Homework 1

- Please submit your answers to all questions.
- We will mark your answers to 3 questions.
- We will provide you with full solutions to all questions.

1. Let $n \in \mathbb{Z}$. Prove that if $3 \mid n+1$ then $3 \nmid n^2 + 5n + 5$.

   Proof. Assume that $3 \mid n+1$ and hence $n + 1 = 3a$, and so $n = 3a - 1$ for some $a \in \mathbb{Z}$.

   Then we see that

   $$\begin{aligned} n^2 + 5n + 5 &= (3a-1)^2 + 5(3a-1) + 5 = 9a^2 - 6a + 1 + 15a - 5 + 5 \\ &= 9a^2 + 9a + 1 \\ &= 3(3a^2 + 3a) + 1. \end{aligned}$$

   Since $3a^2 + 3a \in \mathbb{Z}$, by division algorithm, it follows that $3 \nmid n^2 + 5n + 5$. □

2. If $-1 < x < 2$, then $x^2 - x - 2 < 0$.

   Proof. Since $-1 < x < 2$, we have $x + 1 > 0$ and $x - 2 < 0$. So

   $$x^2 - x - 2 = (x+1)(x-2) < 0.$$

   □

3. Let x and y be real numbers. Suppose that $x < y$ and $y^2 < x^2$. Show that $x + y < 0$.

   Proof. Let x and y be real numbers and assume that $x < y$ and $y^2 < x^2$. Then, we have $y^2 - x^2 < 0$ and $y^2 - x^2 = (y-x)(y+x) < 0$. Since $y > x$, we can multiply on both side by the positive number $\frac{1}{y-x}$, which yields $(y + x) < 0$. □

4. Let $n, a, b, x, y \in \mathbb{Z}$. If $n \mid a$ and $n \mid b$, then $n \mid (ax + by)$.

   Proof. : Assume $n, a, b, x, y \in \mathbb{Z}$, $n \mid a$, and $n \mid b$. This implies that $a = nk$ and $b = nl$ for some $k, l \in \mathbb{Z}$. Thus, $ax + by = nkx + nly = n(kx + ly)$. Since $kx + ly \in \mathbb{Z}$ for $k, x, l, y \in \mathbb{Z}$, then we see $n \mid (ax + by)$. □

5. Let a, b, c, d be integers. Suppose that $a, c, b - d$ are all odd numbers. Prove $ab + cd$ is odd.

Homework 1

Proof. Let a, b, c, d be integers. Suppose that $a, c, b+d$ are all odd numbers. Then, since $a, c, b+d$ are odd, we can write

$$a = 2k_1 + 1$$
$$c = 2k_2 + 1$$
$$b - d = 2k_3 + 1$$

for some $k_1, k_2, k_3 \in \mathbb{Z}$. Then

$$ab+cd = (2k_1+1)b+(2k_2+1)d = 2(k_1b+k_2d)+(b+d) = 2(k_1b+2k_2d+k_3+d)+1$$

is an odd number by noting $k_1b + k_2d + k_3 + d$ is an integer. □

6. We asked our friend to help with the proof of the problem:

   "Let $a \in \mathbb{Z}$. Prove that if $3a + 1$ is odd then $5a + 2$ is even."

   And they gave us their work:

> Proof: Let $a \in \mathbb{Z}$ and assume that $3a + 1$ is odd. Hence $3a + 1 = 2k + 1$ for some $k \in \mathbb{Z}$. Then we see that which implies that $3a = 2k$, that is, $a = \frac{2k}{3}$. Hence,
>
> $$5a + 2 = \frac{10a}{3} + 2 = 2\left(\frac{5a}{3} + 1\right).$$
>
> Therefore $5a + 2$ is even.

   Is your friend's work correct? If not, how would you prove the statement?

   Solution: The provided proof is wrong/incomplete for a couple of reasons.

   Reason one is that in the proof our friend wrote $a = \frac{2k}{3}$, and in the proof wrote $5a+2 = \frac{10a}{3}+2$ which should be $\frac{10k}{3}+2$. We have to be careful with our variables since one variable can be used in many places and such misuse of the variable can cause confusions later on.

   Reason two, at the end of the proof (with the corrected variable) our friend gets $5a + 2 = 2\left(\frac{5k}{3} + 1\right)$ to conclude that $5a + 2$ is even. However, this result would be correct only when $\left(\frac{5k}{3} + 1\right)$ is an integer. Our friend hasn't shown, or justified that $\left(\frac{5k}{3} + 1\right) \in \mathbb{Z}$.

   The way we can prove the statement is as follows:

Proof. Let Let $a \in \mathbb{Z}$ and assume that $3a+1$ is odd. Hence $3a+1 = 2k+1$ for some $k \in \mathbb{Z}$. Then we see that

$$5a + 2 = 3a + 1 + 2a + 1 = 2k + 1 + 2a + 1 = 2(k + a + 1).$$

Then, since $k + a + 1 \in \mathbb{Z}$, we see that $5a + 2$ is even. $\square$

7. **Definition**: Let $n > 0$. We will call n a *funky* number if $n^k = m$ for some $k \in \mathbb{N}$ and $m \in \mathbb{Z}$.

   Which of the following options makes the following result correct? (You only need to pick the one that makes the statement correct, and prove that it works.)

   Result: Given the definition of *funky* numbers above, if $a > 0$ and $b > 0$ are *funky*, then so is __.

   - the algebraic average of a and b, i.e. $\dfrac{a+b}{2}$.
   - the geometric average of a and b, i.e. $\sqrt{ab}$.
   - the harmonic average of a and b, i.e. $\dfrac{2}{\frac{1}{a}+\frac{1}{b}}$.

   Proof. We are going to prove that if $a > 0$ and $b > 0$ are *funky*, then so is the geometric average. Assume that $a, b > 0$ are *funky*, then, by definition, there are natural numbers k, l and integers m, s such that $a^k = n$ and $b^l = m$. Therefore we see that $(\sqrt{ab})^{2kl} = (ab)^{kl} = n^l m^k$. Since $2kl \in \mathbb{N}$, $\sqrt{ab}$ is defined, and $n^l m^k \in \mathbb{Z}$, we see that $\sqrt{ab}$ is also *funky*. $\square$