Mathematics 220 — Homework 11

- Contains 8 questions on 5 pages.
- Please submit your answers to all questions.
- We will mark your answer to 3 questions.
- We will provide you with full solutions to all questions.

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1. Determine if the following sets are countable, and prove your answers.

   (a) The set of all functions $f : \{0, 1\} \to \mathbf{N}$.

   Solution: The set is countable.

   Proof. Let F be the set of all functions from $\{0, 1\}$ to $\mathbb{N}$. Consider the function which sends $f : \{0, 1\} \to \mathbf{N}$ to the tuple $(f(0), f(1))$. Namely, we define $g : F \to \mathbb{N} \times \mathbb{N}$ by

   $$g(f) = (f(0), f(1)).$$

   Then, we see that this function is surjective. Moreover, if $f, h \in F$, and $g(f) = g(h)$, then $(f(0), f(1)) = (h(0), h(1))$, which implies that $f = h$, that is, g is injective. Therefore, since $\mathbb{N} \times \mathbb{N}$ is countable, we see that F is countable as well. $\square$

   (b) The set of all functions $f : \mathbf{N} \to \{0, 1\}$.

   Solution:
   The set is uncountable. We will show that the set is in bijection with $\mathcal{P}(N)$ and then Cantor's theorem tells us that the power set is always strictly larger than the original set.

   Proof. Let $S = \{f : \mathbb{N} \to \{0, 1\}\}$ be the set of functions from $\mathbb{N}$ to $\{0, 1\}$. Consider the function $g : S \to \mathcal{P}(\mathbb{N})$ defined by $g(f) = \{n : f(n) = 1\}$. We claim that g is a bijection. This implies that $|S| = |\mathcal{P}(\mathbb{N})|$ and thus is uncountable.
   To prove the claim, it suffices to show that g is injective and surjective.

   - (Injectivity) Suppose that $f, f' \in S$ such that $g(f) = g(f')$. Then we have $\{n : f(n) = 1\} = \{n : f'(n) = 1\}$ and thus $f = f'$.
   - (Surjectivity) Suppose that $X \subset \mathcal{P}(\mathbb{N})$. Define $f : \mathbb{N} \to \{0, 1\}$ by

     $$f(n) = \begin{cases} 1 & \text{if } n \in X \\ 0 & \text{otherwise.} \end{cases}$$

     Then we have $g(f) = X$ by definition.

   $\square$

2. Prove the following statements

   (a) If A is countable but B is uncountable, then $B - A$ is uncountable.

   Solution:

   Proof. Assume for a contradiction that $B - A$ is countable. Since $A \cap B \subseteq A$ is countable, we see that $B = (A \cap B) \cup (B - A)$ is a union of two countable sets and thus countable. This gives a contradiction and proves the first assertion. $\square$

   (b) Between any real numbers a, b such that $a < b$ there are uncountably many irrationals.

   Solution:

   Proof. Let $A = \mathbb{Q}$ and $B = (a, b)$ (ie the open interval from a to b. Then we know that B is uncountable (it is in bijection with $\mathbb{R}$), while $\mathbb{Q}$ is countable. By the result in (a) above, we know that $B - A$, which is the set of irrational numbers between a and b, is uncountable. $\square$

3. Prove that $\mathbb{R}$ and $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ are equinumerous.

Solution:

Proof. Let $f : \mathbb{R} \to \mathbb{R}^+$ be defined by $f(x) = e^x$.

For any $y > 0$, take $x = \ln y$, so $f(\ln y) = e^{\ln y} = y$. Thus f is surjective.

If $f(x_1) = f(x_2)$, then $e^{x_1} = e^{x_2}$, it follows $x_1 = \ln e^{x_1} = \ln e^{x_2} = x_2$ hence f is injective.

So f is bijective, and $\mathbb{R}$ and $\mathbb{R}^+$ are numerically equivalent. $\square$

4. Let S, T be sets. Prove the following

   (a) If $|S| \leq |T|$ then $|\mathcal{P}(S)| \leq |\mathcal{P}(T)|$.

   Solution:

   Proof. Assume that $|S| \leq |T|$ and hence there is an injection from $f : |S| \to |T|$.
   We must construct an injection from $\mathcal{P}(S) \to \mathcal{P}(T)$.
   Let $g : \mathcal{P}(S) \to \mathcal{P}(T)$ be defined by

   $$g(A) = f(A) = \{f(a) \mid a \in A\} \qquad \text{where } A \in \mathcal{P}(S)$$

   ie - we just apply f to each element of A.
   Now let $A, B \in \mathcal{P}(S)$ and assume that $g(A) = g(B)$. We now show $A = B$.

