Mathematics 220 — Midterm — 45 minutes

23rd & 24th October 2024

- The test consists of 8 pages and 5 questions worth a total of 40 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number								
Section								
Name								
Signature								

Please do not write on this page — it will not be marked.

Additional instructions

General

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor. You must, **on both sides** of any extra pages,
  - put your name and student number, and
  - indicate the test-number and question-number.
- Please do not dismember your test. You must submit all pages.
- Smoking is strictly prohibited during the test.

1. 5 marks

(a) Write the contrapositive of the following statement:

If today is cloudy then I need a raincoat.

Solution: If I don't need a raincoat then today is not cloudy.

(b) Negate the following statement:

$$\exists x \in \mathbb{R} \text{ s.t. } \big((x^2 < 1) \implies (\forall y \in \mathbb{N}, x^n < 1)\big)$$

Note that simply putting a "not" at the front is insufficient.

Solution:

$$\forall x \in \mathbb{R},\ \big((x^2 < 1) \wedge (\exists y \in \mathbb{N} \text{ s.t. } x^n \geq 1)\big)$$

(c) Determine whether the following statement is true or false. Prove your answer.

$$\exists x \in \mathbb{R} \text{ s.t. } \forall y \in \mathbb{R},\ (x > y) \implies (x^2 > y^2)$$

Solution: False: We prove the negation:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } (x > y) \wedge (x^2 \leq y^2)$$

Set $y = -1 - |x|$. Then $x > y$ and $y^2 = |x|^2 + 2|x| + 1 > x^2$ as required.

(d) Let P, Q, R be statements and consider the statement:

$$(P \vee Q) \vee (\sim (P \wedge R))$$

What truth values of P, Q, R make this statement false?

Solution: We must have that both clauses are false.

- Since $\sim (P \wedge R)$ is false we must have that both P, R are true.
- Since $P \vee Q$ is false we must have that both P, Q are false. This cannot happen since we need P to be true.

Hence there are no values of P, Q, R that make the statement false.

2. 10 marks Let P denote the set of all primes. Consider the set

$$A = \{a \in P \mid a \neq 3\}.$$

Determine whether the following two statements are true or false — explain your answers ("true" or "false" is not sufficient).

(a) $\forall\, a \in A, \exists\, b \in A$ s.t. $3 \mid (a + b)$,

(b) $\forall\, a \in A, \forall\, b \in A,\ \exists\, c \in A$ s.t. $3 \mid (a + b + c)$.

Solution: (a) Statement is true.
The only positive integers dividing a prime are 1 and itself. Since $3 \notin A$, we have $3 \nmid a,\ \forall a \in A$. So given $a \in A$, we have two cases:
Case $a \equiv 1 \pmod 3$: Let $b = 2 \in A$. Then $a+b \equiv 0 \pmod 3$, so $3 \mid (a+b)$.
Case $a \equiv 2 \pmod 3$: Let $b = 7 \in A$. Then $a+b \equiv 0 \pmod 3$, so $3 \mid (a+b)$.

(b) Statement is false.
We will prove the negation: $\exists a \in A$ s.t. $\exists b \in A$ s.t. $\forall c \in A,\ \ 3 \nmid (a+b+c)$.
Let $a = 5, b = 7$. Then $a + b \equiv 0 \pmod 3$. Since $c \in A$, $c \not\equiv 0 \pmod 3$ (as explained in part (a)), so we must have $a + b + c \equiv 1$ or $2 \pmod 3$. So $3 \nmid (a + b + c)$.

3. 7 marks Find all real x such that

$$|7 - x| \geq |x + 5|.$$

Solution: Let $x \in \mathbb{R}$. We will consider the following four cases

Case 1: $x < -5$: In this case, $|7 - x| = 7 - x$ and $|x + 5| = -5 - x$. Then since $7 \geq -5$, we have $|7 - x| = 7 - x \geq -5 - x = |x + 5|$, so the inequality holds.

Case 2: $-5 \leq x \leq 7$: In this case, $|7 - x| = 7 - x$ and $|x + 5| = x + 5$. Then $|7 - x| \geq |x + 5|$ if and only if $7 - x \geq x + 5$. Re-arranging, we see that this holds if and only if $2 \geq 2x$, i.e. $x \leq 1$.

Case 3: $x > 7$: In this case, $|7 - x| = x - 7$ and $|x + 5| = x + 5$. Then since $-7 < 5$, we have $|7 - x| = x - 7 < x + 5$, and hence the inequality does not hold.

Combining the three cases analyzed above, we see that $|7 - x| \geq |x + 5|$ if and only if $x \leq 1$.

4. [8 marks] Let $a_1 = 6$, $a_2 = 20$ and

$$a_n = 3a_{n-1} + 4a_{n-2} - 10$$

for $n \geq 3$.

Prove that $a_n \geq 4^n + 2$ for all $n \in \mathbb{N}$.

Solution:

Proof. We proceed by strong induction.

- Base case: When $n = 1$, we have $a_1 = 6 \geq 4^1 + 2$. When $n = 2$, we have $a_2 = 20 \geq 4^2 + 2$. So the result holds for $n = 1, 2$.

- Inductive step: Assume the result holds for $1 \leq k \leq n$, $n \geq 2$. We need to show that the result holds for $n + 1$. We have

  $$a_{n+1} = 3a_n + 4a_{n-1} - 10.$$

  By the inductive assumption, $a_n \geq 4^n + 2$ and $a_{n-1} \geq 4^{n-1} + 2$. Then

  $$\begin{aligned} a_{n+1} &= 3a_n + 4a_{n-1} - 10 \\ &\geq 3(4^n + 2) + 4(4^{n-1} + 2) - 10 \\ &= 3 \cdot 4^n + 6 + 4 \cdot 4^{n-1} + 8 - 10 \\ &= 3 \cdot 4^n + 4^n + 4 \\ &= 4^{n+1} + 4 \geq 4^{n+1} + 2. \end{aligned}$$

  Thus, the result holds for $n + 1$.

Since the base case and inductive step hold, the result follows by induction. □

5. 10 marks Recall that if $a, L \in \mathbb{R}$ if **f** is a real-valued function. We say that the limit of **f** as x approaches a is L when

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ s.t. } (0 < |x - a| < \delta) \implies (|f(x) - L| < \epsilon).$$

Now, let $f(x) = x^2 + x$ (with domain $\mathbb{R}$). Prove, using the rigorous definition of a limit, that

$$\lim_{x \to 3} f(x) = 12.$$

Solution: Let $\epsilon > 0$. Then we can choose $\delta = \min(1, \epsilon/8)$. Then, assuming $0 < |x - 3| < \delta$ we get $2 < x < 4$. This implies that $6 < x + 4 < 8$.

$$\begin{aligned} |f(x) - 12| &= |x^2 + x - 12| \\ &= |x - 3||x + 4| \\ &< \delta \cdot 8 \\ &\leq \epsilon. \end{aligned}$$

Therefore $\lim_{x \to 3} f(x) = 12$.