Lecture 11:
Phase Noise

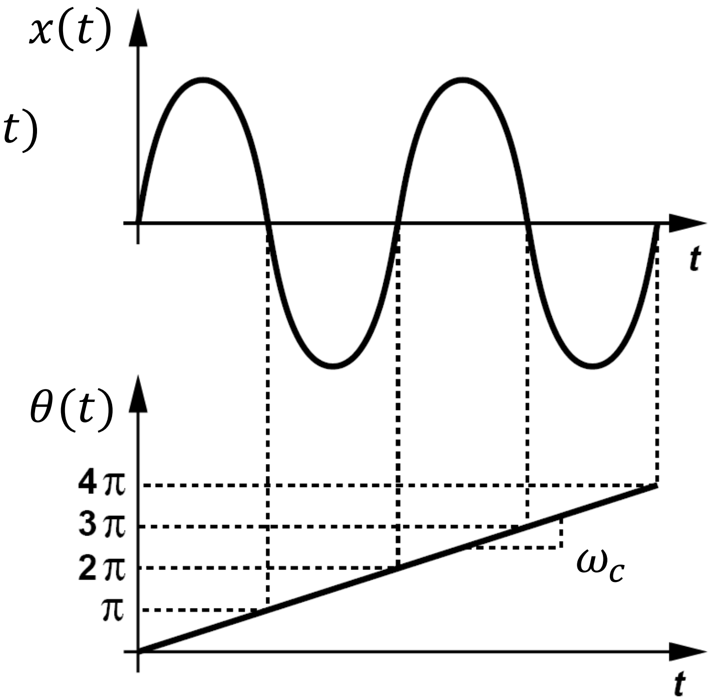
Adapted from *The Design of CMOS Radio-Frequency Integrated Circuits.* T. Lee. Copyright 2004 Cambridge.
Adapted from *RF Microelectronics*. B. Razavi. Copyright 2012 Prentice Hall

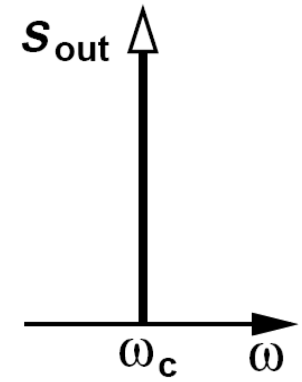
ELEC 404

Ideal Oscillator – Total Phase

$$x(t) = A\cos(\theta(t)) = A\cos(\omega_c t)$$

↑ Total Phase





- $\theta(t)$ sweeps exactly 2π every oscillation cycle.
- PSD is an impulse at ω_c.

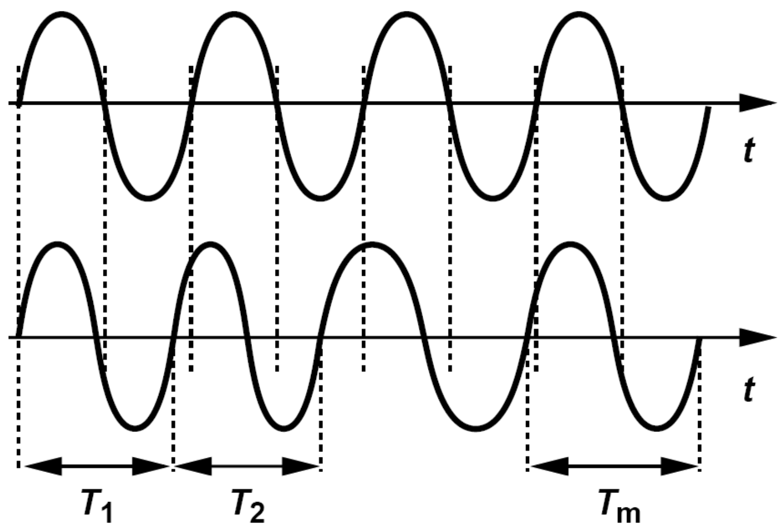
Review: PSD

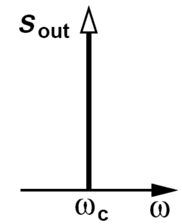
- PSD $S_v(f)$ is defined for a stationary signal v(t)
- Stationary signals are constant in their statistical parameters over time
- Autocorrelation function (R) of a stationary signal depends only on the time difference
  - $R_v(t_1, t_2) = E[v(t_1).v(t_2)] = R_v(t_1 - t_2) = R_v(\tau)$, where E is the expectation
- PSD is the Fourier Transform of the autocorrelation of v(t).
  - $PSD = Sv(f) = \Im\{R_v(\tau)\}$
- PSD creates a two-sided spectrum
- Often, it is easy to use a single-sided spectrum (scaled 2x) if aliasing or noise-folding are not present.

