay

With a time delay:

$$K.L(s) = \frac{20e^{-0.7s}}{(s+1)(s+2)(s+3)}$$



Remarks on Nyquist stability criterion

- **Nyquist stability criterion** gives not only *absolute stability* but also *relative stability*.
  - Absolute stability: Is the closed-loop system stable or not? (Answer is yes or no.)
  - Relative stability: How "much" is the closed-loop system stable? (This addresses the ***margin of safety*** or ***margin of stability***.)
- Relative stability (**stability margin**) is important because a math model is never accurate.
  - Bode plot can also be used for relative stability.
- How to measure relative stability?
  - Use a "distance" from the **critical point -1**.
  - Use **Gain Margin (GM)** & **Phase Margin (PM)**.

Gain margin (GM)

- **Phase crossover frequency**, ω_p:

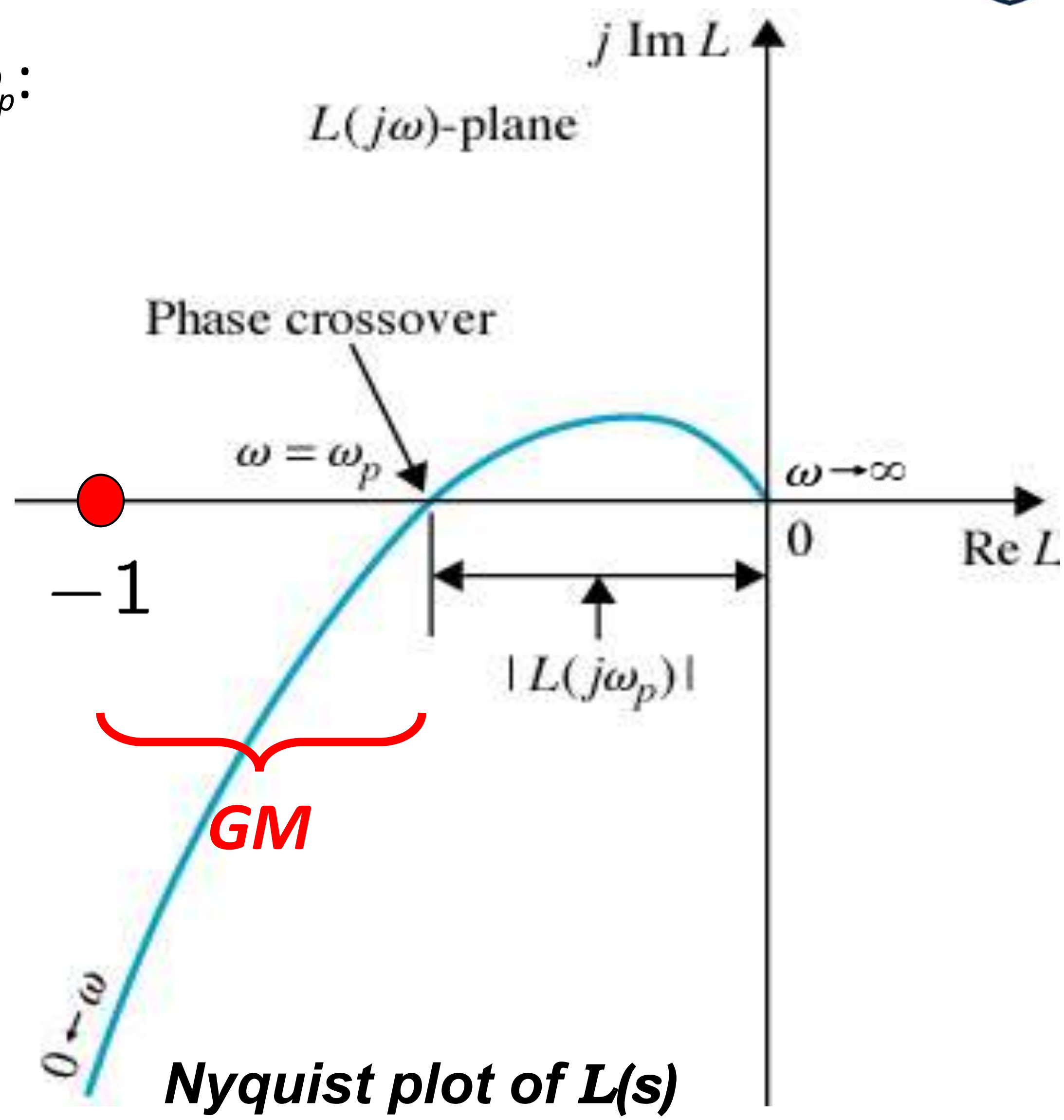
$$\angle L(j\omega_p) = -180^\circ$$

- **Gain margin** (in dB), GM:

$$GM = 20\log_{10}\frac{1}{|L(j\omega_p)|}$$

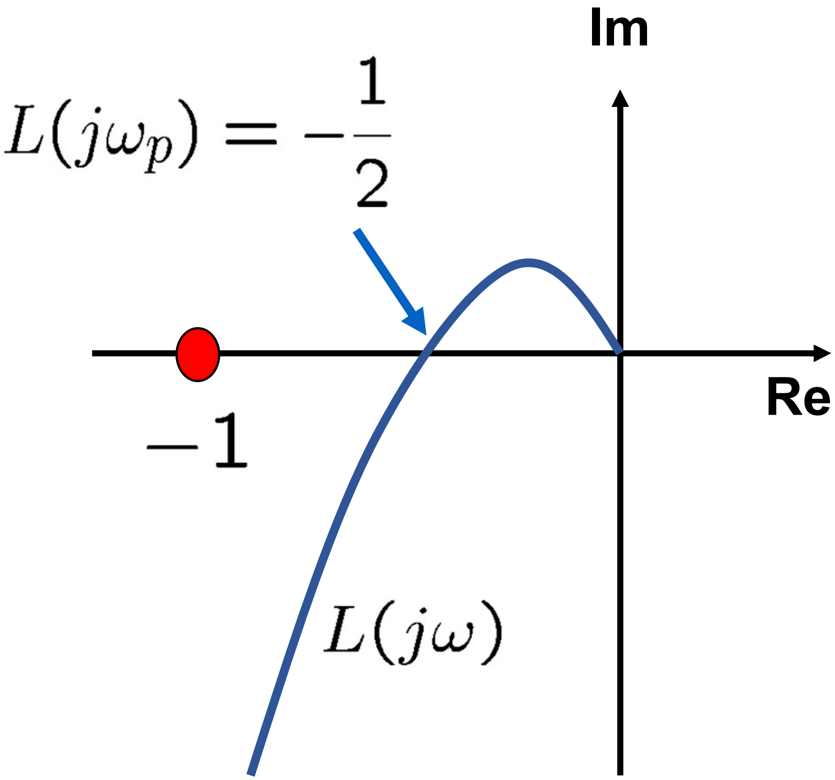
Gain Margin: The amount of additional gain that can be applied to the system before it reaches the verge of instability. Put differently, it is the maximum increase in gain that the system can tolerate before becoming unstable.

Note: For the sake of notation simplicity, from here on, $K.L(s)$ is just shown by $L(s)$.

