The inhomogeneous problem

Recall the ODE:

$$ay'' + by' + cy = f(x)$$

with constants a, b and c, and $f(x)$ a prescribed function.

An example: $y'' + y = e^x$. Notice that if $y \propto e^x$, then all three terms in the ODE have a e^x that therefore can be cancelled out. *i.e.* a potential solution to the ODE is $y(x) = de^x$, for some constant d. Indeed, plugging and chugging implies $d = \frac{1}{2}$, so $y = \frac{1}{2}e^x$ is a solution to the ODE. However, for a general solution we are expecting some additional bits with arbitrary constants in them.

Now let $y(x) = y_h(x) + y_p(x)$ with $y_p(x) = \frac{1}{2}e^x$ and $y_h(x)$ satisfying the corresponding homogeneous ODE, $y_h'' + y_h = 0$. On plugging this combo into the ODE, one realizes that it will still satisfy the equation (because the $y_h(x)$ cancels on the LHS, and $y_p(x)$ gives us the RHS). But we know how to solve for $y_h(x)$, and it, in general, will contain two arbitrary constants!

Thus, the general solution takes the form $y_h(x) + y_p(x)$ where the homogeneous solutions account for the arbitrary parts of the general solution and $y_p(x)$, the "particular solution", is designed to match the inhomogeneous term.

For our example: $y_h(x) = C\cos x + D\sin x$ and $y_p(x) = \frac{1}{2}e^x$. Hence $y(x) = C\cos x + D\sin x + \frac{1}{2}e^x$.

General strategy: (for $ay'' + by' + cy = f(x)$)

1. determine the homogeneous solutions (satisfying $ay'' + by' + cy = 0$)
2. find a particular solution (by posing a trial solution based on $f(x)$ and containing constants to be determined by plugging into the inhomogeneous ODE)
3. apply any initial/boundary conditions, if provided, to fix the arbitrary constants in the homogeneous solutions (do not do this without including the particular solution, as your final answer will not generally then satisfy the initial or boundary conditions; *i.e.* use the full general solution when imposing the initial or boudnary conditions)

To come up with suitable trial particular solutions, the following table should prove helpful:

Inhomogeneous term, $f(x)$:	Trial particular solution (with constants d, d_1, d_2, ... to be determined):
$e^{\eta x}$	$de^{\eta x}$
Polynomial of degree n	Polynomial of degree n, $d_1x^n + d_2x^{n-1} + ...$
$\cos\omega x$ or $\sin\omega x$	$d_1\cos\omega x + d_2\sin\omega x$
$e^{\eta x}\cos\omega x$ or $e^{\eta x}\sin\omega x$	$(d_1\cos\omega x + d_2\sin\omega x)e^{\eta x}$
(Polynomial of degree n)$e^{\eta x}$	$(d_1x^n + d_2x^{n-1} + ...)e^{\eta x}$
Homogeneous solution	$dxf(x)$

One must beware of inhomogeneous terms that are also of the same form as a homogeneous solution. In this case, the trial solution will not work: the trial completely disappears from the left-hand side of the equation because it is a homogeneous solution. A different trial is needed instead; adding an additional factor of x usually works.

e.g. $y'' - y = f(x)$. The homogeneous solutions are $A_1e^x + A_2e^{-x}$.

• For $f(x) = 3e^{2x}$, we use $y_p(x) = de^{2x}$ and (by plugging the trial into the ODE) find that $d = 1$. Thus $y(x) = A_1e^x + A_2e^{-x} + e^{2x}$.

• For $f(x) = 2\cos x$, we use $y_p(x) = d_1\cos x + d_2\sin x$ and find that $d_1 = -1$ and $d_2 = 0$ (note that $d_2 = 0$ is a consequence of the absence of a y' term). Thus $y(x) = A_1e^x + A_2e^{-x} - \cos x$.

• For $f(x) = 5e^x\sin x$, we use $y_p(x) = e^x(d_1\cos x + d_2\sin x)$ and find that $d_1 = 2d_2$ and $2d_1 + d_2 = -5$, so $d_2 = -1$ and $d_1 = -2$. Thus $y(x) = A_1e^x + A_2e^{-x} - e^x(2\cos x + \sin x)$.

