Mathematics 220 — Homework 6

- Contains 5 questions on 5 pages.
- You do not need to submit this homework!
- We will provide you with full solutions to all questions.

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1. Prove or disprove the following:

   (a) If $m, n \in \mathbb{Z}$, then $\{x \in \mathbb{Z} : mn \mid x\} \subseteq \{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$.

   Solution:

   Proof. Let $m, n \in \mathbb{Z}$. Assume that $s \in \{x \in \mathbb{Z} : mn \mid x\}$. Then we see that $mn \mid s$ and thus, $s = mnk$ for some $k \in \mathbb{Z}$. Hence, we see $s = m(nk)$ and $s = n(mk)$. Since $nk, mk \in \mathbb{Z}$, we get that $n \mid s$ and $m \mid s$. Thus, $s \in \{x \in \mathbb{Z} : n \mid x\}$ and $s \in \{x \in \mathbb{Z} : m \mid x\}$.
   This implies, $s \in \{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$.
   Therefore $\{x \in \mathbb{Z} : mn \mid x\} \subseteq \{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$. □

   (b) If A and B are sets, then $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$.

   Solution:

   Disproof. We see that this statement is false. For a counterexample, we can take $A = \{1\}$ and $B = \{2\}$. Then we see that

   $$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}\} \neq \{\emptyset, \{1\}, \{2\}, \{1,2\}\} = \mathcal{P}(\{1,2\}) = \mathcal{P}(A \cup B).$$

   □

   (c) If A, B and C are sets, then $A - (B \cup C) = (A - B) \cup (A - C)$.

   Solution:

   Disproof. We see that this statement is false. For a counterexample we can take $A = B = \{1\}$ and $C = \{2\}$. Then we see that $A - (B \cup C) = \emptyset \neq \{1\} = (A - B) \cup (A - C)$. □

   (d) Suppose A, B and C are sets. If $A = B - C$, then $B = A \cup C$.

   Solution:

   Disproof. This statement is also false. For a counterexample, we can take the same sets as in the previous question, $A = B = \{1\}$ and $C = \{2\}$. Then we see that $A = B - C$, but $A \cup C = \{1, 2\} \neq \{1\} = B$. □

2. For any $k \geq 1$, let A_k be the interval $A_k = \left[\frac{1}{k+1}, 1 + \frac{1}{k+1}\right)$. We recall that

$$\bigcup_{k=1}^{\infty} A_k = \{x \in \mathbb{R} : \exists k \in \mathbb{N} \text{ s.t. } x \in A_k\},$$

and

$$\bigcap_{k=1}^{\infty} A_k = \{x \in \mathbb{R} : \forall k \in \mathbb{N},\ x \in A_k\},$$

(a) Show that $\bigcup_{k=1}^{\infty} A_k = \left(0, \frac{3}{2}\right)$.

Solution:

Proof. Let us show the two inclusions in turn.

- Let $x \in \bigcup_{k=1}^{\infty} A_k$. Then $x \in A_k$ for some $k \in \mathbb{N}$ so that $\frac{1}{k+1} < x < 1 + \frac{1}{k+1}$ from which we get that $0 < x < \frac{3}{2}$, so that $x \in (0, \frac{3}{2})$.
- Let $x \in (0, \frac{3}{2})$. Then let us consider two cases, $x \geq \frac{1}{2}$ or $x < \frac{1}{2}$.
  First case. Suppose that $x \geq \frac{1}{2}$. Then because of the assumption on x we have $\frac{1}{2} \leq x < \frac{3}{2}$, and so $x \in A_1$, so $x \in \bigcup_{k=1}^{\infty} A_k$.
  Second case. Suppose that $x < \frac{1}{2}$. Then let if we let $n = \text{ceiling}(\frac{1}{x})$ then we have $\frac{1}{x} < n + 1$ and so $x > \frac{1}{n+1}$. This is enough to show that $x \in A_n$ with $n \in \mathbb{N}$ and so $x \in \bigcup_{k=1}^{\infty} A_k$.

This finishes the proof. □

(b) Show that $\bigcap_{k=1}^{\infty} A_k = \left[\frac{1}{2}, 1\right]$

Solution:

Proof. Let us show the two inclusions in turn.

- Let $x \in \bigcap_{k=1}^{\infty} A_k$. Then for any $k \in \mathbb{N}$ we have $x \in A_k$ so that the inequality $\frac{1}{k+1} \leq x < 1 + \frac{1}{k+1}$ holds for any $k \in \mathbb{N}$. In particular, $x \geq \frac{1}{2}$. Also since $x < 1 + \frac{1}{k+1}$ then we also have $x \leq \lim_{k \to \infty} 1 + \frac{1}{k+1} = 1$. So in the end $x \in [\frac{1}{2}, 1]$.
- Let $x \in [\frac{1}{2}, 1]$. So $\frac{1}{2} \leq x \leq 1$. Let $k \in \mathbb{N}$, we can check that $\frac{1}{k+1} \leq \frac{1}{2}$ and $1 + \frac{1}{k+1} > 1$, hence $\frac{1}{k+1} \leq x < 1 + \frac{1}{k+1}$ so $x \in A_k$. In the end, for any $k \in \mathbb{N}$ we have $x \in A_k$, so that $x \in \bigcap_{k=1}^{\infty} A_k$.

