Homework 3

- Please submit your answers to all questions.
- We will mark your answers to 3 questions.
- We will provide you with full solutions to all questions.

1. Negate the following statement: There is a real number c, so that for every positive number ϵ there is a positive number M for which

$$\left|\frac{x^2+\sin(x^2)}{x^2+\cos(x)}-c\right|<\epsilon,$$

whenever $x \geq M$.

Proof. First, let us convert this into a statement using quantifiers. The statement becomes

$$\exists\ c \in \mathbb{R} \text{ s.t. } \forall \epsilon > 0,\ \exists M > 0 \text{ s.t. } \forall x \geq M, \left|\frac{x^2+\sin(x^2)}{x^2+\cos(x)}-c\right|<\epsilon.$$

The negation is

$$\forall\ c \in \mathbb{R},\ \exists \epsilon > 0 \text{ s.t. } \forall M > 0, \exists x \geq M \text{ s.t. } \left|\frac{x^2+\sin(x^2)}{x^2+\cos(x)}-c\right| \geq \epsilon.$$

In words: for all real numbers c, there exists a positive number ϵ, so that for all positive numbers M, there is a number $x \geq M$ so that

$$\left|\frac{x^2+\sin(x^2)}{x^2+\cos(x)}-c\right| \geq \epsilon.$$

□

2. Write down the negation of the statement

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{R}, (x \geq y \Rightarrow \frac{x}{y}=1)$$

and determine the statement is true or false.

Proof. Recall that the negation of $P \Rightarrow Q$ is $P \wedge (\sim Q)$. In our case, P is the statement $x \geq y$, and Q is the statement $\frac{x}{y}=1$. It is tempting to write $\sim Q = (\frac{x}{y} \neq 1)$, but we must be careful here, and also account for the possibility that $y=0$. Thus $\sim Q = ((y=0) \vee \frac{x}{y} \neq 1)$.

Thus the negation of the statement is

$$\exists x \in \mathbb{Z}, \forall y \in \mathbb{R}, ((x \geq y) \wedge \big((y=0) \vee (\frac{x}{y} \neq 1)\big).$$

Now let us check whether the statement is true or false. Let x be an integer. Then, taking $y = x + \frac{1}{2}$, we have that $y > x$, so the hypothesis is false which makes the implication true. Thus the statement is true. □

Homework 3

3. Let $A = \{n \in \mathbb{N} : 3 \mid n \text{ or } 4 \mid n\}$. Note that all numbers in A are positive. Determine whether the following four statements are true or false — explain your answers ("true" or "false" is not sufficient).

   (a) $\exists x \in A$ s.t. $\exists y \in A$ s.t. $x + y \in A$.

   (b) $\forall x \in A, \forall y \in A, x + y \in A$.

   (c) $\exists x \in A$ s.t. $\forall y \in A, x + y \in A$.

   (d) $\forall x \in A, \forall y \in A,\ \exists z \in A$ s.t. $x + y + z \in A$.

   Proof. (a) True — Let $x = y = 3 \in A$. Then $x + y = 6 \in A$.

   (b) False — Let $x = 3$ and $y = 4$. Then we see that $x + y = 7 \notin A$

   (c) True — Let $x = 12$ and then let $y \in A$. If $3 \mid y$ then $y = 3k$ and so $x + y = 3(k + 4)$ and so is in A. Similarly, if $4 \mid y$ then $y = 4\ell$ and so $x + y = 4(\ell + 3)$ and so is in A.

   (d) True — Let $x, y \in A$. By Fact 3.0.3 (Euclidean division), there are unique integers q and r so that $x + y = 3q + r$, with $0 \leq r \leq 2$. Let $k = 3 - r$, so $k \geq 1$. Since $4 \equiv 1 \pmod 3$, we have $4k \equiv k \equiv 3 - r \pmod 3$. Let $z = 4k$. We know that $z \in A$, since $4|z$. Finally, by Theorem 5.3.3 we have

   $$x + y + z = 3q + r + 4(3 - r) \equiv r + (3 - r) \equiv 3 \equiv 0 \pmod 3,$$

   This means that $3|(x + y + z)$, and thus $x + y + z \in A$.

