Homework 8 Solutions

- Please submit your answers to all questions.
- We will mark your answers to 3 questions.
- We will provide you with full solutions to all questions.

1. Prove or disprove: If R and S are two equivalence relations on a set A, then $R \cup S$ is also an equivalence relation on A.

   Proof. This statement is false. For a counterexample we can take $A = \{1, 2, 3\}$ and the relations $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ and $S = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$. We see that the relations R and S are equivalence relations, but

   $R \cup S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)\}$ is not an equivalence relation since $(2, 1), (1, 3) \in R \cup S$, but $(2, 3) \notin R \cup S$, that is, $R \cup S$ is not transitive.

   □

2. Define a relation on $\mathbb{Z}$ as aRb if $3 \mid (5a - 8b)$. Is R an equivalence relation? Justify your answer.

   Proof. We need to check whether this relation is reflexive, symmetric, and transitive.

   Reflexive: We see that this relation is reflexive since for any $a \in \mathbb{Z}$, we have $(5a - 8a) = 3(-a)$, which implies $3 \mid (5a - 8a)$, that is, aRa.

   Symmetric: Let $a, b \in \mathbb{Z}$ and assume aRb. Then we see $3 \mid (5a - 8b)$, and so $5a - 8b = 3k$ for some $k \in \mathbb{Z}$. Then

   $5b - 8a = (-3b - 3a) - (5a - 8b) = 3(-b - a - k)$. Since $(-b - a - k) \in \mathbb{Z}$ we see that $3 \mid (5b - 8a)$. Therefore R is symmetric.

   Transitive: Let $a, b, c \in \mathbb{Z}$ and assume aRb and bRc. Then we see $3 \mid (5a - 8b)$ and $3 \mid (5b - 8c)$, so that $5a - 8b = 3k$ and $5b - 8c = 3n$ for some $k, n \in \mathbb{Z}$. Then $5a - 8c = (5a - 8b) + 3b + (5b - 8c) = 3(k + b + n)$. Since $(k + b + n) \in \mathbb{Z}$ we see that $3 \mid (5a - 8c)$. Therefore R is transitive.

   □

3. Determine whether the following relations are reflexive, symmetric and transitive.

   1. On the set X of all functions $\mathbb{R} \to \mathbb{R}$, we define the relation:

      $$fRg \text{ if there exists } x \in \mathbb{R} \text{ such that } f(x) = g(x).$$

   2. Let R be a relation on $\mathbb{Z}$ defined by:

      $$xRy \text{ if } xy \equiv 0 \pmod 4.$$

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Proof. 1. It is reflexive, symmetric but not transitive. For example, let f, g and h such that $f(x) = 0$, $g(x) = x$ and $h(x) = 1$. We have fRg and gRh but it is not true that fRh.

2. (a) We see that $(1,1) \notin R$, since $1 \cdot 1 = 1 \not\equiv 0 \pmod 4$. Therefore, the relation is not reflexive.

   (b) This relation is symmetric since if $xy \equiv 0 \pmod 4$, then $yx = xy \equiv 0 \pmod 4$, that is, if $(x, y) \in R$, then $(y, x) \in R$.

   (c) This relation is not transitive. For a counterexample, we can take, $a = 1, b = 4, c = 1$. Then, we see that $(1, 4), (4, 1) \in R$, whereas, $(1, 1) \notin R$.

□

4. Let A be a non-empty set and $\mathcal{S} \subseteq \mathcal{P}(A)$ and $\mathcal{T} \subseteq \mathcal{P}(A)$ partitions of A. Show that $\mathcal{R}$ defined as

$$\mathcal{R} = \{S \cap T : S \in \mathcal{S},\ T \in \mathcal{T}\} \setminus \{\emptyset\}$$

is a partition of A.

Proof. The set $\mathcal{R}$ is a set of non-empty subsets of A by definition. Let $x \in A$. Since $\mathcal{S}$ and $\mathcal{T}$ are partitions, there exists $S \in \mathcal{S}$ and $T \in \mathcal{T}$ such that $x \in S$ and $x \in T$. This entails that $x \in S \cap T$ and $S \cap T \in \mathcal{R}$.

Let $U_1, U_2 \in \mathcal{R}$. By definition $U_1 = S_1 \cap T_1$ for $S_1 \in \mathcal{S}$ and $T_1 \in \mathcal{T}$ and $U_2 = S_2 \cap T_2$ for $S_2 \in \mathcal{S}$ and $T_2 \in \mathcal{T}$. Then $U_1 \cap U_2 = S_1 \cap T_1 \cap S_2 \cap T_2 = (S_1 \cap S_2) \cap (T_1 \cap T_2)$. From there, either $S_1 = S_2$ and $T_1 = T_2$, in which case $U_1 = U_2$. Or we have $S_1 \cap S_2 = \emptyset$ or $T_1 \cap T_2 = \emptyset$, which entails that $U_1 \cap U_2 = \emptyset$.