□

Be careful with the open and closed endpoints of the intervals.

3. Let A, B, C be sets. Prove that

$$(A - B) \cup (A - C) = A - (B \cap C)$$

Hint — some careful thoughts about contrapositives may help you through some tricky parts.

Solution:

Proof. We prove each inclusion in turn.

- Let $x \in LHS$. Then either $x \in A - B$ or $x \in A - C$.
  - Assume that $x \in A - B$, then $x \in A$ and $x \notin B$. Since $x \notin B$ it follows that $x \notin B \cap C$ (to see this, think of the contrapositive: if $x \in B \cap C$ then $x \in B$). Since $x \in A$ and $x \notin B \cap C$ we know that $x \in RHS$.
  - Assume that $x \in A - C$, then $x \in A$ and $x \notin C$. Since $x \notin B$ it follows that $x \notin B \cap C$. Since $x \in A$ and $x \notin B \cap C$ we know that $x \in RHS$.
- Now let $x \in RHS$. Hence $x \in A$ and $x \notin B \cap C$. Since $x \notin B \cap C$ it follows that $x \notin B$ or $x \notin C$ (again to see this, think of the contrapositive: if $x \in B$ and $x \in C$ then $x \in B \cap C$).
  - Assume that $x \notin B$. Then since we also know that $x \in A$ it follows that $x \in A - B$.
  - Now assume that $x \notin C$, then by similar reasoning we know that $x \in A - C$.

  In both cases, it follows that $x \in LHS$ as required.

□

4. Let A, B, C be sets. Prove that

$$A - (B - C) = (A \cap C) \cup (A - B)$$

Hint — you will need to negate carefully in a couple of places.

Solution:

Proof. We prove each inclusion in turn.

- Let $x \in LHS$. Hence $x \in A$ but $x \notin (B - C)$. Since

$$x \notin (B - C) \equiv \sim (x \in B \wedge x \notin C) \equiv x \notin B \vee x \in C$$

  it follows that $x \notin B$ or $x \in C$.

  - Assume that $x \notin B$ then, since $x \in A$, it follows that $x \in A - B$ and so $x \in RHS$.
  - Now assume that $x \in C$, then by similar reasoning, $x \in A \cap C$ and so $x \in RHS$.

  In both cases, $x \in RHS$ as required.

- Now assume that $x \in LHS$, and thus either $x \in A \cap C$ or $x \in A - B$.
  - Assume that $x \in A \cap C$. Hence we know that $x \in A$ and $x \in C$. Now since $x \in C$ it follows that $x \notin B - C$ (to see this consider the contrapositive carefully: $x \in B - C$ implies that $x \notin C$). Then since $x \in A$ we can conclude that $x \in LHS$.
  - Now assume that $x \in A - B$. Then $x \in A$ and $x \notin B$. Now since $x \notin B$ it follows that $x \notin B - C$ (again, the contrapositive gives us $x \in B - C$ implies $x \in B$). And since $x \in A$ we know that $x \in LHS$.

  In both cases $x \in LHS$ as required.

□

5. Suppose $x, y \in \mathbb{R}$ and $k \in \mathbb{N}$ satisfying, $x, y > 0$ and $x^k = y$. Then prove that

$$\{x^a : a \in \mathbb{Q}\} = \{y^a : a \in \mathbb{Q}\}.$$

Solution:

Proof. This is a set equality. Thus, we need to prove

$$\{x^a : a \in \mathbb{Q}\} \subseteq \{y^a : a \in \mathbb{Q}\}$$

and

$$\{y^a : a \in \mathbb{Q}\} \subseteq \{x^a : a \in \mathbb{Q}\}.$$

We prove each in turn.

- Proof of $\{x^a : a \in \mathbb{Q}\} \subseteq \{y^a : a \in \mathbb{Q}\}$: Let $z \in \{x^a : a \in \mathbb{Q}\}$. Then we know that $z = x^a$ for some $a \in \mathbb{Q}$. Since $a \in \mathbb{Q}$ we know that $a = \dfrac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then, since $x^k = y$, we see that $z = x^{\frac{p}{q}} = y^{\frac{p}{qk}}$. Thus, we see that $z \in \{y^a : a \in \mathbb{Q}\}$ since $\dfrac{p}{qk} \in \mathbb{Q}$.

- Proof of $\{y^a : a \in \mathbb{Q}\} \subseteq \{x^a : a \in \mathbb{Q}\}$: Let $u \in \{y^a : a \in \mathbb{Q}\}$. Then we see that $u = y^a$ for some $a \in \mathbb{Q}$. Since $a \in \mathbb{Q}$ we know that $a = \dfrac{s}{t}$ for some $s \in \mathbb{Z}$ and $t \in \mathbb{N}$. Then, since $x^k = y$, we see that $z = y^{\frac{s}{t}} = x^{\frac{sk}{t}}$. Thus, we see that $z \in \{x^a : a \in \mathbb{Q}\}$ since $\dfrac{sk}{t} \in \mathbb{Q}$.

$\square$