   □

4. Negate the following statements and determine whether the original statements are true or false. Justify your answer.

   (a) $\forall n \in \mathbb{Z},\ \exists y \in \mathbb{R} - \{0\}$ such that $y^n \leq y$.

   (b) $\exists y \in \mathbb{R} - \{0\}$ such that $\forall n \in \mathbb{Z},\ y^n \leq y$.

   (c) $\forall x \in \mathbb{R}$ where $x \neq 0$, we have $x \leq 1$ or $\dfrac{1}{x} \leq 1$.

   (d) $\forall x \in \mathbb{R}$ where $x \neq 0$, we have $x \geq 1$ or $\dfrac{1}{x} \geq 1$.

   Proof. (a) The statement is true and its negation is "$\exists n \in \mathbb{Z}$, such that $\forall y \in \mathbb{R} - \{0\}$ we have $y^n > y$". We see that we have different cases for any $n \in \mathbb{Z}$, we can take $y = 1$, and get $y^n = 1 = y$. Hence the statement is true.

   (b) The statement is true and its negation is "$\forall y \in \mathbb{R} - \{0\}$, $\exists n \in \mathbb{Z}$ such that $y^n > y$.". We can see that for $y = 1$, we get $y^n = 1^n = 1 = y$ for all $n \in \mathbb{Z}$.

(c) The statement is true and its negation is "$\exists x \in \mathbb{R}$ where $x \neq 0$, such that $x > 1$ and $\frac{1}{x} > 1$". To justify the original statement, we can look at different cases on x. We see that if $x < 0$, then $x \leq 1$ and $\frac{1}{x} \leq 1$. Moreover, if $0 < x \leq 1$, then we see that the statement $x \leq 1$ or $\frac{1}{x} \leq 1$ is automatically satisfied. Finally, for $x > 1$, we see that $\frac{1}{x} < 1$. Thus, again, the statement $x \leq 1$ or $\frac{1}{x} \leq 1$ is true. This means that the statement is true for all $x \in \mathbb{R}$, $x \neq 0$.

(d) The statement is false and its negation is "$\exists x \in \mathbb{R}$ where $x \neq 0$, such that $x < 1$ and $\frac{1}{x} < 1$". To see that the negated statement is true, consider $x = -1$. Then $x < 1$ and $-1 = 1/x < 1$.

□

Homework 3

5. After cleaning your basement, you find a set of keys K and a set of locks L. For every one of the following statements (a), (b) and (c),

   1. re-express the statement in a mathematical form using quantifiers $\forall$ and/or $\exists$,
   2. negate this mathematical statement,
   3. re-express the negation in standard english.

   E.g.: All keys unlock all locks.

   - Reformulated statement: $\forall k \in K, \forall l \in L, k$ unlocks l.
   - Negation: $\exists k \in K, \exists l \in L, k$ does not unlock l.
   - Reformulated negation: Some key does not unlock some lock.

   (a) At least one of the keys unlocks one of the locks.

   (b) Some key unlocks all the locks.

   (c) Some lock is not unlocked by any key.

   Proof. (a)At least one of the keys unlocks one of the locks.

   - Reformulated statement: $\exists k \in K, \exists l \in L, k$ unlocks l.
   - Negation: $\forall k \in K, \forall l \in L, k$ does not unlock l.
   - Reformulated negation: No key unlocks any lock.

   (b) Some key unlocks all the locks.

   - Reformulated statement: $\exists k \in K, \forall l \in L, k$ unlocks l.
   - Negation: $\forall k \in K, \exists l \in L, k$ does not unlock l.
   - Reformulated negation: For any key, there is a lock that it cannot unlock.

   (c) Some lock is not unlocked by any key.

   - Reformulated statement: $\exists l \in L, \forall k \in K, k$ does not unlock l.
   - Negation: $\forall l \in L, \exists k \in K, k$ unlocks l.
   - Reformulated negation: Every lock is unlocked by some key.

   □