Mathematics 220 — Midterm — 45 minutes

23rd & 24th October 2024

- The test consists of 8 pages and 5 questions worth a total of 40 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number								
Section								
Name								
Signature								

Please do not write on this page — it will not be marked.

Additional instructions

General

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor. You must, **on both sides** of any extra pages,
  - put your name and student number, and
  - indicate the test-number and question-number.
- Please do not dismember your test. You must submit all pages.
- Smoking is strictly prohibited during the test.

1. 5 marks

(a) Write the converse of the following statement:

If you like sweet food, you like baklava.

Solution: If you like baklava then you like sweet food.

(b) Negate the following statement:

$$\forall\, x \in \mathbb{R},\ \exists\, y \in \mathbb{Z} \text{ s.t. } \forall z \in \mathbb{Z}, (x > y) \wedge (x^z > y^z).$$

Note that simply putting a "not" at the front is insufficient.

Solution:

$$\exists\, x \in \mathbb{R} \text{ s.t. } \forall y \in \mathbb{Z}, \exists z \in \mathbb{Z}, \text{ s.t. } (x \leq y) \vee (x^z \leq y^z).$$

(c) Determine whether the following statement is true or false. Prove that your answer is correct.

If $a, b, c, d \in \mathbb{Z}$, $a|b$, and $c|d$, then $ac|bd$.

Solution: This statement is true. Let $a, b, c, d \in \mathbb{Z}$ and assume that $a|b$, and $c|d$. Then, we see that $b = am$ and $d = cn$ for some $n, m \in \mathbb{Z}$. Hence, $bd = ac(mn)$, and since $mn \in \mathbb{Z}$, we see that $ac \mid bd$.

(d) Let P, Q, R be statements and assume that

$$(P \wedge Q) \implies (P \wedge R)$$

is false. What are the possible truth values of P, Q, R?

Solution: Since the implication is false, we know that the hypothesis is true, while the conclusion must be false.

- Since the hypothesis is true, we know that P, Q are both true.
- Since the conclusion is false, we know that R must be false (we already know P is true).

Thus we have $(P, Q, R) = (T, T, F)$.

2. 10 marks Let $A = \{n \in \mathbb{N} : 3 \mid n \text{ and } 4 \nmid n\}$. Note that all numbers in A are positive.
Determine whether the following two statements are true or false — explain your answers ("true" or "false" is not sufficient).

(a) $\exists x \in A$ s.t. $\forall y \in A, x + y \in A$.

(b) $\forall x \in A, \forall y \in A,\ \exists z \in A$ s.t. $x + y + z \in A$.

Solution: (a) Statement is false.
We will prove the negation: $\forall x \in A,\ \exists y \in A$ such that $x + y \notin A$.
Given $x \in A$, let $y = 3x$. Since $3 \mid 3x$, and $4 \nmid x$ (so $4 \nmid 3x$) we have $y \in A$.
Now $x + y = 4x$, so $4 \mid (x + y)$, therefore $x + y \notin A$.

(b) Statement is true.
Note that for any $z \in A$, since $3 \mid x, y, z$, we have $3 \mid x + y + z$.
Given $x, y \in A$, write $x + y \equiv b \pmod 4$.
Cases $b = 0, 2, 3$: Let $z = 3$. Then $x + y + z \equiv 3, 1, 2 \pmod 4$ (respectively), and so $x + y + z \in A$.
Case $b = 1$: Let $z = 6$. Then $x + y + z \equiv 3 \pmod 4$, and so $x + y + z \in A$.

3. 8 marks Let $x \in \mathbb{R}$. Prove that if $x \notin (-5, 2)$, then

$$|x + 5| + |2 - x| = |2x + 3|.$$

Solution: Let $x \in \mathbb{R}$ and assume $x \notin (-5, 2)$. Then, we have two cases: either $x \leq -5$, or $x \geq 2$.

Case 1: If $x \leq -5$, then $x + 5 \leq 0$ so that $|x + 5| = -x - 5$. Second, we have $-x \geq 5$ so that $2 - x \geq 7 \geq 0$ and hence $|2 - x| = 2 - x$. Third, we have $-2x - 3 \geq 0$, and thus $|2x + 3| = -2x - 3$. We can then rewrite the left and right side of the equation without absolute values to get

$$|x + 5| + |2 - x| = (-x - 5) + (2 - x) = -2x - 3 = |2x + 3|.$$

This completes this case of the proof.

