Homogeneous systems of ODEs

We now deal with the system of first-order equations,

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x},$$

where $\mathbf{x}(t)$ is a (column) vector of length n and A is an $n \times n$ matrix.

Practically, we will consider A to have constant entries, in which case, by a process of successive elimination of all but one of the components of $\mathbf{x}(t)$, we may reduce the system to an $n^{th}-$order, linear, constant-coefficient ODE.

e.g. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, implying the two coupled first-order ODEs,

$$\dot{x}_1 = x_1 + 2x_2 \qquad \& \qquad \dot{x}_2 = 3x_1 + 4x_2$$

We have $x_2 = \frac{1}{2}(\dot{x}_1 - x_1)$ from the first equation, and so the substitution of this relation into the second equation gives the second-order ODE,

$$\ddot{x}_1 - 5\dot{x}_1 - 2x_1 = 0$$

The reverse is also true: an $n^{th}-$order linear ODE can be recast as a system.

e.g. $a\ddot{x} + b\dot{x} + cx = 0$. Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$. We therefore have the two equations, $\dot{x}_1 = 0x_1 + 1x_2$ and $\dot{x}_2 = -\frac{c}{a}x_1 - \frac{b}{a}x_2$, which are equivalent to the system

$$\frac{d}{dt}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}\mathbf{x}.$$

Solution strategy: Pose $\mathbf{x} = \mathbf{v}e^{mt}$, where m is a constant exponent to be determined from an auxiliary equation and $\mathbf{v}$ is a constant vector (of length n). Plugging in to the system and cancelling off the exponential factor e^{mt} gives

$$m\mathbf{v} = A\mathbf{v}$$

i.e. the eigenvalue equation of A. Thus, m must be an eigenvalue of A and $\mathbf{v}$ is the corresponding eigenvector. The general solution is then a linear combination of all the eigensolutions:

$$\mathbf{x} = A_1\mathbf{v}_1e^{m_1t} + A_2\mathbf{v}_2e^{m_2t} + ... + A_n\mathbf{v}_ne^{m_nt},$$

with a set of arbitrary constants $\{A_1, A_2, ..., A_n\}$. Note that here we use the facts that

(i) an $n \times n$ matrix has n eigenvalues and eigenvectors (except in some special situations that we will not consider),

(ii) the eigenvalues follow from setting $\det(A - mI) = 0$ where "det" is short for "determinant" and I is the $n \times n$ identity matrix,

(iii) the eigenvectors are computed by solving the linear system $(A - mI)\mathbf{v} = \mathbf{0}$ once each of the eigenvalues m is found,

(iv) the eigenvectors are not unique, but can be multiplied by any arbitrary constant. Practically when one computes the eigenvectors one often make a convenient choice to fix the solution. However, the general solution to the system of ODEs requires as many arbitrary constants as possible. In the general solution above, we have explicitly included these arbitrary constants, denoting them by A_1, A_2, ... A_n, with the understanding that the eigenvectors have been computed without including any further free constants.

e.g. $d\mathbf{x}/dt = A\mathbf{x}$ with

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \longrightarrow \quad \det(A - mI) = (1-m)^2 - 4 = m^2 - 2m - 3 = (m-3)(m+1).$$

Hence $m = 3$ or -1. The corresponding eigenvectors satisfy

$$m = 3: \qquad \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{or} \qquad v_1 = v_2$$

and

$$m = -1 : \qquad \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{or} \qquad v_1 = -v_2$$

We may therefore take $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $m = 3$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $m = -1$, then write the general solution

$$\mathbf{x} = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

Initial conditions: To fix the arbitrary constants in the general solution, we need an initial condition such as $\mathbf{x}(0) = \mathbf{x}_0$. This is an $n \times n$ linear system for A_1, A_2, ... A_n.