• For $f(x) = 2e^x$, there is a hitch as this is a homogeneous solution and so de^x will not work as a trial particular solution. Instead, let $y_p(x) = dxe^x$. Plugging into the ODE now gives $d = 1$, and so $y(x) = A_1e^x + A_2e^{-x} + xe^x$.

Potpouri of examples:

• $y' + 3y = 9x$ with $y(0) = 0$.
This is a linear first-order ODE with integrating factor $I = e^{3x}$. Furthermore, $\int qIdx = 9\int xe^{3x}dx = 3xe^{3x} - e^{3x}$. Hence the general solution is $y(x) = Ce^{-3x} + 3x - 1$. Applying $y(0) = 0$ gives $C = 1$.

• $y' = (y + y^{-1})\cos x$ with $y(0) = 0$.

The RHS is separable and so

$$\int \frac{ydy}{1+y^2} = C + \int \cos x dx \qquad \text{or} \qquad \log(1+y^2) = 2(C + \sin x) \qquad \text{or} \qquad y = \pm\sqrt{Ae^{2\sin x} - 1}$$

with arbitrary C or A (note that there is no need for any absolute value as $1 + y^2 > 0$). Imposing the starting value gives $A = 1$.

• $y'' + 3y' + 2y = 0$ with $y(0) = 0$ and $y'(0) = 1$. Auxiliary equation is $m^2 + 3m + 2 = (m+1)(m+2) = 0$, so $y(x) = A_1e^{-x} + A_2e^{-2x}$. Applying the initial conditions gives $y(x) = e^{-x} - e^{-2x}$.

• $y'' + 6y' + 13y = 0$ with $y(0) = 2$ and $y'(\pi) = 0$. Auxiliary equation is $m^2 + 6m + 13 = (m+3)^2 + 4 = 0$, so $y(x) = e^{-3x}(C\cos 2x + D\sin 2x)$. Applying the boundary conditions gives $C = 2$ and $-3C + 2D = 0$, or $D = 3$.

• $y'' + 6y' + 9y = 0$. Auxiliary equation is $m^2 + 6m + 9 = (m+3)^2 = 0$, so $y(x) = e^{-3x}(ax + b)$.

• $y'' - 4y' + 3y = f(x)$, for $f(x) = e^{2x}$, $\cos x$, x^2 and e^x.

Homogeneous solutions: Auxilliary equation, $m^2 - 4m - 3 = (m-3)(m-1) = 0$, implying $m = 1$ or 3. Hence the homogeneous solutions are $Ae^x + Be^{3x}$, where A and B are arbitrary constants.

Particular solutions:

For $f(x) = e^{2x}$, try $y = de^{2x}$. Plugging and chugging implies that this is a solution if $4d - 8d + 3d = 1$; that is, if $d = -1$.

For $f(x) = \cos x$, try $y = d_1\cos x + d_2\sin x$. Plug and chug; the solution works if $-d_1 - 4d_2 + 3d_1 = 1$ and $-d_2 + 4d_1 + 3d_2 = 0$, which gives $d_1 = 1/10$ and $d_2 = -1/5$.

For $f(x) = x^2$, try $y = d_1x^2 + d_2x + d_3$. Plug and chug to find that $3d_1 = 1$, $3d_2 - 8d_1 = 0$ and $3d_3 - 4d_2 + 2d_1 = 0$, or $d_1 = 1/3$, $d_2 = 8/9$ and $d_3 = 26/27$.

For $f(x) = e^x$, the trial solution $y = de^x$ will not work because the inhomogeneous term has the form of a homogeneous solution. Now we must try the solution dxe^x. Plugging and chugging implies that $-2d = 1$, or $d = -1/2$.