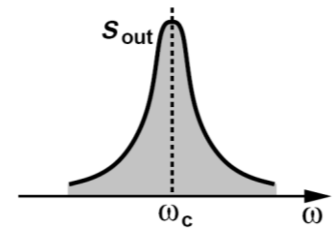
Ideal vs. Real Oscillator – Phase Noise

$$x(t) = Acos(\omega_c t)$$

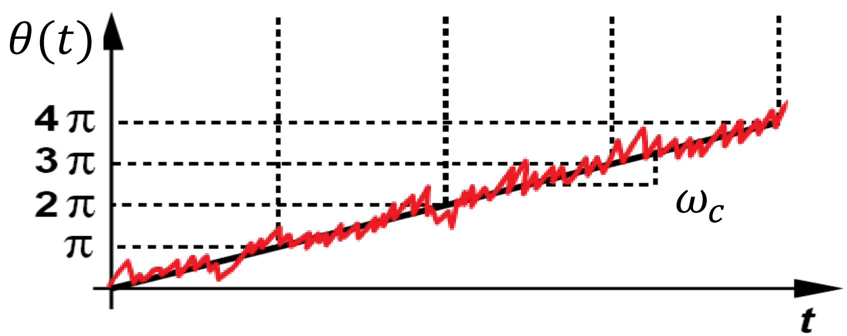
$$x(t) = Acos(\omega_c t + \Phi_n(t))$$







- The noise of the oscillator devices randomly perturbs the zero crossings.
- ***$\Phi_n(t)$*** is called the "phase noise" ←→ random jitter (RJ) in time domain.
- PSD experiences random variations → departs from ***ω_c*** occasionally.



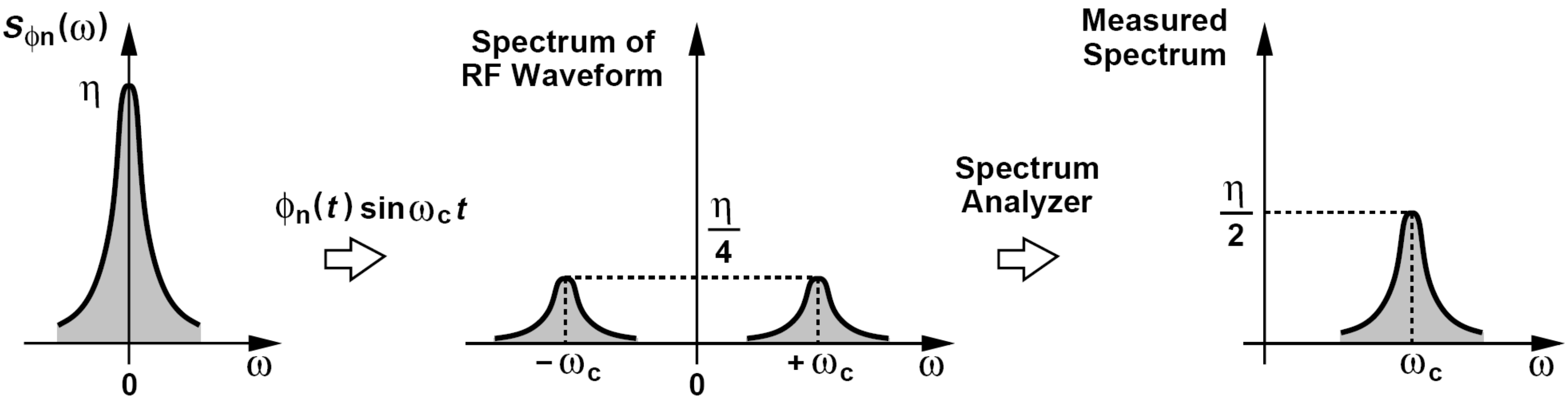
Real Oscillator – Amplitude Noise

$$x(t) = [A + a_n(t)]cos(\omega_c t + \Phi_n(t))$$

- **Assume the oscillator is followed by hard limiting stages such as switching transistors in a mixer, dividers, etc.**
- **Most of the LO amplitude noise is "masked" – neglect the effect.**