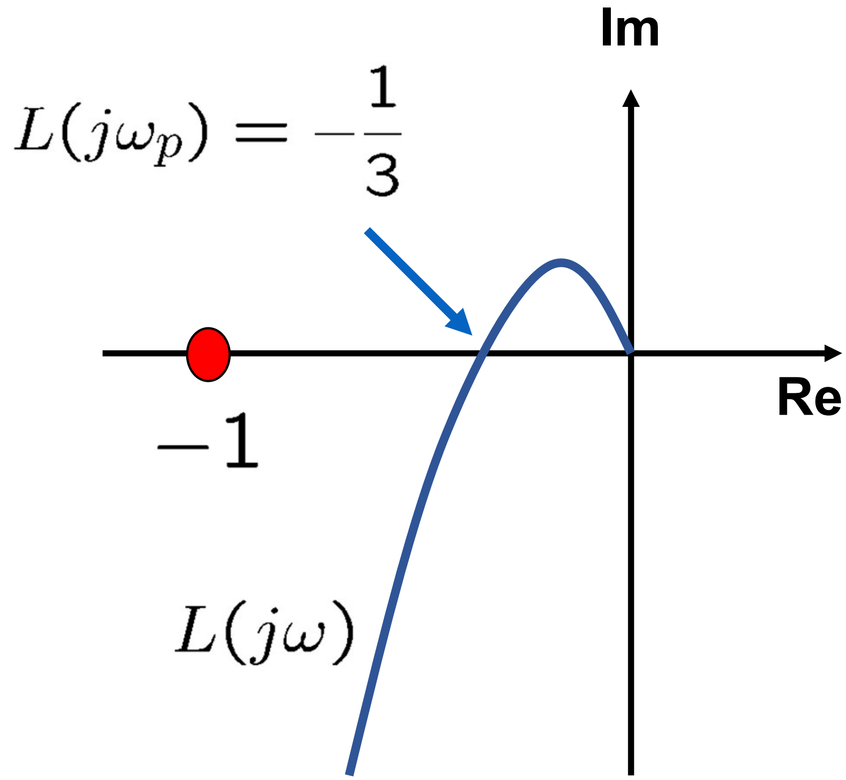


Nyquist plot of L(s)

Example 4: GM

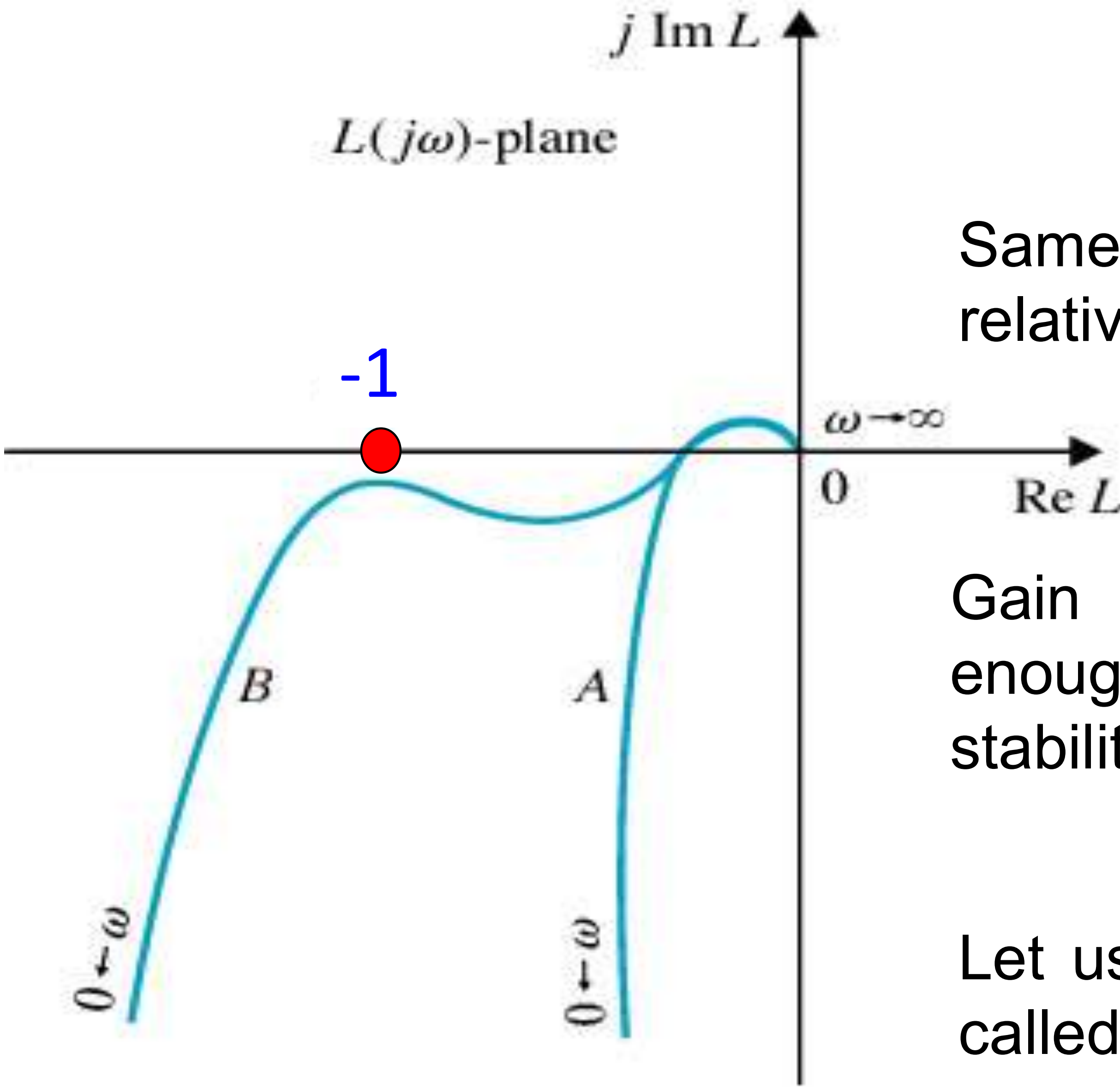


$$GM = 20\log_{10}\overbrace{\frac{1}{\underbrace{|L(j\omega_p)|}_{2}}}^{1} \approx +6\text{dB}$$



$$GM = 20\log_{10}\overbrace{\frac{1}{\underbrace{|L(j\omega_p)|}_{3}}}^{1} \approx +9.5\text{dB}$$

Reason why using only GM is sometimes not enough



Same gain margin, but different relative stability.

Gain margin is sometimes not enough for analyzing relative stability.

Let us look into another metric called **phase margin**!

Phase margin (PM)

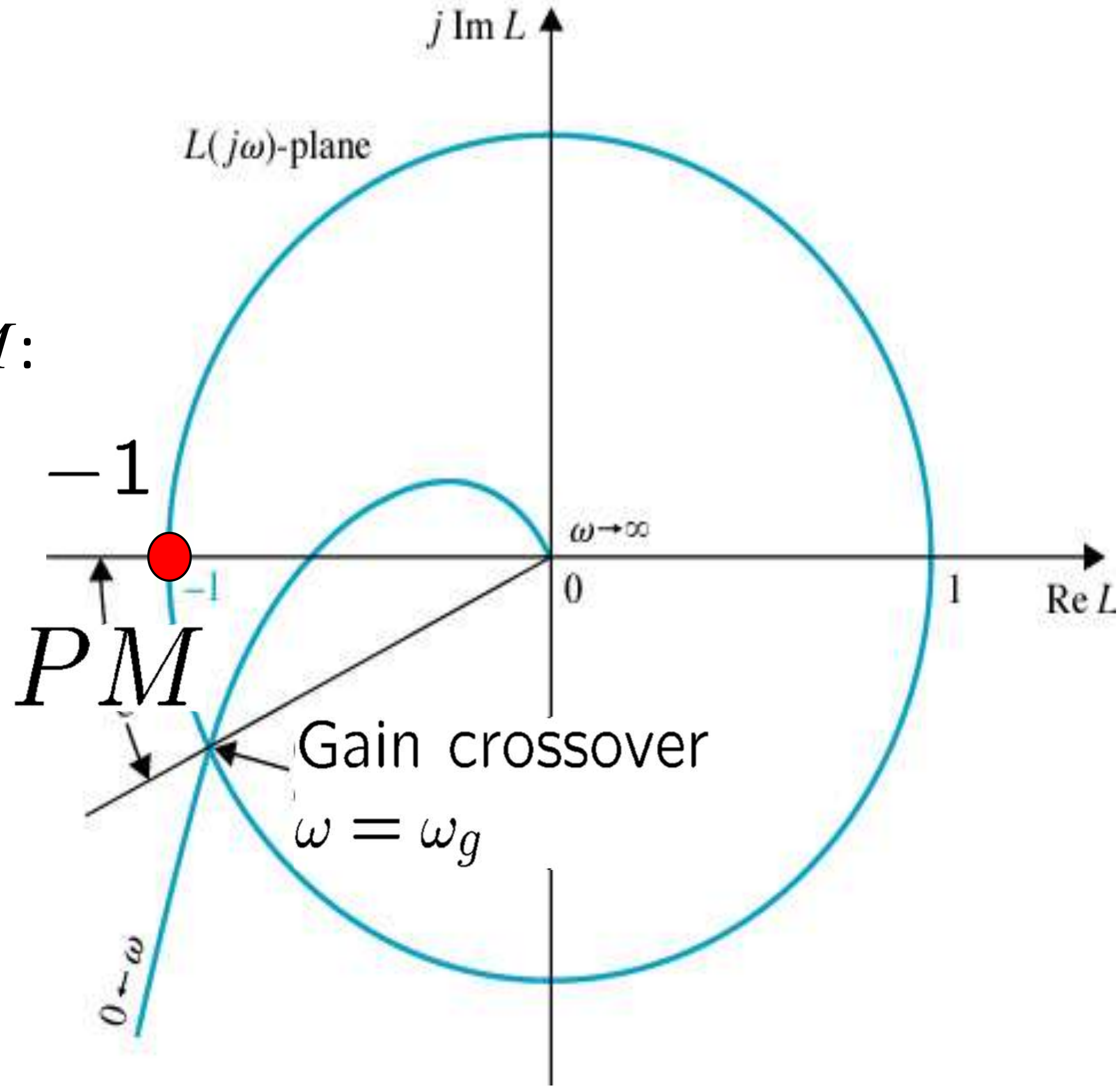
- **Gain crossover frequency**, ω_g:

$$|L(j\omega_g)| = 1$$

- **Phase margin** (in degrees), PM:

$$PM = \angle L(j\omega_g) + 180^\circ$$

Phase Margin: The amount of additional phase lag the system can tolerate before reaching the verge of instability. Put differently, it is the maximum phase lag that can be introduced into the system before it becomes unstable.



Nyquist plot of L(s)

Example 5: PM and GM (analytically)

$$L(s) = \frac{2500}{s(s+5)(s+50)}$$

Step 1: Find ω_g $|L(j\omega_g)| = 1$

$$L(j\omega) = \frac{2500}{(j\omega)(j\omega+5)(j\omega+50)} \to |L(j\omega)| = \frac{2500}{|j\omega||j\omega+5||j\omega+50|} = \frac{2500}{\omega\sqrt{\omega^2+2}\ \sqrt{\omega^2+2500}} \to$$

$$|L(j\omega_g)| = \frac{2500}{\omega_g\sqrt{\omega_g{}^2+25}\ \sqrt{\omega_g{}^2+2500}} = 1 \to$$

$$\boxed{\omega_g = 6.22\ \frac{\text{rad}}{\text{s}}}$$

Step 2: Find PM $PM = \angle L(j\omega_g) + 180^\circ$

$$\angle L(j\omega_g) = \angle\frac{2500}{(j\times 6.22)(j\times 6.22+5)(j\times 6.22+50)} = \angle 2500 - \angle\{(j\times 6.22)(j\times 6.22+5)(j\times 6.22+50)\} = 0^\circ -$$

$$\left\{\tan^{-1}\left(\frac{6.22}{0}\right) + \tan^{-1}\left(\frac{6.22}{5}\right) + \tan^{-1}\left(\frac{6.22}{50}\right)\right\} = 0^\circ - \{90^\circ + 51.2^\circ + 7.09^\circ\} \to \angle L(j\omega_g) = -148.29^\circ \to$$

$$PM = -148.29^\circ + 180^\circ \to$$

$$\boxed{PM = +31.71^\circ}$$

Step 3: Find ω_p $\angle L(j\omega_p) = -180^\circ$

$$\angle L(j\omega_p) = -180^\circ \to \angle\frac{2500}{(j\omega_p)(j\omega_p+5)(j\omega_p+50)} = -180^\circ \to$$

$$-\left\{90^\circ + \tan^{-1}\left(\frac{\omega_p}{5}\right) + \tan^{-1}\left(\frac{\omega_p}{50}\right)\right\} = -180^\circ \quad \to \tan^{-1}\left(\frac{\omega_p}{5}\right) + \tan^{-1}\left(\frac{\omega_p}{50}\right) = 90^\circ \to$$

$$\boxed{\omega_p = 15.88\ \frac{\text{rad}}{\text{s}}}$$

Step 4: Find GM $GM = 20\log_{10}\frac{1}{|L(j\omega_p)|}$

$$|L(j\omega_p)| = |L(15.88j)| = \frac{2500}{15.88\sqrt{15.88^2+25}\ \sqrt{15.88^2+2500}} \to |L(15.88j)| = 0.1802 \to$$

$$GM = 20\log\frac{1}{|L(15.88j)|} = 20\log\frac{1}{0.1802} \to$$

$$\boxed{GM = +14.80\ \text{dB}}$$

Example 5 (cont'd): PM and GM (analytically)

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How to solve an equation using "**Ans**" key in a regular calculator:

Example: Find ω_g in $\dfrac{2500}{\omega_g\sqrt{{\omega_g}^2+25}\,\sqrt{{\omega_g}^2+2500}} = 1.$

Step 1: Isolate ω_g:

$$\omega_g = \frac{2500}{\sqrt{{\omega_g}^2+25}\,\sqrt{{\omega_g}^2+2500}}$$

Step 2: Type in your initial guess for ω_g and then press the key "**=**". Let us say we use an initial guess of 3:

Type in "3" and then press "**=**".

Step 3: Type in the right hand side of the equation in Step 1, but replace all the "ω_g" with "**Ans**":

2500×(**Ans**^{2} + 25)^(-0.5)×(**Ans**^{2} + 2500)^(-0.5) and then press "**=**".

Step 4: Continue pressing "**=**" until the "**Ans**" does not change. This is the ω_g your were looking for.

Example 5 (cont'd): PM and GM (analytically)

How to use "**TABLE**" feature in a regular calculator to solve any equation:

Example: Find ω_p in $\tan^{-1}\left(\frac{\omega_p}{5}\right) + \tan^{-1}\left(\frac{\omega_p}{50}\right) = 90°.$

Step 1: Press "**MODE**" and then select "**TABLE**" (usually it is "3: TABLE").

MODE SETUP

Step 2: Type in the left-hand side of the above equation (replace ω_p with X):

$$\mathrm{f(X)} = \tan^{-1}\left(\frac{\mathrm{X}}{5}\right) + \tan^{-1}\left(\frac{\mathrm{X}}{50}\right)$$

Step 3: You need to enter the range of X for which you want to have in your table. Also, you need to enter the interval between two consecutive X's. **Start?**, **End?**, **Step?**. The table will be generated as below:

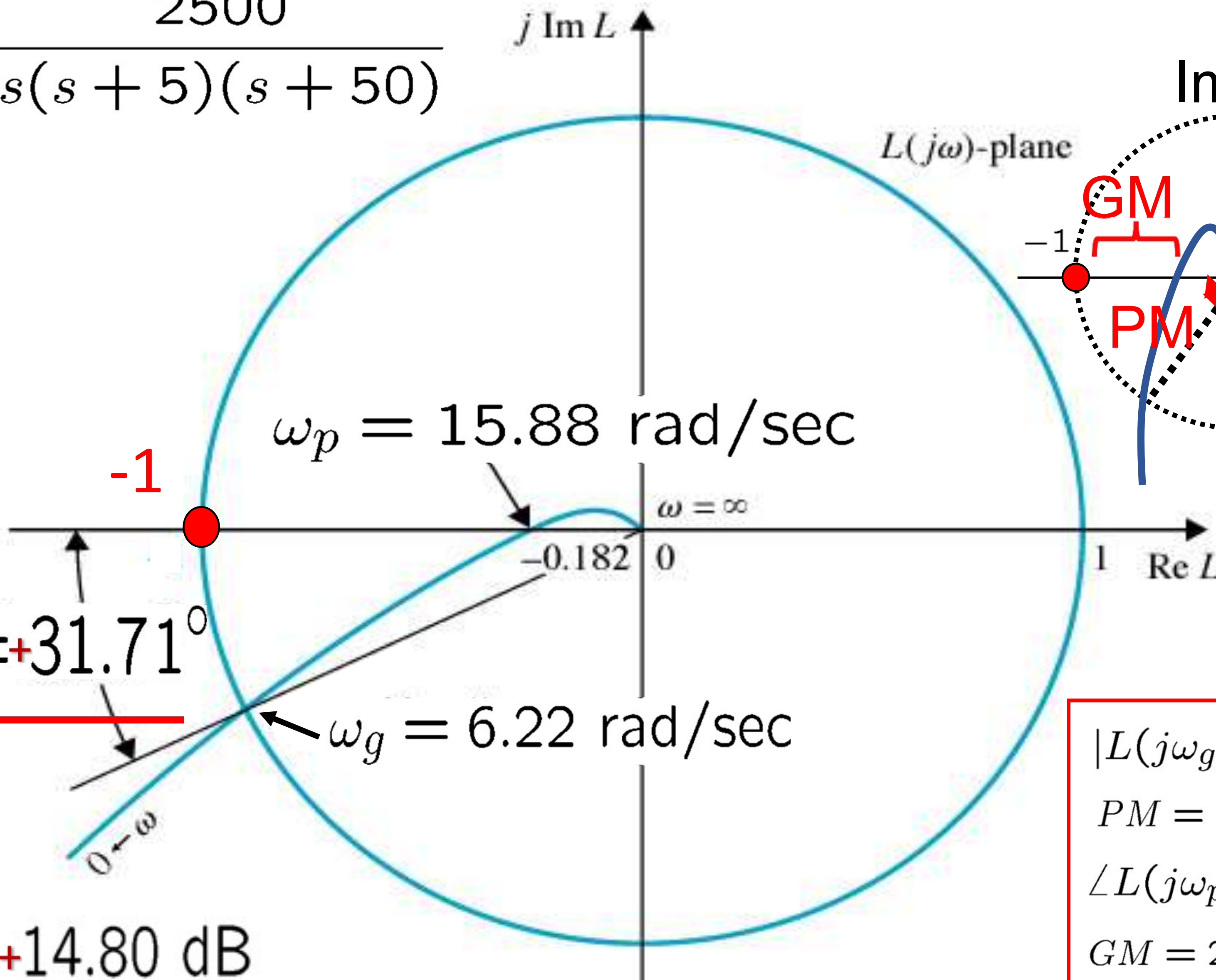
X	f(X)
⋮	⋮
15.8	89.976
15.9	90.184
⋮	⋮

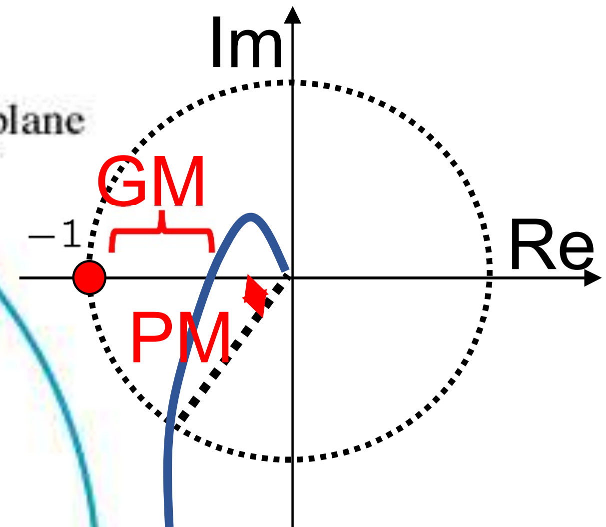
Step 4: From the table, read the value of X (here, ω_p) that has f(X) equal to or close to 90 (depending on accuracy).

$$\mathrm{X} = \omega_p \approx 15.9\ \mathrm{rad/s}$$

Example 5 (cont'd): PM and GM (analytically)