Case 2: If $x \geq 2$, then $2 - x \leq 0$, but also $x + 5 \geq 7 \geq 0$. We also have $2x + 3 \geq 0$, and thus $2x + 3 = |2x + 3|$. We can then remove the absolute values from the left and right side of the equation to get

$$|x + 5| + |2 - x| = (x + 5) + (-2 + x) = 2x + 3 = |2x + 3|.$$

We conclude that in both cases, the equation holds.

4. $\boxed{\text{7 marks}}$ Recall that for $n \in \mathbb{N}$, we define $n! = n(n-1)(n-2)\cdots 2 \cdot 1$.
Find, with proof, all $n \in \mathbb{N}$ such that $(n+1)! \geq 2^{(n+2)}$.

Solution: First, let us compute $(1+1)! = 2 < 8 = 2^{1+2}$, $(2+1)! = 6 < 16 = 2^{2+2}$, $(3+1)! = 24 < 32 = 2^{3+2}$, $(4+1)! = 120 \geq 64 = 2^{4+2}$. Thus $n = 4$ is the first integer for which $(n+1)! \geq 2^{n+2}$.

We will prove by induction that for all $n \geq 4$, we have $(n+1)! \geq 2^{n+2}$. The base case $n = 4$ was verified above. We will now prove the induction step: suppose that for some $n \in \mathbb{N}$ we have $(n+1)! \geq 2^{n+2}$. Then

$$(n+2)! = (n+1)!(n+2) \geq 2^{n+2}(n+2) \geq 2^{n+2} \cdot 2 = 2^{(n+1)+2}.$$

The first equality used the definition of $(n+2)!$; the second inequality used the induction hypothesis. The third inequality used the fact that $n \in \mathbb{N}$ and thus $n \geq 1$.

This completes the induction step. We conclude that $(n+1)! \geq 2^{n+2}$ for all $n \geq 4$. We already verified that the inequality is false for $n = 1, 2, 3$. Thus $(n+1)! \geq 2^{n+2}$ if and only if $n \geq 4$.

5. $\boxed{\text{10 marks}}$ We say that a sequence (x_n) has a **limit** $L \in \mathbb{R}$ as $n \to \infty$ when

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N \text{ with } n \in \mathbb{N}, \text{ we have } (|x_n - L| < \epsilon).$$

Prove that the sequence

$$(x_n) = \left(\frac{3n^3}{4n^3 + 5} \right)$$

has limit $L = \frac{3}{4}$, as $n \to \infty$.

Solution:

Proof: Let $\epsilon > 0$. Then, we know that for all $n \in \mathbb{N}$, since $n \geq 1$ we have

$$\begin{aligned} 16n^3 + 20 &> 16n^3 \\ &> 15n^3 \\ &> 15n. \end{aligned}$$

This implies that $16n^3 + 20 > 15n$ for all $n \in \mathbb{N}$.

Hence, for $N = \left\lceil \frac{1}{\epsilon} \right\rceil$, where $\lceil . \rceil$ denotes the ceiling function. Then, $\forall n > N$, we have

$$\left| \frac{3n^3}{4n^3+5} - \frac{3}{4} \right| = \frac{12n^3 - 12n^3 - 15}{16n^3 + 20} \leq \frac{15}{15n^3} < \frac{1}{n} < \frac{1}{N} \leq \epsilon.$$

Therefore the sequence is convergent.

Alternate proof: Let $\epsilon > 0$. Then, for $N = \left\lceil \sqrt[3]{\frac{15}{16\epsilon}} \right\rceil$, where $\lceil . \rceil$ denotes the ceiling function. Then, $\forall n > N$, we have

$$\left| \frac{3n^3}{4n^3+5} - \frac{3}{4} \right| = \left| \frac{12n^3 - 12n^3 - 15}{16n^3 + 20} \right| \leq \frac{15}{16n^3} < \frac{15}{16N^3} \leq \epsilon.$$

Therefore the sequence is convergent.