PN Representations – I

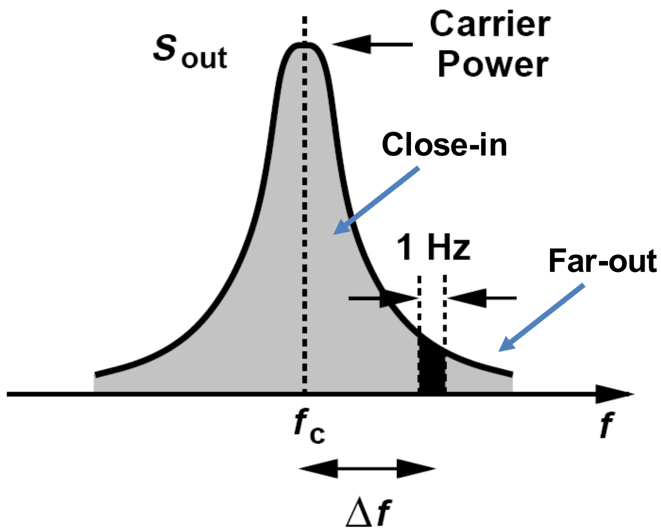
$$x(t) = A\cos[\omega_c t + \phi_n(t)]$$
$$\approx A\cos\omega_c t - A\sin\omega_c t\sin[\phi_n(t)]$$
$$\approx A\cos\omega_c t - A\phi_n(t)\sin\omega_c t.$$



- $\Phi_n(t)$ multiplied by $\sin\omega_c t$ → its PSD, $S_{\Phi n}$, multiplied by 1/4 as it is translated to $\pm\omega_c$
- A spectrum analyzer measuring the resulting spectrum folds the –ve frequency spectrum atop the +ve frequency spectrum, raising the spectral density by 2x.

PN Representation – II

- PN specified at a certain frequency offset with respect to ω_c.
- Consider a 1Hz BW of the spectrum at an offset of Δf, measure the power in this BW, and normalize the result to the "carrier power", called "dB with respect to the carrier".

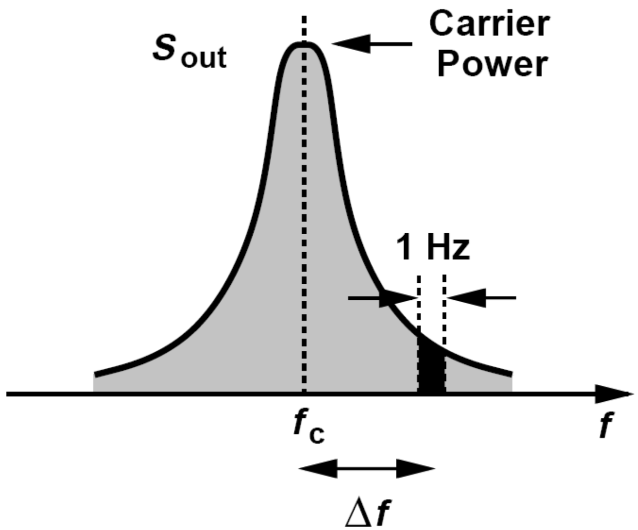


$$\mathcal{L}(\Delta\omega) = 10log_{10}\left\{\frac{P_{1Hz}(\omega_c + \Delta\omega)}{P_s}\right\}$$

- PN reaches a constant floor at large frequency offsets (beyond a few MHz).
- Remember the 2X factor for SSB vs. DSB PN.

