Homework 7

- Please submit your answers to all questions.
- We will mark your answers to 3 questions.
- We will provide you with full solutions to all questions.

1. We define a relation $\mathcal{R}$ on $\mathcal{P}(\{1,2\})$ (the power set of $\{1,2\}$) by

$$S\mathcal{R}T \iff S \cap T = \emptyset.$$

Write down the all the elements in $\mathcal{R}$.

Proof. We see that R can be written as

$$R = \big\{(\emptyset,\emptyset),(\emptyset,\{1\}),(\{1\},\emptyset),(\emptyset,\{2\}),(\{2\},\emptyset),\\ (\emptyset,\{1,2\}),(\{1,2\},\emptyset),(\{1\},\{2\}),(\{2\},\{1\})\big\}$$

□

2. Let R be a relation on a nonempty set A. Then $\overline{R} = (A \times A) - R$ is also a relation on A. Prove or disprove each of the following statements:

   1. If R is reflexive, then $\overline{R}$ is reflexive.
   2. If R is symmetric, then $\overline{R}$ is symmetric.
   3. If R is transitive, then $\overline{R}$ is transitive.

Proof.
   1. This statement is not true. Let $a \in A$. Then, we see that if R is reflexive, then $(a,a) \in R$. Thus, $(a,a) \notin \overline{R}$. Therefore $\overline{R}$ is not reflexive.
   2. Let $a,b \in A$. Assume R is symmetric and assume that $(a,b) \in \overline{R}$. By definition, we see that $(a,b) \notin R$. Thus, since R is symmetric we know that whenever $(b,a) \in R$ we have $(a,b) \in R$. Moreover, since we know that $(a,b) \notin R$ we see $(b,a) \notin R$. Hence, $(b,a) \in \overline{R}$. Therefore $\overline{R}$ is symmetric.
   3. This statement is false. For a counterexample, we can take $A = \{1,2\}$ and $R = \{(1,1),(2,2)\}$. Then we see that R is the "equal" relation on A and is transitive. We also see that $\overline{R} = \{(1,2),(2,1)\}$. Since $(1,2) \in \overline{R}$ and $(2,1) \in \overline{R}$ but $(1,1) \notin \overline{R}$, we see that $\overline{R}$ is not transitive.

□

3. Let R be a relation on a set A. Suppose that R is reflexive and satisfy $(aRc \wedge bRc) \implies aRb$ for any $a,b,c \in A$. Prove that R is symmetric and transitive.

Proof. Since R is reflexive, bRb. If aRb, then from $(bRb \wedge aRb)$ and the assumption, we know bRa, so R is symmetric.

If aRb, bRc, then symmetry implies cRb. Now, from $cRb \wedge aRb$ and the assumption, we have cRa, so R os transitive. □

4. We define a relation $\mathcal{R}$ on the real numbers as

$$\mathcal{R} = \{(x, x+n) : x \in \mathbb{R}, n \in \mathbb{N}\}.$$

Determine (with proof) whether the following holds:

(a) If $x_1 \mathcal{R} y_1$ and $x_2 \mathcal{R} y_2$ then $(x_1 + x_2)\mathcal{R}(y_1 + y_2)$.

(b) If $x_1 \mathcal{R} y_1$ and $x_2 \mathcal{R} y_2$ then $(x_1 \cdot y_1)\mathcal{R}(x_2 \cdot y_2)$.

Proof. Point (a) holds true. Let x_1, x_2, y_1, y_2 be such that $x_1 \mathcal{R} y_1$ and $x_2 \mathcal{R} y_2$. Then by definition we have $y_1 = x_1 + n_1$ and $y_2 = x_2 + n_2$ with $n_1, n_2 \in \mathbb{N}$. So $(y_1 + y_2) = (x_1 + x_2) + (n_1 + n_2)$ and $n_1 + n_2 \in \mathbb{N}$ so $(x_1 + x_2)\mathcal{R}(y_1 + y_2)$.

Point (b) is false. Take $x_1 = 0, y_1 = 1, x_2 = \frac{1}{2}, y_2 = \frac{3}{2}$. Then we have $x_1 \mathcal{R} y_1$ and $x_2 \mathcal{R} y_2$, but $x_1 \cdot y_1 = 0$ is not in relation with $x_2 \cdot y_2 = \frac{3}{2}$. □

5. We define a relation T on $\mathbb{R} - \{0\}$ by

$$aTb \iff \frac{a}{b} \in \mathbb{Q}.$$

Show that T is symmetric, reflexive, and transitive.

Proof.

- (symmetric) Let $a \in \mathbb{R} - \{0\}$. Then $\frac{a}{a} = 1 \in \mathbb{Q}$.
- (reflexive) Let $a, b \in \mathbb{R} - \{0\}$ such that $\frac{a}{b} \in \mathbb{Q}$. Then $\frac{b}{a} = 1/\frac{a}{b} \in \mathbb{Q}$.
- (transitive) Let $a, b, c \in \mathbb{R} - \{0\}$ such that $\frac{a}{b} \in \mathbb{Q}$ and $\frac{b}{c} \in \mathbb{Q}$. Then $\frac{a}{c} = \frac{a}{b} \times \frac{b}{c} \in \mathbb{Q}$.

□

6. For this question, you may use the following fact without proving it: Let e be Euler's number $e = 2.718....$ Then for every $n \in \mathbb{N}$, e^n is irrational.

Let T be the relation on $\mathbb{R} - \{0\}$ from the previous question. In the previous question, you proved that T is an equivalence relation.

Prove that this equivalence relation defines infinitely many distinct equivalence classes.

Proof. For each $n \in \mathbb{N}$, consider the equivalence class $[e^n]$. We will show that each of these equivalence classes are distinct.

To see this, let $n, m \in \mathbb{N}$ with $n > m$. Since $\frac{e^n}{e^m} = e^{n-m}$, and $n - m \in \mathbb{N}$, we can use the fact that $e^{n-m} \notin \mathbb{Q}$ to conclude that $e^m \notin [e^n]$. On the other hand, we clearly have $e^m \in [e^m]$, and thus $[e^m] \neq e^{[}n]$. We conclude that for each pair of distinct natural numbers $n, m \in \mathbb{N}$, the equivalence classes $[e^n]$ and $[e^m]$ are distinct.

Since there are infinitely many distinct equivalence classes of the form $[e^n]$, $n \in \mathbb{N}$, we conclude that the equivalence relation T on $\mathbb{R} - \{0\}$ has infinitely many distinct equivalence classes. $\square$