Homework 5 solutions

- Please submit your answers to all questions.
- We will mark your answers to 3 questions.
- We will provide you with full solutions to all questions.

1. Prove that for all $n \in \mathbb{N}$, $\sum_{k=1}^{n}(2k-1).2^k = 6 + 2^n(4n-6)$.

   Proof. We are going to prove this statement using induction.

   Base case: We see that for $n = 1$, we have

   $$\sum_{k=1}^{1}(2k-1).2^k = 2 = 6 + 2^1(4 \cdot 1 - 6),$$

   which is a true. Thus, the statement is true for $n = 1$.

   Inductive step: Assume that the statement is true for $a = n$ for some $n \geq 1$, that is,
   $\sum_{k=1}^{n}(2k-1).2^k = 6 + 2^n(4n-6)$. Then, by adding $\big(2(n+1)-1\big) \cdot 2^{n+1}$, to both sides, we get,

   $$\begin{aligned}\sum_{k=1}^{n+1}(2k-1).2^k &= \sum_{k=1}^{n}(2k-1).2^k + \big(2(n+1)-1\big) \cdot 2^{n+1}\\ &= 6 + 2^n(4n-6) + (2n+1) \cdot 2^{n+1}\\ &= 6 + 2^n(4n-6) + 2^n(4n+2)\\ &= 6 + 2^n\big(4n-6+4n+2\big)\\ &= 6 + 2^n\big(8n-4\big)\\ &= 6 + 2^{n+1}\big(4n-2\big)\\ &= 6 + 2^{n+1}\big(4(n+1)-6\big)\end{aligned}$$

   Therefore, by mathermatical induction, we see that the statement is true for all $n \in \mathbb{N}$.

   $\square$

2. Let $n \in \mathbb{N}$. Prove that if $a_{n+2} = 5a_{n+1} - 6a_n$ and $a_1 = 1, a_2 = 5$, then $a_n = 3^n - 2^n$ for all $n \geq 3$.

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Proof. We use strong induction.

Base case: $n = 3$, we have $a_3 = 5 \cdot 5 - 6 \cdot 1 = 19 = 3^3 - 2^2$. When $n = 4$, $a_4 = 5a_3 - 6a_2 = 5 \cdot 19 - 6 \cdot 5 = 65 = 3^4 - 2^4$.

Inductive step: Assume $a_i = 3^i - 2^i$ for $4 \leq i \leq k$. Now

$$\begin{aligned} a_{k+1} &= 5a_k - 6a_{k-1} \\ &= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1}) \\ &= (5-2)3^k - (5-3)2^k \\ &= 3^{k+1} - 2^{k+1}. \end{aligned}$$

By the strong math induction, we conclude that if $a_{n+2} = 5a_{n+1} - 6a_n$ and $a_1 = 1, a_2 = 5$, then $a_n = 3^n - 2^n$ for all $n \geq 3$. □

3. Let $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for each $n \in \mathbb{N}, n \geq 2$. Show that $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ for every $n \in \mathbb{N}$.

   Proof. We prove by induction.

   Base case: $n = 1$, we have$f_2 \cdot f_0 - f_1^2 = 1 \cdot 0 - 1^2 = (-1)^1$.

   Inductive step: Assume statement holds for $f_{k+1}f_{k-1} - f_k^2 = (-1)^k$. Then

$$\begin{aligned} f_{k+2}f_k - f_{k+1}^2 &= (f_{k+1} + f_k)f_k - (f_k + f_{k-1})f_{k+1} \\ &= f_k^2 - f_{k+1}f_{k-1} \\ &= -(-1)^k \\ &= (-1)^{k+1}. \end{aligned}$$

   □

4. Prove that $7^{4n+3} + 2$ is a multiple of 5 for all non-negative integers n.

   Proof. We will use mathematical induction to prove the statement.

   Base case: When $n = 0$, we have $7^3 + 2 = 343 + 2$ which is a multiple of 5.
   Inductive step: Assume, for some non-negative integer k, that

$$5 \mid 7^{4k+3} + 2.$$

   Consider $7^{4(k+1)+3} + 2 = 7^4(7^{4k+3} + 2) + 2(1 - 7^4)$. We know that, on the right-hand side, the first summand is divisible by 5 due to the induction assumption and the second summand is divisible by 5 by inspection. So the induction step holds.

   Therefore, by induction, the statement holds true for all non-negative integers n. □

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5. Let x be a real number. We let $u_0 = 1$, $u_1 = \cos x$ and define (u_n) in such a way that for all $n \geq 0$, $u_{n+2} = 2u_1 u_{n+1} - u_n$. Show that for all $n \geq 0$ we have

$$u_n = \cos(nx).$$

Note: In the induction step, you can use the trigonometric formulas $\cos(a+b) = \cos a \cos b - \sin a \sin b$ *and* $\cos(a-b) = \cos a \cos b + \sin a \sin b$*.*

Proof. We will prove the result by strong induction.

Base case: it is clear that the property is true for $n = 0$ (check that $u_0 = \cos(0x) = 1$) and $n = 1$ (check that $u_0 = \cos(1x) = \cos x$.

Inductive step: Assume that the result is true for n and $n+1$. Then we write

$$\cos((n+2)x) = \cos((n+1)x)\cos x - \sin((n+1)x)\sin x$$

and

$$\cos(nx) = \cos((n+1)x - x) = \cos((n+1)x)\cos x + \sin((n+1)x)\sin x.$$

Summing the two last display we get

$$\cos((n+2)x) + \cos(nx) = 2\cos((n+1)x)\cos x.$$

Now let us use the induction hypothesis and write

$$\begin{aligned} u_{n+2} &= 2u_{n+1}u_1 - u_n \\ &= 2\cos((n+1)x)\cos x - \cos(nx) \\ &= \cos((n+2)x). \end{aligned}$$

This prove $P(n+2)$.

Conclusion: using a two step induction, $P(n)$ is proved for all $n \geq 0$. $\square$

6. Find, with proof, all positive integers n so that $n^3 > 2n^2 + n$.

Proof. Check some small n cases

- When $n = 1$, we have $1 \geq 3$ which is obviously false.
- When $n = 2$, we have $8 \geq 10$ which is obviously false.
- When $n = 3$, we have $27 \geq 21$ which is true.
- When $n = 4$, we have $64 \geq 36$ which is true.

So it appears to be true for $n \geq 3$. We prove this by induction.

- **Base case:** — when $n = 3$, the statement is $27 \geq 18 + 3 = 21$ ✓.

- **Inductive step**: — assume that the statement is true for $n = k$. Hence

  $$k^3 \geq 2k^2 + k$$

  We need to show that

  $$\begin{aligned}(k+1)^3 &\geq 2(k+1)^2 + (k+1)\text{equivalently}\\ k^3 &\geq -k^2 + 2k + 1\end{aligned}$$

  It suffices to show that $2k^2+k \geq -k^2+2k+2$, or equivalently $3k^2 \geq k+2$. By assumption $k \geq 3 \geq 2$, and so $2k \geq k+2$. By similarly reasoning $2k^2 \geq 2k \geq k+2$, and finally $3k^2 \geq 2k^2 \geq k+2$. Thus we know $3k^2 \geq k+2$.

  Now by assumption $k^3 \geq 2k^2 + k$, and we have show that $3k^2 \geq k+2$ and so $k^3 \geq 2k^2 + k \geq -k^2 + 2k + 2$ as required.

Thus by induction, the statement is true for all integer $n \geq 3$.

$\square$