Homework 4 Solutions

- Please submit your answers to all questions.
- We will mark your answers to 3 questions.
- We will provide you with full solutions to all questions.

1. Recall Bézout's identity: Let $a, b \in \mathbb{Z}$ such that a and b are not both zero. Then there exists $x, y \in \mathbb{Z}$ such that $ax + by = \gcd(a, b)$.

   Use this result to prove the following result:

   Let $a, b, c \in \mathbb{Z}$ such that $\gcd(a, b) = 1$. Then

   $$(a \mid bc) \implies (a \mid c).$$

   Proof. Assume $a|bc$. Then $bc = ka$ for some $k \in \mathbb{Z}$. Since $\gcd(a, b) = 1$, using Bézout's identity there exists $x, y \in \mathbb{Z}$ such that $ax + by = 1$. Multiplying the last equality by c yields $c = acx + bcy = acx + kay = a(cx + ky)$. Since $(cx + ky) \in \mathbb{Z}$, we have $a|c$. □

2. Let $P \subset \mathbb{N}$ be the set of prime numbers $P = \{2, 3, 5, 7, 11, \ldots\}$. Determine whether the following statements are true or false. Prove your answers ("true" or "false" is not sufficient).

   1. $\forall x \in P$, $\forall y \in P$, $x + y \in P$.
   2. $\forall x \in P$, $\exists y \in P$ such that $x + y \in P$.
   3. $\exists x \in P$ such that, $\forall y \in P$, $x + y \in P$.
   4. $\exists x \in P$ such that, $\exists y \in P$, $x + y \in P$.

   Proof.
   1. **Disproof**: This statement is false. As a counterexample, we can take any $x = 3 \in P$ and $y = 3 \in P$. Then we see that $x + y = 6 \notin P$.
   2. **Disproof**: This statement is false. For a counterexample, let $x = 7 \in P$. Then, we see that if $y \in P$ is odd, then $x + y > 2$ is even and thus, $x + y \notin P$. Moreover, if y is even, then this means $y = 2$, and hence, $x + y = 9 \notin P$.
   3. **Disproof**: This statement is false. To disprove it we are going to prove its negation:

      "$\forall x \in P$, $\exists y \in P$, such that $x + y \notin P$."

      Let $x \in P$. Then we have two cases to consider:

      Case 1: $x = 2$: In this case, we can pick $y = 7 \in P$, and get $x + y = 9 \notin P$.

      Case 2: x **is odd**: In this case, we can take $y = x$. This implies that $x + y > 2$ is even and thus, $x + y \notin P$.

      Therefore the negation of the statement is true, and hence, the original statement is false.

4. **Proof**: This statement is true. As an example, we can take any $x = 2 \in P$, and $y = 3 \in P$. Then we see that $x + y = 5 \in P$.

$\square$

3. Prove the following statement: For every positive number ϵ there is a positive number M such that

$$\left|\frac{2x^2}{x^2+1} - 2\right| < \epsilon$$

whenever $x \geq M$.

Proof. Given $\epsilon > 0$, let $M = 2/\epsilon'$, where $\epsilon' = \min\{\epsilon, 1\}$. Then for $x \geq M$, we have

$$x^2 + 1 > 4/(\epsilon')^2 > 2/\epsilon'.$$

For the second inequality, we used the fact that $\epsilon' \leq 1$. Thus

$$\epsilon \geq \epsilon' > \frac{2}{x^2+1} = \left|\frac{2x^2}{x^2+1} - 2\right|.$$

$\square$

4. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if $\lim_{x\to a} f(x) = f(a)$. Let

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Is f continuous at $x = 0$?

Note: Make sure to use the definition of a limit to justify your answer. You may also use the fact that $\forall x \in \mathbb{R}, |\sin(x)| \leq 1$.

Proof. Yes. It suffices to show that $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$. Given $\epsilon > 0$, we let $\delta = \sqrt{\epsilon}$. Then for each x satisfying $0 < |x| < \delta$, we have

$$|x^2 \sin(\frac{1}{x}) - 0| = x^2 |\sin(\frac{1}{x})| \leq x^2 < \sqrt{\epsilon}^2 = \epsilon,$$

where the first inequality follows from the fact that $|\sin(x)| \leq 1$. This proves $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$. $\square$

5. We say that a sequence (x_n) is bounded if

$$\exists M \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N}, |x_n| \leq M.$$

Prove that if a sequence (x_n) converges to 0, then (x_n) is bounded.

Proof. Assume that the sequence (x_n) converges to 0. Set $\epsilon = 1$. From the definition of convergence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|x_n - 0| \leq 1$. Then let $M = \max(1, |x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|)$ be the maximum absolute value of the $N-1$ first values of the sequence and 1. Let $n \in \mathbb{N}$. Then either $n \leq N-1$ or $n \geq N$. Let us consider the two cases separately.

- Case 1: Assume $n \leq N-1$. Then

  $$|x_n| \leq \max(|x_1|, |x_2|, |x_3|, \ldots, |x_N|) \leq M$$

  by definition of M.

- Assume $n \geq N$. Then $|x_n| = |x_n - 0| \leq 1 \leq M$ thanks to the definition of N and of M.

In all cases we have $|x_n| \leq M$, which finishes the proof. □

6. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to L if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, (n > N) \implies \big(|x_n - L| < \epsilon\big).$$

Using the definition, prove that the sequence (x_n) with $x_n = (-1)^n + \dfrac{1}{n}$ does not converge to any $L \in \mathbb{R}$.

Proof. Case 1: $L \geq 0$. For $\epsilon = 1/2$, $\forall N \in \mathbb{N}$, take $n = 2N+1 > N$. Then

$$|x_n - L| = |-1 + \frac{1}{n} - L| = 1 - \frac{1}{n} + L > 1 - \frac{1}{2N+1} + L \geq 1 - \frac{1}{3} > \frac{1}{2},$$

where the second inequality is because $-1 + \frac{1}{n} - L \leq 0$ for all $n \in \mathbb{N}$ and $L \geq 0$

Case 2: $L < 0$. For $\epsilon = 1/3$, $\forall N \in \mathbb{N}$, take $n = 2N > N$. Then

$$|x_n - L| = |1 + \frac{1}{2N+1} - L| = 1 + \frac{1}{2N+1} - L > 1 + \frac{1}{3} > \frac{1}{3}.$$

□