Mechanical oscillators

Consider a mass M hanging vertically down at position $X(t)$ on the end of a spring with constant k. Air drag, proportional to $\dot{X}$ with drag coefficient D, slows down the mass. There is a motor at the other end of the spring that shakes this end up a down at frequency ω with amplitude Y_0, so that its position is $Y(t) = Y_0 \cos \omega t$. Newton's law gives

$$M\ddot{X} = Mg - k(X - Y) - D\dot{X} = Mg - kX - D\dot{X} + kY_0 \cos \omega t$$

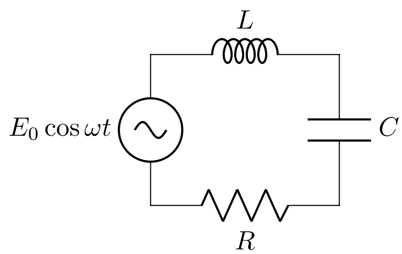
where g is gravity. Let $y = X - Mg/k$, $\gamma = D/(2M)$, $\Omega^2 = k/M$ and $a = kY_0/M$. Then,

$$\ddot{y} + 2\gamma\dot{y} + \Omega^2 y = a \cos \omega t$$

Note that ω has the units of radians per second. The cyclic frequency f, which has units of cycles per second (*i.e.* Hertz) is related by $\omega = 2\pi f$.

Electrical circuits

The mechanical oscillator problem is the same as that for an electrical circuits in which an EMF drives an alternating current through a resistor, impedance and capacitor (in series).



The various laws of electrical circuits demand that

$$Lq'' + Rq' + \frac{q}{C} = E_0 \cos \omega t,$$

where $q(t)$ is the charge in the circuit at time t, L is the impedance, R is the resistance, C is the capacitance and E_0 is the amplitude of the EMF, which has frequency ω. If $\gamma = R/(2L)$, $\Omega = (LC)^{-1/2}$ and $a = E_0/L$, we arrive at exactly the same problem as considered above with $q(t)$ replacing $y(t)$.

Solution of the ODE and resonance

The general solution to this ODE is $y(t) = y_h(t) + y_p(t)$, where $y_h(t)$ are the homogeneous solutions (free oscillations) and $y_p(t)$ is the particular solution (forced response). The homogeneous solutions can be found from looking at the auxiliary equation, $m^2 + 2\gamma m + \Omega^2 = 0$, giving $m = -\gamma \pm \sqrt{\gamma^2 - \Omega^2}$.
• For $\gamma > \Omega$, the solutions are real and unequal; the homogeneous solutions decay exponentially in time, This is called the "over-damped" case.
• For $\gamma = \Omega$, the roots are equal and $m = -\gamma$. This gives the homogeneous solutions $y_h(t) = (At+B)e^{-\gamma t}$, which still decays because of the exponential factor $e^{-\gamma t}$. This is the "critically damped" case.
• For $\gamma < \Omega$ we have decaying oscillations with $y_h = e^{-\gamma t}[C \cos(\sqrt{\Omega^2 - \gamma^2}t) + D \sin(\sqrt{\Omega^2 - \gamma^2}t)]$. This is called the "under-damped" case.
For all three cases, if we wait for long enough, $y_h(t)$ decays away, leaving the particular solution, which we will now concentrate on.

We pose the trial $y_p(t) = c \cos \omega t + d \sin \omega t$. After some algebra, we find

$$c = \frac{(\Omega^2 - \omega^2)a}{(\Omega^2 - \omega^2)^2 + 4\omega^2\gamma^2}, \qquad d = \frac{2\omega\gamma a}{(\Omega^2 - \omega^2)^2 + 4\omega^2\gamma^2},$$

and with a little more effort,

$$y_p(t) = S(\omega) \cos(\omega t - \delta), \qquad S(\omega) = \frac{a}{\sqrt{(\Omega^2 - \omega^2)^2 + 4\gamma^2\omega^2}}, \qquad \tan \delta = \frac{2\gamma\omega}{\Omega^2 - \omega^2}.$$

i.e. a pure oscillation at frequency ω, with a phase lag δ behind the forcing. The amplitude $S(\omega)$ has a maximum of $S_{max} = a/\sqrt{\Omega^4 - \omega^4} = a/[2\gamma\sqrt{\Omega^2 - \gamma^2}]$ when $\omega^2 = \Omega^2 - 2\gamma^2$. Thus, as the damping ($\gamma$) becomes small, the forced response becomes large at the "resonant" frequency $\omega \approx \Omega$. For $\omega \to 0$, $S(\omega) \to a/\Omega^2$, and for $\omega \to \infty$, $S(\omega) \to 0$. The "response curve", S against ω, illustrates the forced behaviour.