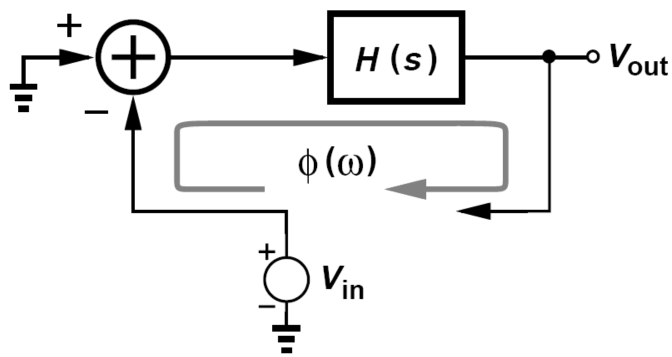
PN Example

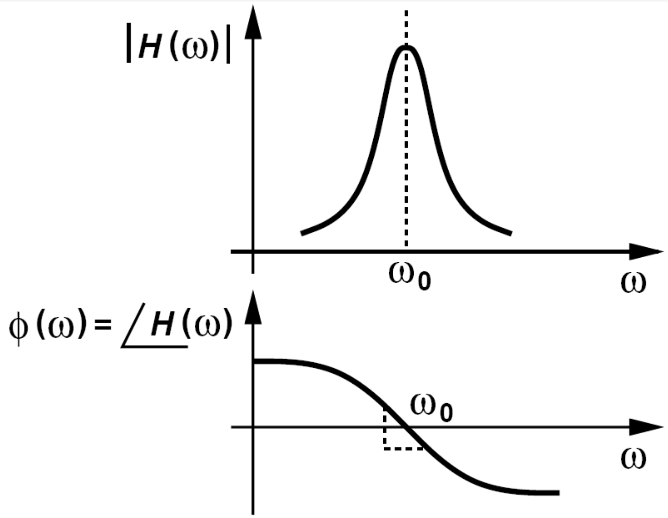
At high carrier frequencies, it is difficult to measure the noise power in a 1Hz BW. Suppose a spectrum analyzer measures a noise power of –70dBm in a 1kHz BW @ 1MHz offset. How much is the PN at this offset if the average oscillator output power is –2dBm?



- 1kHz BW carries 10 log(1000) = 30dB higher noise than a 1Hz BW.
  - → PN_{1Hz} = –100 dBm.
  - → $\mathcal{L}$ (1MHz) = -100 – (–2) = –98dBc/Hz.

Quality Factor (Q) of an Oscillator





- **Consider the oscillator circuit as a feedback system and examine the phase of the OL xfn at the resonance frequency.**

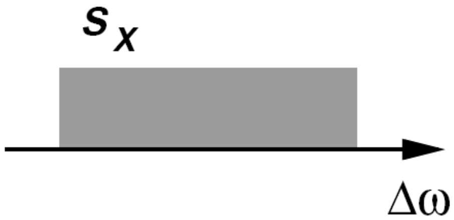
$$Q = \frac{\omega_0}{2}\left|\frac{d\phi}{d\omega}\right|$$

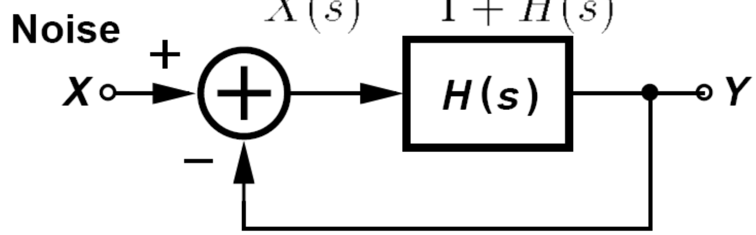
- **Noise injected by the devices attempts to deviate the frequency from ω_0, changing the total phase shift around the loop, violating Barkhausen's criterion and forcing the oscillator to return to ω_0.**
- **The larger the slope of $\Phi(j\omega)$, the greater the restoration force.**
- **Oscillators with a high open-loop Q tend to spend less time at frequencies other than ω_0.**

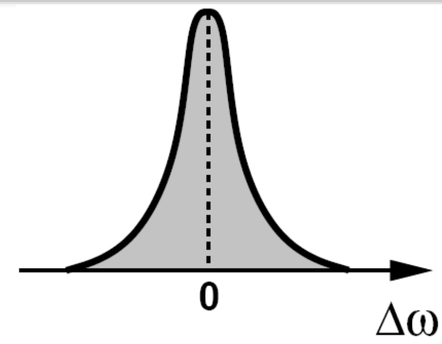
B. Razavi, "A study of phase noise in CMOS oscillators," JSSC Mar. 1996.