$$L(s) = \frac{2500}{s(s+5)(s+50)}$$





$PM = +31.71^0$

$GM = +14.80$ dB

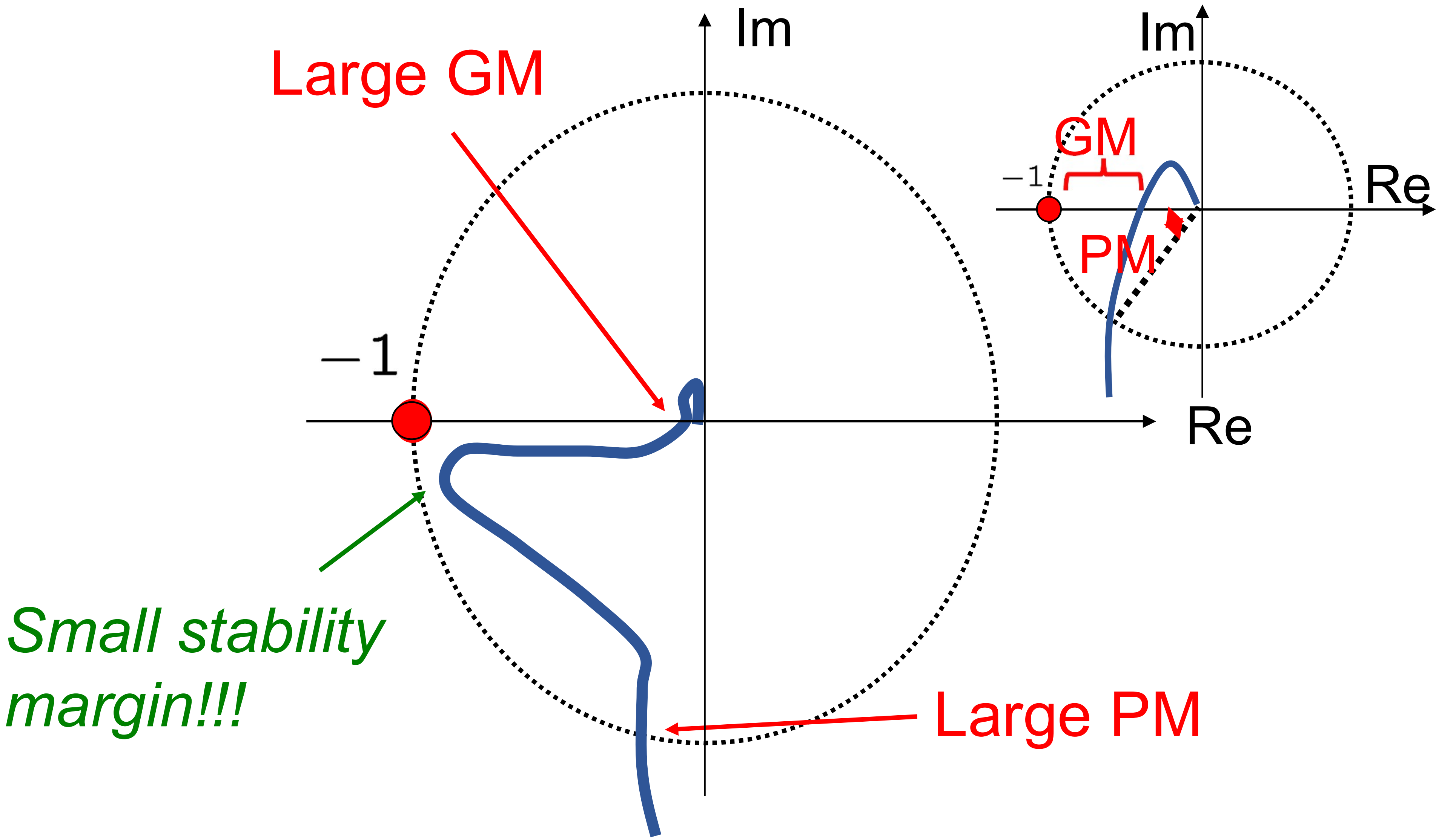
$$|L(j\omega_g)| = 1$$

$$PM = \angle L(j\omega_g) + 180^\circ$$

$$\angle L(j\omega_p) = -180^\circ$$

$$GM = 20\log_{10}\frac{1}{|L(j\omega_p)|}$$

GM and PM: An extreme case



Notes on Nyquist plot

- **Advantages:**
  - Nyquist plot can be used for analysis of the closed-loop stability using open-loop transfer function (in the form of frequency response function). Even if the FRF is unstable and includes time-delay, we can still use Nyquist plot to determine closed-loop stability.
- **Disadvantage:**
  - Controller design on Nyquist plot is somewhat difficult.
    - ➢ Controller design on Bode plot is much simpler.

We will translate GM and PM defined in Nyquist plot into those in Bode plot for designing suitable controller!

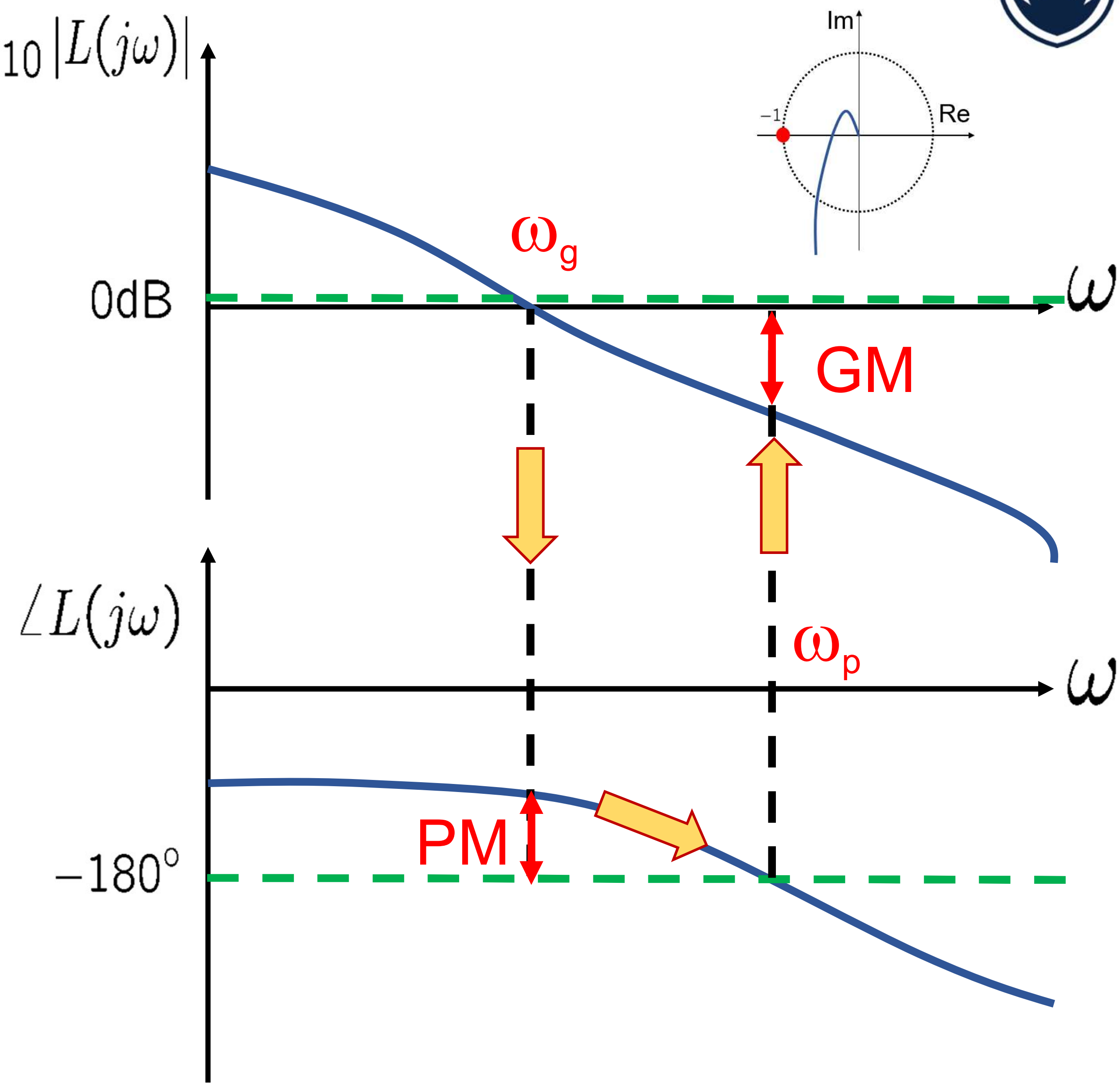
Relative stability on Bode plot (graphically)

Step 1: Find the intersection of the Gain Curve with the horizontal **0 dB line**. The ω of this intersection point is ω_g.

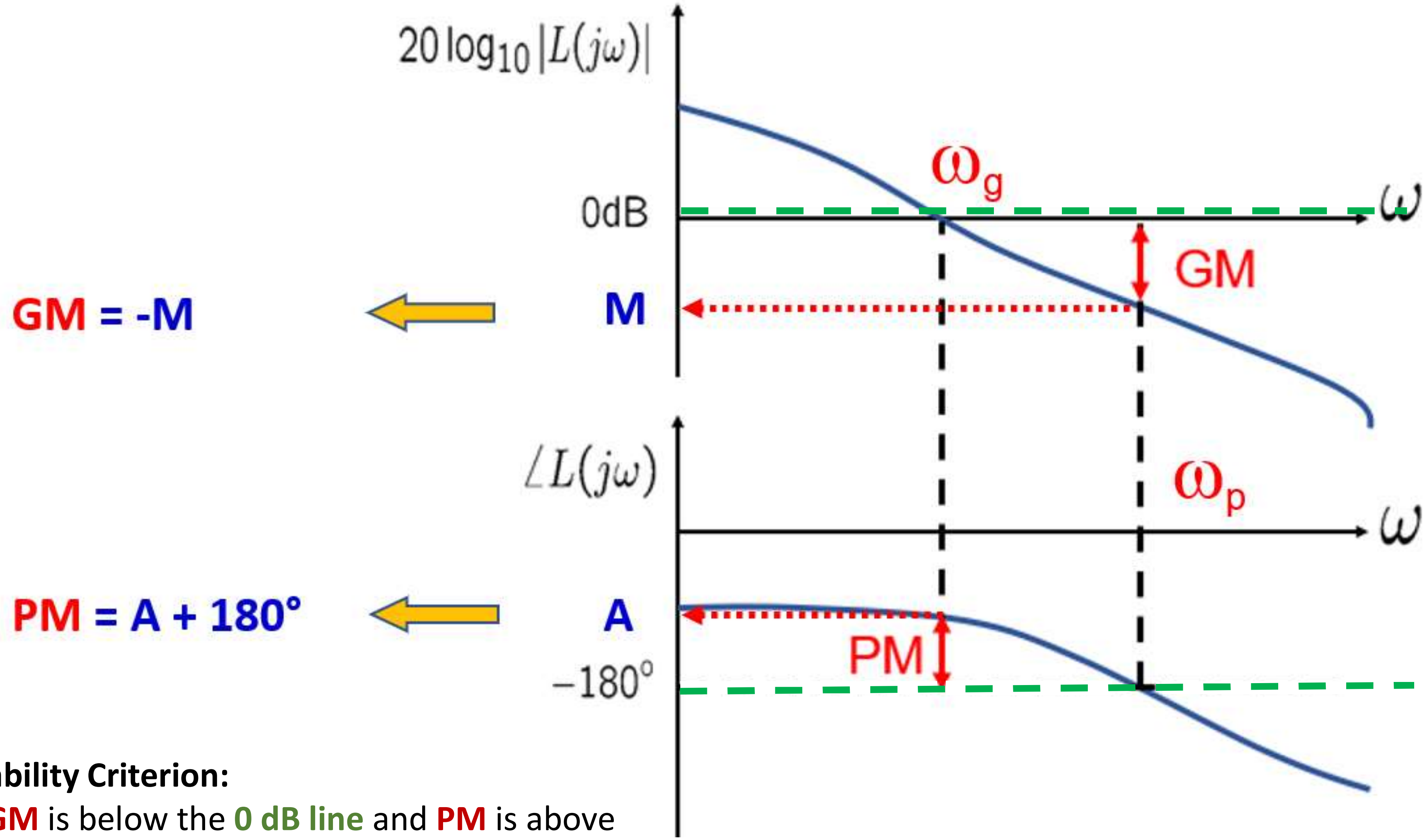
Step 2: From ω_g in the Gain Curve draw a vertical line and find the intersection of this vertical line with the Phase Curve. The vertical distance between this intersection point and the horizontal **-180° line** is PM.