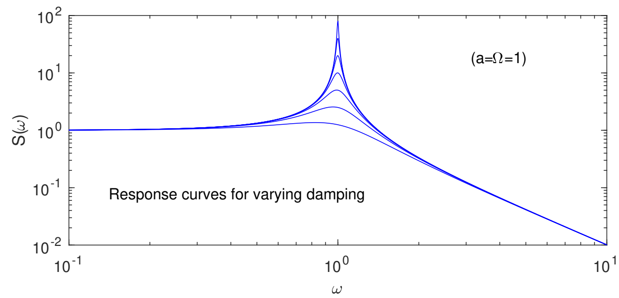


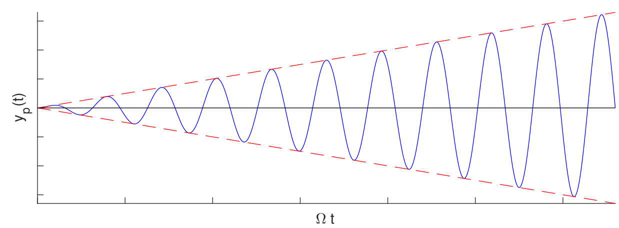
Figure 1: RESONANCE: the near divergence of the forced amplitude when damping is small and an oscillator is driven at a frequency close to the natural frequency (Ω).

No damping

Now consider what happens without any damping, $\gamma = 0$. The ODE is $\ddot{y} + \Omega^2 y = a\cos\omega t$. The particular solution is now

$$y_p(t) = S\cos\omega t, \quad S = \frac{a}{\Omega^2 - \omega^2}, \quad \text{for } \omega \neq \Omega \qquad \text{and} \qquad y_p(t) = \frac{at}{2\Omega}\sin\Omega t, \quad \text{for } \omega = \Omega.$$

The latter (the driven resonant response without damping) is an oscillation with an amplitude that grows linearly with time. This can be destructive (*e.g.* the Tacoma Narrows suspension bridge; shattering a wine glass with sound) or exploited (quartz clock; lasers).



Beats

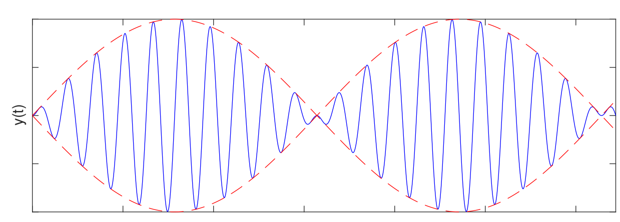
Consider now the situation in which the oscillator is driven close to (but not at) resonance, with initial condition $y(0) = \dot{y}(0) = 0$. The solution is

$$y(t) = S(\cos\omega t - \cos\Omega t)$$

If $\omega = \Omega + \epsilon$ with $|\epsilon| \ll \Omega$, and with the help of some trig formulae, we find

$$y(t) = 2S\sin\left[\frac{1}{2}(\omega - \Omega)t\right]\sin\left[\frac{1}{2}(\Omega + \omega)t\right] \approx 2S\sin\left(\frac{1}{2}\epsilon t\right)\sin\Omega t$$

This is a fast oscillation at frequency Ω, but with an amplitude that varies sinusoidally in time over a much longer period $4\pi/\epsilon$. *i.e.* The phenomenon of BEATING.



Euler equations

Euler equations (of second order) have the form

$$\alpha x^2 y'' + (\beta + \alpha) x y' + \gamma y = 0,$$

for three constants α, β and γ. That is, for each derivative, there is a corresponding factor of x. The general solutions to this equation can be found by posing the solution $y(x) = Ax^m$. Plugging this trial solution into the ODE gives

$$[\alpha m(m-1) + (\beta + \alpha)m + \gamma]Ax^m = (\alpha m^2 + \beta m + \gamma)Ax^m = 0.$$

We once more demand that the auxiliary equation $\alpha m^2 + \beta m + \gamma = 0$ is satisfied, giving the two options $m = m_1$ and $m = m_2$ with $m_{1,2} = (-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma})/(2\alpha)$. The general solution is then $y(x) = A_1 x^{m_1} + A_2 x^{m_2}$.

e.g. $x^2 y'' + xy' - y = 0$. Plugging in $y(x) = Ax^m$ gives $m(m-1) + m - 1 = 0$ or $m = \pm 1$. Hence, $y(x) = A_1 x + A_2 x^{-1}$.