Noise Shaping in Oscillators

$$\frac{Y(s)}{X(s)} = \frac{H(s)}{1 + H(s)}$$







In the vicinity of the ω_0, approximate *H*(*jω*) with the first two terms in its Taylor series:

$$H(j\omega) \approx H(j\omega_0) + \Delta\omega \frac{dH}{d\omega}$$

-1   <<1

$$\frac{Y}{X}(j\omega_0 + j\Delta\omega) \approx \frac{-1}{\Delta\omega \frac{dH}{d\omega}}$$

The noise spectrum is "shaped" by

$$|\frac{Y}{X}(j\omega_0 + j\Delta\omega)|^2 = \frac{1}{\Delta\omega^2 |\frac{dH}{d\omega}|^2}$$

B. Razavi, "A study of phase noise in CMOS oscillators," JSSC Mar. 1996.

Noise Shaping in Oscillators – II

Write *H*(*jω*) in polar form, and differentiate with respect to *ω*,

$$\frac{dH}{d\omega} = \left(\frac{d|H|}{d\omega} + j|H|\frac{d\phi}{d\omega}\right)\exp(j\phi)$$

$$|\frac{dH}{d\omega}|^2 = |\frac{d|H|}{d\omega}|^2 + |\frac{d\phi}{d\omega}|^2|H|^2$$

← **Leads to a general definition of *Q***

negligible comparatively (first term); **~1** ($|H|$) — **for an LC oscillator near ω_0**

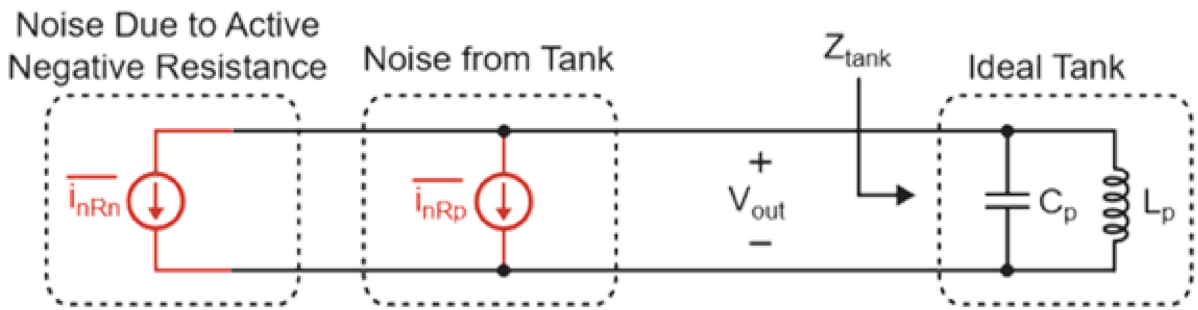
$$|\frac{Y}{X}(j\omega_0 + j\Delta\omega)|^2 = \frac{1}{\frac{\omega_0^2}{4}|\frac{d\phi}{d\omega}|^2}\frac{\omega_0^2}{4\Delta\omega^2}$$

$$|\frac{Y}{X}(j\omega_0 + j\Delta\omega)|^2 = \frac{1}{4Q^2}\left(\frac{\omega_0}{\Delta\omega}\right)^2$$

- **Same as Leeson's Equation**
- **Open-loop *Q* signifies how much the oscillator rejects the noise.**

B. Razavi, "A study of phase noise in CMOS oscillators," JSSC Mar. 1996.

Noise Factor, F



Ack: M. Perrott

- **Assuming uncorrelated noise sources, total noise spectral density impacting the amplitude and phase of the oscillator:**

$$\frac{\overline{v_{out}^2}}{\Delta f} = \left(\frac{\overline{i_{nRp}^2}}{\Delta f} + \frac{\overline{i_{nRn}^2}}{\Delta f}\right)|Z_{tank}(\Delta f)|^2 = \frac{\overline{i_{nRp}^2}}{\Delta f}\underline{\left(1 + \frac{\overline{i_{nRn}^2}}{\Delta f} \Big/ \frac{\overline{i_{nRp}^2}}{\Delta f}\right)}|Z_{tank}(\Delta f)|^2$$