- Let $x \in A$. Hence $f(x) = y \in f(A) = g(A)$. Since $g(A) = g(B)$ we must have $y \in g(B)$. Thus there is some $z \in B$ so that $y = f(z)$. But then since f is injective, $x = z$ and so $x \in B$. Hence $A \subseteq B$.
- The reverse inclusion is similar. Let $z \in B$. Hence $f(z) = y \in f(B) = g(B)$. Since $g(A) = g(B)$ we must have $y \in g(A)$. Thus there is some $x \in A$ so that $y = f(x)$. But then since f is injective, $x = z$ and so $z \in A$. Hence $B \subseteq A$.

Thus g is injective and the result follows. □

(b) If $|S| = |T|$ then $|\mathcal{P}(S)| = |\mathcal{P}(T)|$.

Solution:

Proof. Assume $|S| = |T|$. Hence there is a bijection $f : S \to T$. We must construct an injection from $\mathcal{P}(S) \to \mathcal{P}(T)$. We use the same function as above. Let $g : \mathcal{P}(S) \to \mathcal{P}(T)$ be defined by

$$g(A) = f(A) = \{f(a) \mid a \in A\} \qquad \text{where } A \in \mathcal{P}(S)$$

By the previous question we know that g is injective. It suffices to prove that g is also surjective. Let $B \in \mathcal{P}(T)$ and since f is a bijection, its inverse exists and we may set

$$A = \left\{f^{-1}(b) \mid b \in B\right\}$$

We must now prove that $g(A) = B$.

- Let $x \in g(A)$. Then $x = f(a)$ for some $a \in A$. But then $a = f^{-1}(b)$ for some $b \in B$. Hence $x = f(f^{-1}(b)) = b$. So $x \in B$.
- Now let $x \in B$. By construction $f^{-1}(x) \in A$. Hence $f(f^{-1}(x)) \in f(A) = g(A)$. But $f(f^{-1}(x)) = x$, so $x \in g(A)$

Thus g is surjective and we are done. □

5. Show that there exist infinitely many pairs of distinct natural numbers a, b such that $17^a - 17^b$ is divisible by 2023.

   Hint: Pigeons can help.

Solution:

Proof. Consider the set

$$\{17^1, 17^2, \ldots, 17^{2024}\}.$$

If we consider their remainders when dividing by 2023, we see that there are at most 2023 possible remainders, but 2024 numbers in the set. So by the pigeonhole principle, there exists distinct integers a and b such that $17^a \equiv 17^b \pmod{2023}$. This implies that their difference is divisible by 2023.

We then repeat this process on sets of the form

$$S_k := \{17^{1+2024k}, 17^{2+2024k}, \ldots, 17^{2024+2024k}\},$$

for each natural number k, so as to obtain infinitely many such pairs. □

6. Prove that $(-\infty, -\sqrt{29})$ and $\mathbb{R}$ are equinumerous by constructing an explicit bijection.

Solution:

Proof. Let $f(x) = \log(-x - \sqrt{29})$. We then construct a function $g(y) = -e^y - \sqrt{29}$. Note that f has domain $(-\infty, -\sqrt{29})$ and g has domain $\mathbb{R}$.

Now

$$f(g(y)) = \log(e^y + \sqrt{29} - \sqrt{29}) = y$$

and

$$g(f(x)) = -e^{\log(-x-\sqrt{29})} - \sqrt{29} = -(-x - \sqrt{29}) - \sqrt{29} = x.$$

We conclude that $f \circ g$ and $g \circ f$ are both identity functions (on $\mathbb{R}$ and $(-\infty, -\sqrt{29})$, respectively), and so f has a two-sided inverse, which means that it is a bijection.

Therefore, $(-\infty, -\sqrt{29})$ and $\mathbb{R}$ are equinumerous. □

7. Prove or disprove: for any non-empty sets A, B, C, if $|A \times B| = |A \times C|$ then $|B| = |C|$.

Solution:

Proof. The statement is false. Take $A = \mathbb{N}$, $B = \{1\}$ and $C = \{1, 2\}$. Then we have $|A \times B| = |\mathbb{N} \times \{1\}| = |\mathbb{N}|$ and $|A \times C| = |\mathbb{N} \times \{1, 2\}| = |\mathbb{N}|$ but $|B| < |C|$. □

8. Let A be a finite set and $f : \mathbb{R} \to A$. Show that there exists some $a \in \mathbb{A}$ such that $f^{-1}(\{a\})$ is uncountable.

Solution:

Proof. Let us argue by contradiction and assume that for all $a \in A$ the set $f^{-1}(\{a\})$ is countable. We can write

$$\mathbb{R} = \bigcup_{a \in A} f^{-1}(\{a\})$$

Since $\mathbb{R}$ is a finite union of countable sets, then $\mathbb{R}$ is countable. This is a contradiction. $\square$