Step 3: While in the Phase plot, find the intersection of the Phase Curve with the horizontal **-180° line**. The ω of this intersection point is ω_p.

Step 4: From the intersection point in Step 3, draw a vertical line and find the intersection of this line with the Gain Curve. The vertical distance between this intersection point and the horizontal **0 dB line** is GM.



How to find GM and PM graphically



Stability Criterion:
If **GM** is below the **0 dB line** and **PM** is above the **-180 line**, the CL system is **stable**.

Notes on Bode plot as Related to PM and GM

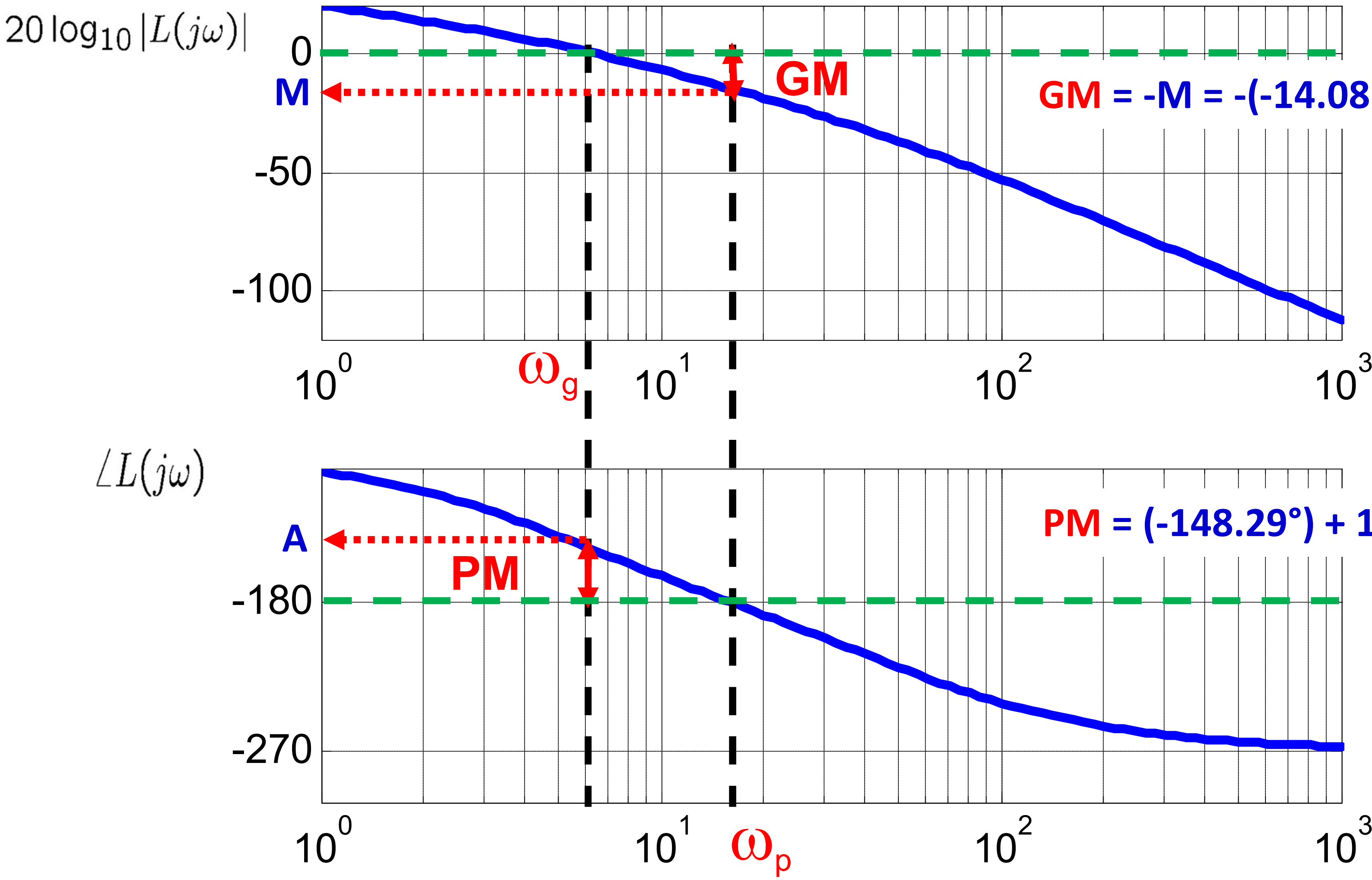
- **Advantages:**
  - Without computer, Bode plot can be sketched easily by using straight-line approximations. It is a good method for determining **Relative Stability**.
  - GM, PM, crossover frequencies are easily determined on Bode plot.
    - If straight-line approximations are used for the Bode plot, then the GM and PM will also be approximate. However, if the accurate Bode plot is obtained using software tools, the calculated GM and PM will be more precise.
  - Controller design on Bode plot is simpler. (Next lecture)

- **Disadvantage:**
  - If OL (open-loop) system has poles in open right half plane, it may sometimes become more complicated to use Bode plot for closed-loop stability analysis.
    - While you can estimate gain and phase margins from a Bode plot, these margins do not guarantee stability if the open-loop system has RHP poles.
- **Note:**
  - Gain margin is related to phase crossover frequency, while phase margin is related to gain crossover frequency.
    - If both GM and PM are positive, then the closed-loop system is stable. If even one of them is negative, the closed-system is unstable. ***We normally emphasize more on PM in controller design than on GM.***

Example 6: Find GM and PM Graphically

We are revisiting Example 5.