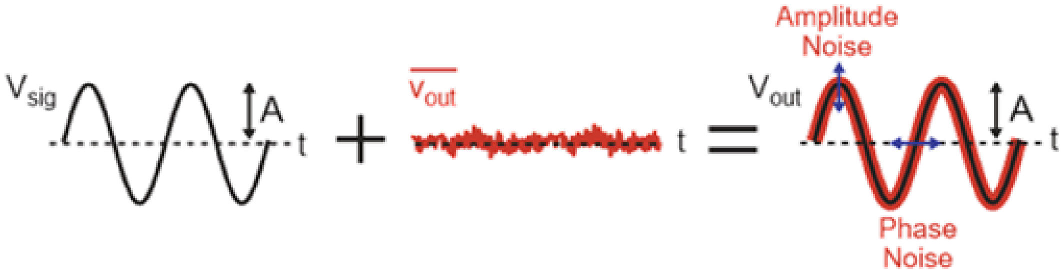
$$\mathbf{F(\Delta f)} = \frac{\text{total noise in tank at frequency } \Delta f}{\text{noise in tank due to tank loss at frequency } \Delta f}$$

- **Tank noise spectral density (single-sided spectrum):** $\frac{\overline{i_{nRp}^2}}{\Delta f} = 4kT\frac{1}{R_p}$
- **Tank impedance,** $|\frac{Y}{X}(j\omega_0 + j\Delta\omega)|^2 = \frac{1}{4Q^2}\left(\frac{\omega_0}{\Delta\omega}\right)^2 \Rightarrow |Z_{tank}(\Delta f)|^2 \approx \left(\frac{R_p}{2Q}\frac{f_o}{\Delta f}\right)^2$

- **Total o/p noise spectral density (single-ended)**

$$\frac{\overline{v_{out}^2}}{\Delta f} = 4kT\frac{1}{R_p}F(\Delta f)\left(\frac{R_p}{2Q}\frac{f_o}{\Delta f}\right)^2 = \boxed{4kTF(\Delta f)R_p\left(\frac{1}{2Q}\frac{f_o}{\Delta f}\right)^2}$$

Leeson's Equation for Oscillator



$$P_{sig} = \frac{V^2_{sig,rms}}{R_p} = \frac{(A/\sqrt{2})^2}{R_p}, \quad S_{noise}(\Delta f) = \frac{1}{R_p}\frac{\overline{v^2_{out}}}{\Delta f}$$

Referred to R_p

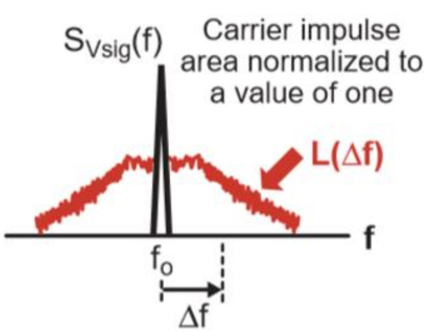
- **Equipartition theorem of thermodynamics → noise on a sinusoidal V_{sig} splits evenly between amplitude and phase, and amplitude variations suppressed by feedback in the oscillator**

$$\left.\frac{\overline{v^2_{out}}}{\Delta f}\right|_{phase} = \boxed{2kTF(\Delta f)R_p\left(\frac{1}{2Q}\frac{f_o}{\Delta f}\right)^2} \text{ (single-sided)}$$

$$L(\Delta f) = 10\log\left(\frac{\text{Spectral density of noise}}{\text{Power of carrier}}\right)$$

$$= 10\log\left(\frac{S_{noise}(\Delta f)}{P_{sig}}\right) = \boxed{10\log\left(\frac{2kTF(\Delta f)}{P_{sig}}\left(\frac{1}{2Q}\frac{f_o}{\Delta f}\right)^2\right)}$$

