Homework 9

- Please submit your answers to all questions.
- We will mark your answers to 3 questions.
- We will provide you with full solutions to all questions.

1. Suppose $f : A \to A$ such that $f \circ f$ is bijective. Is f necessarily bijective? Prove that your answer is correct.

   Proof. Yes.

   - f is injective
     Suppose $a, a' \in A$ satisfying $f(a) = f(a')$. Then applying f to both sides we get $f(f(a)) = f(f(a'))$. Then, since $f \circ f$ is injective, we get $a = a'$. Hence, f is injective.
   - f is surjective
     Suppose $a \in A$. Since $f \circ f$ is surjective, there exists $b \in A$ such that $f \circ f(b) = a$. This means that, there exists $c = f(b) \in A$ such that $f(c) = a$. Hence f is surjective.

   □

2. Let $f : A \to A$ be a function. Prove that f is a symmetric and transitive relation on A if and only if f is the identity function.

   Proof. Let f be a function from A to A and assume that $f = \{(a, f(a) : a \in A)\}$ is a symmetric and transitive relation. Let $a \in A$. Then, since f is a function, $\exists f(a) \in A$, such that $(a, f(a)) \in f$. Thus, we see that since $(a, f(a)) \in f$, by symmetry, we get $(f(a), a) \in f$. Since f is transitive, this implies $(a, a) \in f$. Moreover, since f is a function, and $(a, f(a)), (a, a) \in f$, we see $f(a) = a$. Thus, f is the identity function.

   Moreover, if f is the identity function, we see that $f = \{(a, a) : a \in A\}$. Then, by definition, we see that f is the identity relation on A, and hence it is an equivalence relation, in particular, it is symmetric and transitive. □

   For the next problem, we need the following definition. Let $f\colon A \to B$ be a function, and let $D \subseteq A$. We define the *restriction of f to D*, denoted f_D, to be the function $f_D\colon D \to B$ given by $f_D(x) = f(x)$ for all $x \in D$.

3. Let X, Y, B be sets, and define $A = X \cup Y$. Let $f\colon A \to B$ be a function. Prove or disprove the following

   a) If f_X and f_Y are injective, then f is injective.
   Disproof: We see that this statement is false. For a counterexample we can take the function $f : \mathbb{R} \to \mathbb{R}$, defined as $f(x) = x^2$ and $X = (-\infty, 0]$ and $Y = [0, \infty)$ then see that f_X and f_Y are injective, but f is not injective.

b) If f is surjective, then $f(X) \cup f(Y) = B$.

Proof: Assume that f is surjective. We know that since $X, Y \subseteq A$, then $f(X), f(Y) \subseteq B$. Thus, we see $f(X) \cup f(Y) \subseteq B$. Moreover, if $b \in B$, then since f is surjective, $\exists a \in A$ such that $f(a) = b$. Thus, since $X \cup Y = A$, we see $a \in X \cup Y$. Hence, $a \in X$, in which case $f(a) \in f(X)$, or $a \in Y$, in which case $f(a) \in f(Y)$. In both cases, we see that $f(a) = b \in f(X) \cup f(Y)$. Thus, $B \subseteq f(X) \cup f(Y)$.
Therefore $f(X) \cup f(Y) = B$.

4. Consider the function

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & [-1, +\infty) \\ x & \longmapsto & x^2 + 2x \end{array}$$

a) Show that f is well defined, namely that: $\forall x \in \mathbb{R}$, we have $x^2 + 2x \geq -1$.

b) What is $f^{-1}(\{0\})$? $f^{-1}(\{-4\})$? $f^{-1}(\{-1\})$?

c) Show that $\text{Range}(f) = [-1, +\infty)$.

d) Is the function f injective, surjective, bijective?

e) Now we consider the function

$$g: \begin{array}{ccc} \mathbb{R} & \longrightarrow & [-1, +\infty) \\ x & \longmapsto & f(e^x) \end{array}$$

Show that the function g is not surjective.

Proof. a for any $x \in \mathbb{R}$, we have $(x+1)^2 = x^2 + 2x + 1 \geq 0$ so $x^2 + 2x \geq -1$.

b $f^{-1}(\{0\}) = \{0, -2\}$. $f^{-1}(\{-4\}) = \emptyset$. $f^{-1}(\{-1\}) = \{-1\}$.

c Let $y \in [-1, +\infty)$. We want to show that there exists $x \in \mathbb{R}$ such that $f(x) = y$ namely $x^2 + 2x - y = 0$. We compute $\Delta = 4 + 4y$ which is greater than or equal to 0 since $y \geq -1$. Therefore, this equation has a solution ($x = -1 + \sqrt{1+y}$ or $x = -1 - \sqrt{1+y}$). This proves that $y \in \text{Range}(f)$.

d Surjective, not injective (see above questions).

e Let x be such that $g(x) = -1$. It means that $f(e^x) = -1$, namely $e^{2x} + 2e^x = -1$. Contradiction.

$\square$

5. Let R be the equivalence relation on $\mathbb{R}^2$ defined as

$$(x, y)\ R\ (s, t) \quad \text{if and only if} \quad x^2 - y^2 = s^2 - t^2.$$

Now, let $\mathcal{S}$ be the set of equivalence classes of the relation R. Let $F: \mathcal{S} \to (0, \infty)$ be the function defined as

$$F\big([(x, y)]\big) = e^{x^2 - y^2}.$$

a) Given $(x, y) \in \mathbb{R}^2$, determine $[(x, y)]$, i.e. write down an explicit description of the set $[(x, y)]$.

b) Prove that the function F is bijective.

(You can use without proof that R is an equivalence relation.)

Proof. a) Let $(x, y) \in \mathbb{R}^2$. Then, we see that

$$\begin{aligned}[(x,y)] &= \left\{(a,b) \in \mathbb{R}^2 \mid (x,y)\ R\ (a,b)\right\} \\ &= \left\{(a,b) \in \mathbb{R}^2 \mid x^2 - y^2 = a^2 - b^2\right\}.\end{aligned}$$

Therefore we see that $[(x, y)]$ is a rectangular hyperbola.

b) **Show F is bijective:** We need to show that F is injective and surjective.

- **Injective:** Let $(x, y), (s, t) \in \mathbb{R}^2$, and suppose that

$$F\big([(x,y)]\big) = F\big([(s,t)]\big).$$

This means that $e^{x^2-y^2} = e^{s^2-t^2}$. Since $z \mapsto e^z$ is injective (this follows, for example, from that fact that $z \mapsto e^z$ is increasing), this in turn implies that

$$x^2 - y^2 = s^2 - t^2.$$

Thus, we get $(x, y)\ R\ (s, t)$, that is,

$$[(x,y)] = [(s,t)].$$

Hence F is injective.

- **Surjective:** Let $r \in (0, \infty)$, and let $R = \log(r)$. If $R \geq 0$, we take $[(x, y)] = [(\sqrt{R}, 0)] \in \mathcal{S}$, and see that

$$F\big([(x,y)]\big) = F\big([(\sqrt{R},0)]\big) = e^{\sqrt{R}^2} = e^R = r.$$

If $R < 0$, we take $[(x, y)] = [(0, \sqrt{-R})] \in \mathcal{S}$, and see that

$$F\big([(x,y)]\big) = F\big([(0,\sqrt{-R})]\big) = e^{0-\sqrt{-R}^2} = e^{-(-R)} = e^R = r.$$

Hence, F is surjective.

Therefore, we see that $F : \mathcal{S} \to [0, \infty)$ is bijective.

$\square$