In the end, $\mathcal{R}$ is a partition of A. □

5. Let E be a non-empty set and $x \in E$ be a fixed element of E. Consider the relation R on $\mathcal{P}(E)$ defined as

$$ARB \iff (x \in A \cap B) \vee (x \in \overline{A} \cap \overline{B}),$$

where for any set $S \subseteq E$, we write $\overline{S} = E \setminus S$ for the complement of S in E. Prove or disprove that R an equivalence relation.

Proof. Let us prove that R is an equivalence relation.

- Reflexivity: Let $A \in \mathcal{P}(E)$. Then $(x \in A) \vee (x \in \overline{A})$ which we can rewrite as $(x \in A \cap A) \vee (x \in \overline{A} \cap \overline{A})$. Hence, ARA.
- Symmetry: The symmetry is immediate from the symmetry of the intersection of sets.

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- Transitivity: Let $A, B, C \in \mathcal{P}(E)$ and assume that ARB and BRC so that

$$\big((x \in A \cap B) \vee (x \in \overline{A} \cap \overline{B})\big) \wedge \big((x \in B \cap C) \vee (x \in \overline{B} \cap \overline{C})\big)\,.$$

  Now we can study 4 cases in turn:

  - Case 1: $(x \in A \cap B) \wedge (x \in B \cap C)$. Then $x \in A \cap B \cap C$ so $x \in A \cap C$ so ARC.
  - Case 2: $(x \in A \cap B) \wedge (x \in \overline{B} \cap \overline{C})$, which entails that $x \in B \cap \overline{B}$ so this case never happens.
  - Case 3: $(x \in \overline{A} \cap \overline{B}) \wedge (x \in B \cap C)$. This case does not happen for the same reason as above.
  - Case 4: $\big(x \in \overline{A} \cap \overline{B})\big) \wedge (x \in \overline{B} \cap \overline{C})$. From there $x \in \overline{A} \cap \overline{C}$ and so ARC.

□

6. Suppose that $n \in \mathbb{N}$ and $\mathbb{Z}_n$ is the set of equivalence class of congruent modulo n on $\mathbb{Z}$ (in Sections 101 and 103, this was called $\mathbb{Z}/n\mathbb{Z}$). In this question we will call an element $[u]_n$ invertible if it has a multiplicative inverse.

   Now, define a relation R on $\mathbb{Z}_n$ by xRy iff $xu = y$ for some invertible $[u]_n \in \mathbb{Z}_n$.

   (a) Show that R is a equivalence relation.

   (b) Compute the equivalence classes of this relation for $n = 6$.

   Hint: First find the invertible elements in $\mathbb{Z}_6$

   Proof. To prove (a), we have to show that R is reflexive, symmetric and transitive.

   - (reflexive) We have xRx since $[a]_n[1]_n = [a]_n$ for all $n \in \mathbb{Z}$.
   - (symmetric) Suppose xRy, that is $xu = y$ for some $u \in \mathbb{Z}_n$ which admits a multiplicative inverse. Write $v \in \mathbb{Z}_n$ to be a inverse of u, i.e. $uv = [1]_n$. Then we have $yv = xuv = x[1]_n = x$ and thus yRx.
   - (transitive) Suppose xRy and yRz, that is $xu = y$ and $yv = z$ with u, v both admitting multiplicative inverse. Then we have $xuv = yv = z$. Write u' for the multiplicative inverse of u and v' for that of v. Then we see that $uvv'u' = u[1]_nu' = uu' = [1]_n$ and thus uv admits a multiplicative inverse. Therefore, we have xRz.

   For (b), we first note that the set of elements in $\mathbb{Z}_6$ with a multiplicative inverses are $U = \{[1]_6, [5]_6\}$. Thus we may list the equivalence classes defined by R:

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- We see that

$$\begin{aligned}[[0]_6] &= \{[y]_6 \in \mathbb{Z}_6 \colon [y]_6 = [0]_6[u]_6 \text{ for some invertible } [u]_6 \in \mathbb{Z}_6\} \\ &= \{[0]_6 u : u \in U\} = \{[0]_6\}.\end{aligned}$$

  Then similarly,

- $[[1]_6] = \{[1]_6 u : u \in U\} = \{[1]_6, [5]_6\}$.
- $[[2]_6] = \{[2]_6 u : u \in U\} = \{[2]_6, [4]_6\}$.
- $[[3]_6] = \{[3]_6 u : u \in U\} = \{[3]_6\}$.

□