Output Spectrum



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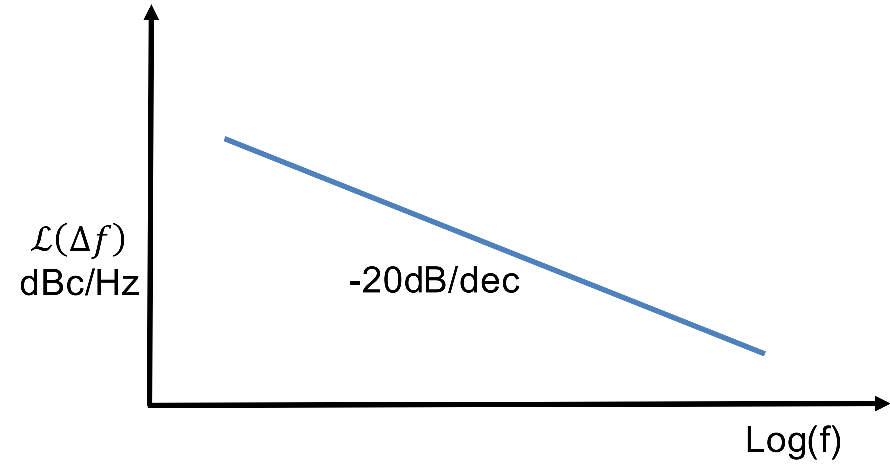
Leeson's PN Model – I

$$\mathcal{L}(\Delta\omega) = 10log_{10}\left[\frac{2FkT}{P_s}\left(\frac{f_0}{2Q\Delta f}\right)^2\right]$$

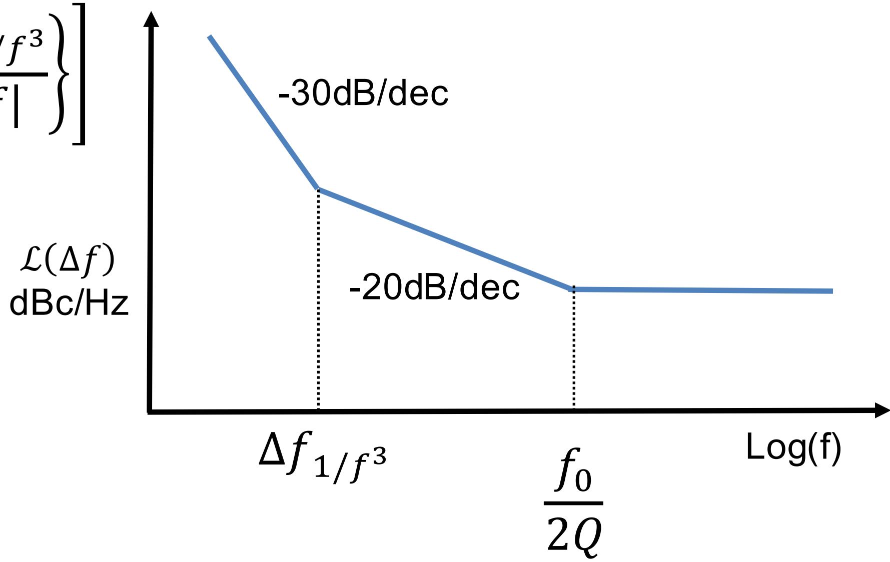


- **F = empirical factor**
- **Noise increases as oscillator frequency increases**
- **Higher Q → lower noise**
- **Higher carrier signal power (output voltage swing) → lower noise**

D. B. Leeson, "A simple model of feedback oscillator noise spectrum," Proc. IEEE, Feb. 1966.

Leeson's PN Model – II

$$\mathcal{L}(\Delta f) = 10log_{10}\left[\frac{2kTF}{P_s}\left\{1+\left(\frac{f_0}{2Q\Delta f}\right)^2\right\}\left\{1+\frac{\Delta f_{1/f^3}}{|\Delta f|}\right\}\right]$$



- **F = Empirical factor**
- **PN depends on the amount of noise that different devices inject and the point in time during a cycle at which the devices inject noise (some parts of the waveform are more sensitive than others)**
- $\Delta f_{1/f^3}$ **= 1/f noise oscillator corner**
- **Accounts for trends in all regions of PN, but provides no guidelines on how to reduce F or** $\Delta f_{1/f^3}$

D. B. Leeson, "A simple model of feedback oscillator noise spectrum," Proc. IEEE, Feb. 1966.

Other PN Models

- **J. Rael and A. Abidi, CICC 2000**
- **A. Hajimiri and T. Lee, JSSC 1998**