$$L(s) = \frac{2500}{s(s+5)(s+50)}$$



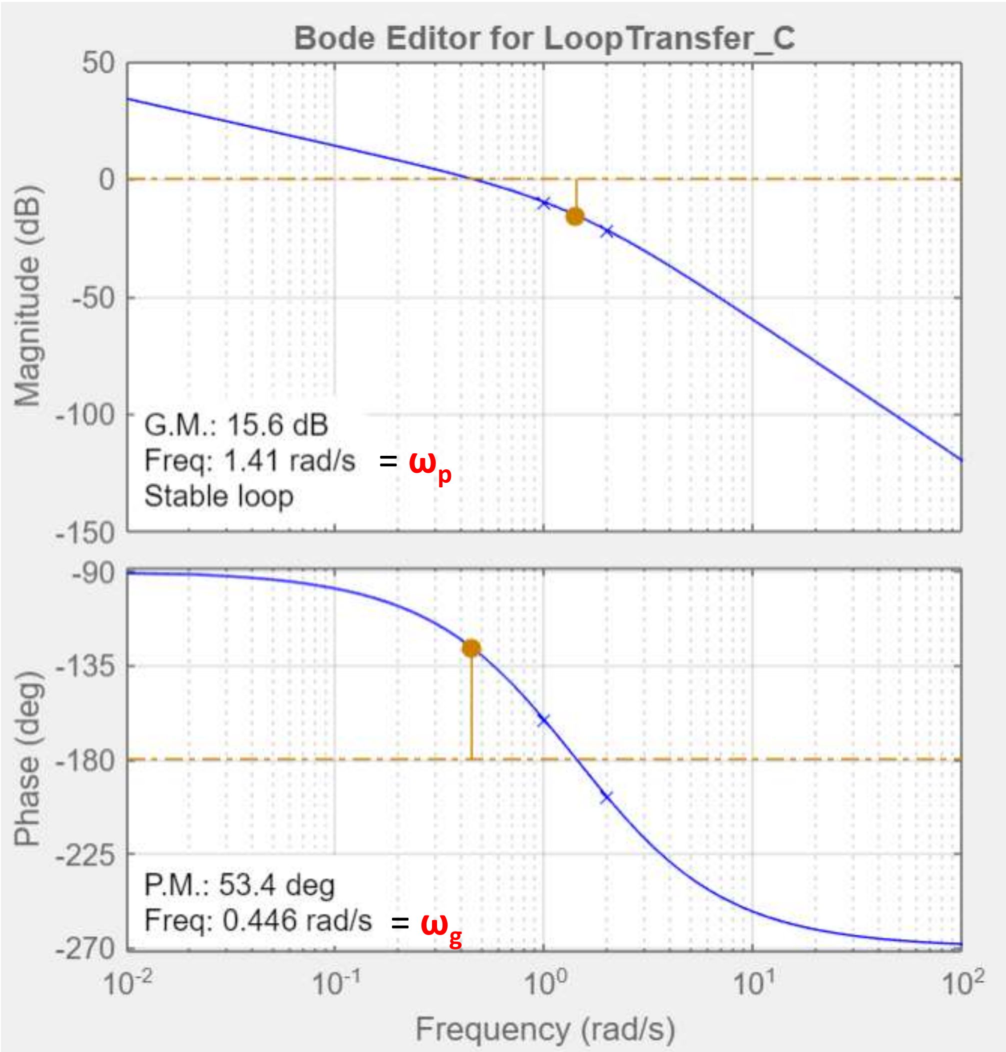
GM = -M = -(-14.08) = +14.08 dB

PM = (-148.29°) + 180° = +31.71°

Example 7: Relative stability with time delay

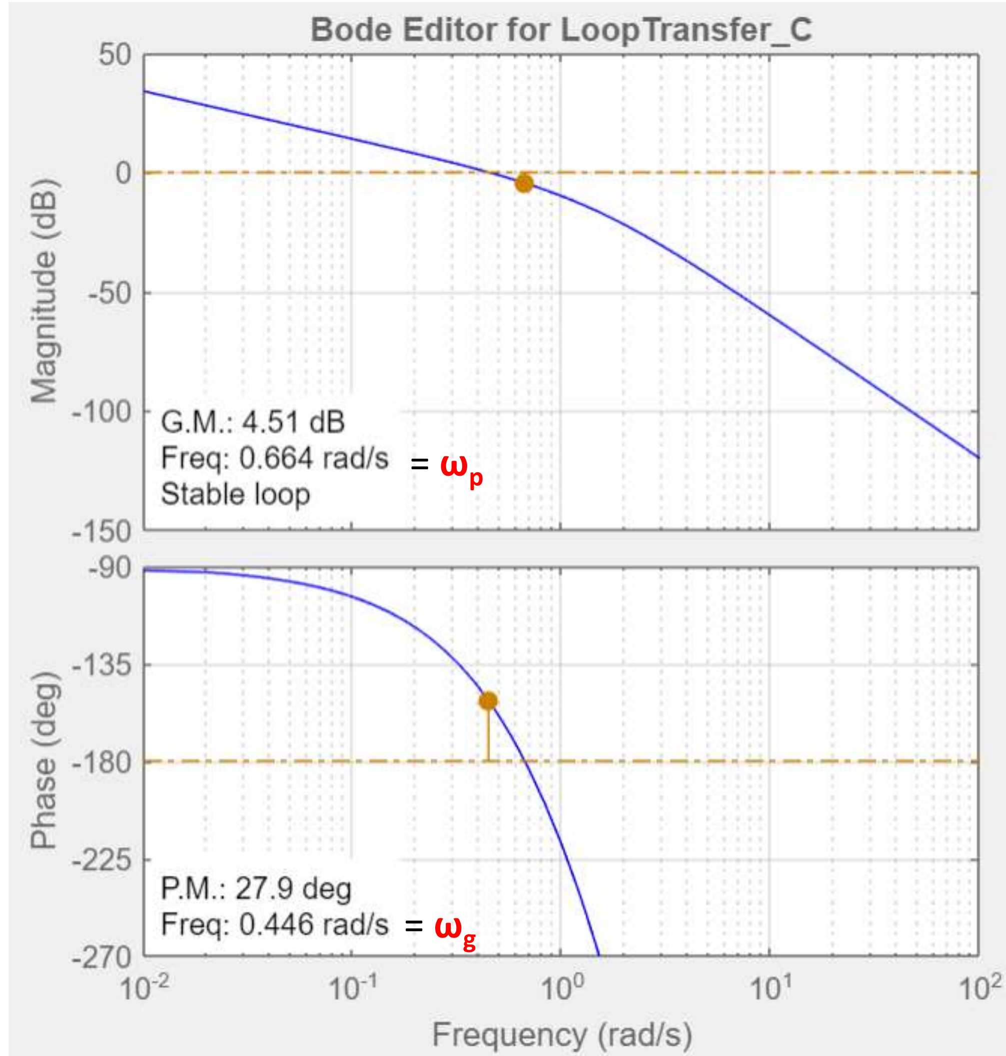
Without a time delay:

$$L(s) = \frac{1}{s(s+1)(s+2)}$$

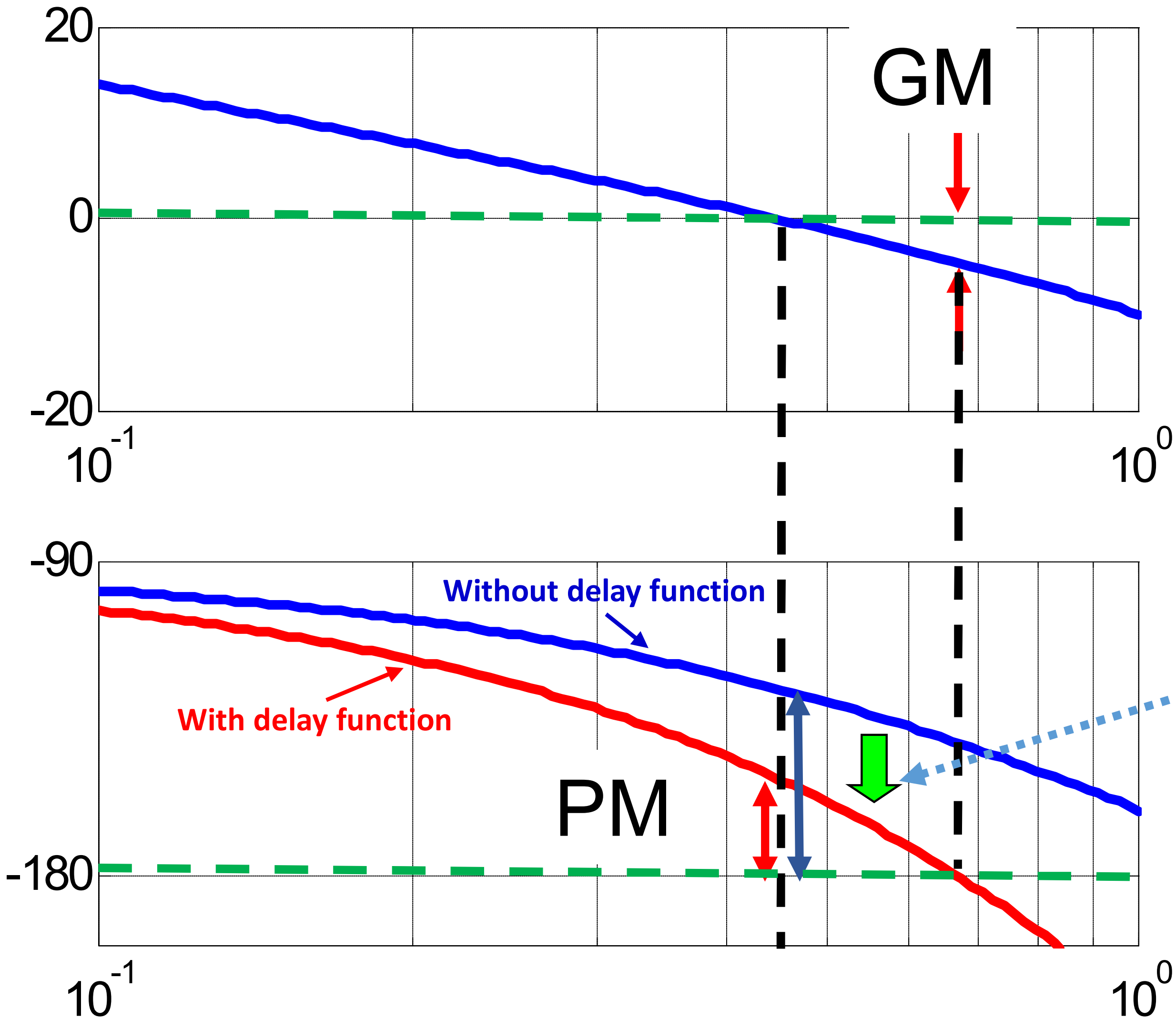


With a time delay:

$$L(s) = \frac{e^{-s}}{s(s+1)(s+2)}$$



Example 7 (cont'd): Relative stability with time delay



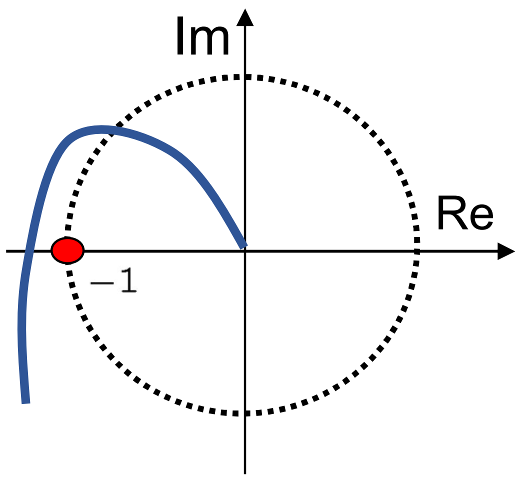
$$L(s) = \frac{1}{s(s+1)(s+2)}$$

$$L(s) = \frac{e^{-s}}{s(s+1)(s+2)}$$

Time delay reduces relative stability!

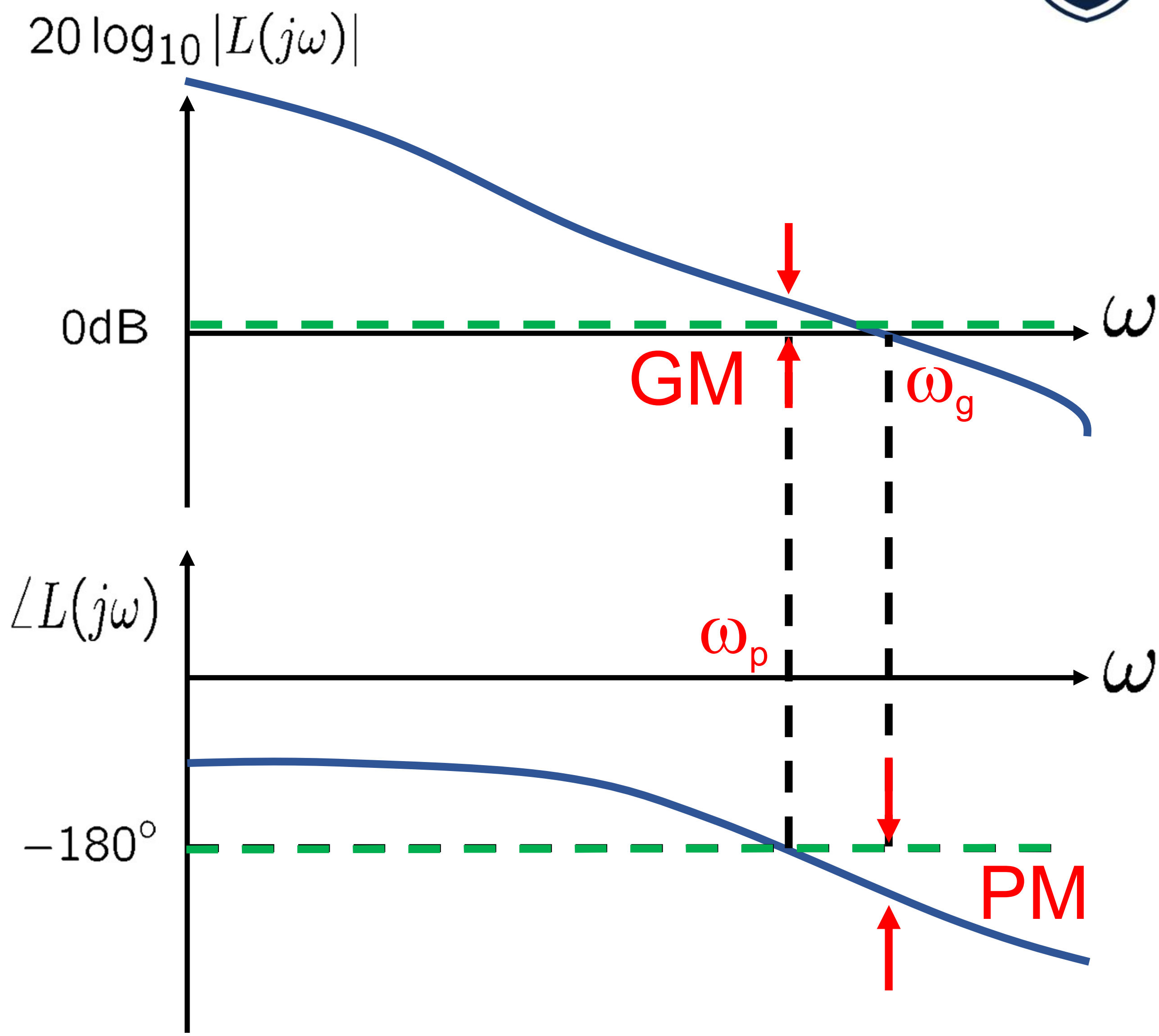
Unstable closed-loop case

Assume that $P = 0$.



Note: In the above case, both GM and PM are negative. Therefore, the closed-loop system is **unstable**. Even if one of the margins (PM or GM) is negative, the closed-loop system is considered unstable.

Stability Criterion:
If **GM** is below the **0 dB line** (i.e., positive) and **PM** is above the **-180 line** (i.e., positive), the CL system is **stable**.



Summary

- Relative stability: Closeness of Nyquist plot to the critical point **-1**.
  - Gain margin is related to phase crossover frequency
  - Phase margin is related to gain crossover frequency
- Relative stability on Bode plot
- We normally emphasize PM in controller design.
- Next
  - Frequency domain specifications (i.e., the use of Bode plot in controller design)