Mathematics 220 — Midterm — 45 minutes

23rd & 24th October 2024

- The test consists of 8 pages and 5 questions worth a total of 40 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number								
Section								
Name								
Signature								

Please do not write on this page — it will not be marked.

Additional instructions

General

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor. You must, **on both sides** of any extra pages,
  - put your name and student number, and
  - indicate the test-number and question-number.
- Please do not dismember your test. You must submit all pages.
- Smoking is strictly prohibited during the test.

1. 5 marks

(a) Write the contrapositive of the following statement:

If you like horror movies then you like Halloween.

Solution: If you don't like Halloween then you don't like horror movies.

(b) Negate the following statement:

$$\forall x \in \mathbb{R},\ \big((\forall y \in \mathbb{N}, x^y < 1) \implies (x^2 < 1)\big)$$

Note that simply putting a "not" at the front is insufficient.

Solution:

$$\exists x \in \mathbb{R} \text{ s.t. } \big((\forall y \in \mathbb{N}, x^y < 1) \wedge (x^2 \geq 1)\big)$$

(c) Determine whether the following statement is true or false. Prove your answer.

$$\exists x \in (0, \infty) \text{ s.t. } \forall y \in \mathbb{R}, \ \big((|x| > |y|) \implies (x^2 > y^2)\big)$$

Solution: True: If $|x| > |y|$, then

$$x^2 = |x|^2 = |x||x| > |x||y| > |y||y| = |y|^2 = y^2.$$

(d) Let P, Q, R be statements and consider the statement:

$$(P \wedge R) \implies (P \vee Q)$$

What truth values of P, Q, R make this statement false.

Solution: The statement is false if $(P \wedge R)$ is true and $(P \vee Q)$ is false. However, if $(P \wedge R)$ is true, then in particular P is true, and thus $(P \vee Q)$ is true. We conclude that there are no values of P, Q, R that make the statement false.

2. 10 marks Let P denote the set of all primes. Consider the set

$$A = \{a \in P \mid a > 3\}.$$

Determine whether the following two statements are true or false — explain your answers ("true" or "false" is not sufficient).

(a) $\forall a \in A, \exists b \in A$ s.t. $6 \mid (a+b)$,

(b) $\forall a \in A, \forall b \in A, \exists c \in A$ s.t. $6 \mid (a+b+c)$.

Solution: (a) Statement is true.
The only positive integers dividing a prime are 1 and itself. Since $2, 3 \notin A$, we have $2, 3 \nmid a$, $\forall a \in A$. So $a \not\equiv 0, 2, 3, 4 \pmod 6$ (otherwise 2 or 3 would divide a).
So given $a \in A$, we have two cases:
Case $a \equiv 1 \pmod 6$: Let $b = 5 \in A$. Then $a+b \equiv 0 \pmod 6$, so $6 \mid (a+b)$.
Case $a \equiv 5 \pmod 6$: Let $b = 7 \in A$. Then $a+b \equiv 0 \pmod 6$, so $6 \mid (a+b)$.

(b) Statement is false.
We will prove the negation: $\exists a \in A$ s.t. $\exists b \in A$ s.t. $\forall c \in A,\ \ 6 \nmid (a+b+c)$.
As explained in part (a), any element in A must be odd.
So $a+b$ is even (e.g., choose $a = b = 5$). Therefore, $a+b+c$ is odd, which means that $2 \nmid (a+b+c)$, and so $6 \nmid (a+b+c)$.

3. 7 marks Find all real x such that

$$|6 - x| \geq |x + 4|.$$

Solution: Let $x \in \mathbb{R}$. We will consider the following four cases

Case 1: $x < -4$: In this case, $|6 - x| = 6 - x$ and $|x + 4| = -4 - x$. Then since $6 \geq -4$, we have $|6 - x| = 6 - x \geq -4 - x = |x + 4|$, so the inequality holds.

Case 2: $-4 \leq x \leq 6$: In this case, $|6 - x| = 6 - x$ and $|x + 4| = x + 4$. Then $|6 - x| \geq |x + 4|$ if and only if $6 - x \geq x + 4$. Re-arranging, we see that this holds if and only if $2 \geq 2x$, i.e. $x \leq 1$.

Case 3: $x > 6$: In this case, $|6 - x| = x - 6$ and $|x + 4| = x + 4$. Then since $-6 < 4$, we have $|6 - x| = x - 6 < x + 4$, and hence the inequality does not hold.

Combining the three cases analyzed above, we see that $|6 - x| \geq |x + 4|$ if and only if $x \leq 1$.

4. 8 marks Let $a_1 = 1$, $a_2 = 4$, and $a_n = 5a_{n-1} - 4a_{n-2}$ for $n \geq 3$. Prove that for all natural numbers n, we have $a_n = 4^{n-1}$.

Solution: We are going to use strong mathematical induction to prove this statement.

Base case: We see that for $n = 1$, we have $a_1 = 1 = 4^{1-1}$, and for $n = 2$, we have $a_2 = 4 = 4^{2-1}$. This implies that the statement is true for $n = 1$ and $n = 2$.

Inductive case: Assume that this statement is true for all $n \leq k$ for some $k \geq 2$, that is $a_n = 4^{n-1}$ for all $n \leq k$. Then, we see that

$$a_{k+1} = 5a_k - 4a_{k-1} = 5 \cdot 4^{k-1} - 4 \cdot 4^{k-2} = 4^{k-2}(20 - 4) = 4^2 \cdot 4^{k-2} = 4^k.$$

Hence, the statement is true for $n = k + 1$.

Therefore, by mathematical induction, we see that $a_n = 4^{n-1}$ for all $n \in \mathbb{N}$.

5. 10 marks Recall that if $a, L \in \mathbb{R}$ if **f** is a real-valued function. We say that the limit of **f** as x approaches a is L when

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } (0 < |x - a| < \delta) \implies (|f(x) - L| < \epsilon).$$

Now, let $f(x) = x^2 + 2x$ (with domain $\mathbb{R}$). Prove, using the rigorous definition of a limit, that

$$\lim_{x \to 2} f(x) = 8.$$

Solution: Let $\epsilon > 0$. Then we can choose $\delta = \min(1, \epsilon/7)$. Then, assuming $0 < |x - 2| < \delta$ we get $1 < x < 3$. This implies that $5 < x + 4 < 7$.

$$\begin{aligned} |f(x) - 12| &= |x^2 + 2x - 8| \\ &= |x - 2||x + 4| \\ &< \delta \cdot 7 \\ &\leq \epsilon. \end{aligned}$$

Therefore $\lim_{x \to 2} f(x) = 8$.