Reduction of Order

"Reduction of order" is a technique that furnishes a second solution to a homogeneous linear second-order ODE if a first solution is known. *e.g.* $x^2 y'' + xy' - y = 0$ has solution $y(x) = Ax$ for any constant A; what is a second solution? Note that this equation does not have constant coefficients!

Strategy: Take the arbitrary constant in the known solution and make it into a function of x. *i.e.* if $y = Ay_1(x)$ is the known solution, put $y = A(x)y_1(x)$. Plugging into the ODE:

$$ay'' + by' + cy = a(Ay_1'' + 2A'y_1' + y_1 A'') + b(Ay_1' + y_1 A') + cAy_1 = 0,$$

where the coefficients a, b and c may not be functions of x. But $ay_1'' + by_1' + cy_1 = 0$, and so

$$a(2A'y_1' + y_1 A'') + by_1 A' = 0, \qquad \text{or} \qquad ay_1 Z' + (2ay_1' + by_1)Z = 0 \text{ if } Z = A'.$$

This is a linear, separable first-order ODE for $Z(x)$ that can be solved:

$$Z = A' = C \exp\left[-\int (2ay_1' + by_1)\frac{dx}{ay_1}\right] = \frac{C}{y_1^2}\exp\left[-\int \frac{b}{a}dx\right]$$

A further integral then gives A.

For our example: inserting $y = xA(x)$ into $x^2 y'' + xy' - y = 0$ gives $x^3 A'' + 3x^2 A' = 0$ or $Z' = -3Z/x$. Hence, $Z = Cx^{-3}$. Integrating again: $A = A_1 + A_2 x^{-2}$ for two new arbitrary constants A_1 and A_2 (with A_2 related to C). Thus, $y = A_1 x + A_2 x^{-1}$.

The method can also be applied to an inhomogeneous ODE to generate a particular solution without posing a trial. For example, for a second-order, constant-coefficient ODE (with constants a, b and c), $ay'' + by' + cy = f(x)$, the homogeneous solutions are $y = A_1 e^{m_1 x} + A_2 e^{m_2 x}$. Now try a solution to the inhomogeneous ODE using reduction of order, $y(x) = A(x)e^{mx}$ where m is either of m_1 or m_2. Then,

$$[a(A'' + 2mA' + m^2 A) + b(A' + mA) + cA]e^{mx} = f(x).$$

But $am^2 + bm + c = 0$, and so

$$A'' + \left(2m + \frac{b}{a}\right)A' = \frac{1}{a}e^{-mx}f(x) \qquad \longrightarrow \qquad A' = e^{-(2m+b/a)x}\left[C + \frac{1}{a}\int e^{(m+b/a)x}f(x)dx\right],$$

using the integrating factor $I = e^{(2m+b/a)x}$. Note that the first term corresponds to the other homogeneous solution and the integral to the particular solution; the original homogeneous solution appears when we integrate A' and add a second constant of integration.

e.g. $y'' - y = f(x)$. We set $y(x) = A(x)e^x$ (since $m^2 = 1$). Then, $A'' + 2A' = e^{-x}f(x)$ giving $A' = e^{-2x}[C + \int f(x)e^x dx]$ (the integrating factor is e^{2x}). At this stage we need an $f(x)$ to make further progress. For $f(x) = 4xe^x$, we find $A' = Ce^{-2x} + 2x - 1$. Integrating a second time: $A = A_1 + A_2 e^{-2x} + x^2 - x$ for two new arbitrary constants A_1 and A_2 (with A_2 related to C). Thus, $y(x) = A_1 e^x + A_2 e^{-x} + x(x-1)e^x$.