e.g. For the previous example, we impose the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{or} \qquad 1 = A_1 + A_2 \quad \& \quad 0 = A_1 - A_2 \qquad \to \qquad A_1 = A_2 = \frac{1}{2},$$

giving

$$\mathbf{x} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} = \frac{1}{2} \begin{pmatrix} e^{3t} + e^{-t} \\ e^{3t} - e^{-t} \end{pmatrix}$$

Inhomogeneous problems: If an inhomogeneous terms also appears in the system, then the general solution is composed of homogeneous solutoins, constructed as above, and a particular solution, found by posing a suitable trial solution. *e.g.*

$$\dot{\mathbf{x}} = A\mathbf{x} + e^{3t}\mathbf{c},$$

where $\mathbf{c}$ is a constant vector. We pose the trial $\mathbf{x}_p = \mathbf{d}e^{3t}$, to obtain

$$3\mathbf{d}e^{3t} = (A\mathbf{d} + \mathbf{c})e^{3t}.$$

The exponential function is along for the ride at this stage, so we may cancel it, leaving

$$(A - 3I)\mathbf{d} = -\mathbf{c},$$

where I is the $n \times n$ identity matrix. This is a linear system of the form $M\mathbf{d} = -\mathbf{c}$ with matrix $M = A - 3I$. As long as we can invert this matrix, we can write the formal solution,

$$\mathbf{d} = -(A - 3I)^{-1}\mathbf{c}, \quad \text{or} \quad \mathbf{x}_p = -(A - 3I)^{-1}\mathbf{c}e^{3t},$$

and there is some linear algebra to be performed to write the solution more explicitly. The general solution is then

$$\mathbf{x} = \sum_{j=1}^{n} A_j \mathbf{v}_j e^{m_j t} - (A - 3I)^{-1}\mathbf{c}e^{3t}. \tag{1}$$

Note that the construction fails if $M = A - 3I$ is not invertible. This arises when $\det(M) = \det(A - 3I) = 0$. But $\det(A - \lambda I) = 0$ gives the eigenvalues. So the failure will occur when the exponential on the RHS corresponds to one of the homogeneous solutions, which ought to sound familiar by now. One can guess what the strategy might be to fix this! In any event, to avoid digressing too much into a lot of linear algebra, we'll usually leave the solution in a formal form like (1). *i.e.* don't compute those inverses!

Second-order systems: the methodology also works for second-order systems of the form

$$\ddot{\mathbf{x}} = A\mathbf{x},$$

which is typical of coupled oscillator systems. Again, we introduce $\mathbf{x} = \mathbf{v}e^{mt}$, to arrive at

$$m^2\mathbf{v} = A\mathbf{v}.$$

This time the eigenvalues of A are m^2; the corresponding eigenvectors are still $\mathbf{v}$.

$$e.g. \quad A = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix} \quad \longrightarrow \quad \det(A - m^2 I) = (m^2+3)^2 - 1 = m^4 + 6m^2 + 8 = (m^2+2)(m^2+4).$$

Hence $m^2 = -2$ or -4. The corresponding eigenvectors satisfy

$$m^2 = -2: \qquad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{or} \qquad v_1 = -v_2$$

and

$$m^2 = -4: \qquad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{or} \qquad v_1 = v_2$$

We may therefore take $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $m^2 = -2$, or $m = \pm\sqrt{2}i$, and $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $m^2 = -4$, or $m = \pm 2i$. Hence the general solution is

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} [A_1 \cos(\sqrt{2}t) + B_1 \sin(\sqrt{2}t)] + \begin{pmatrix} 1 \\ 1 \end{pmatrix} [A_2 \cos(2t) + B_2 \sin(2t)].$$

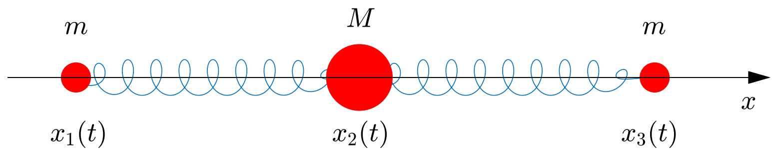
Note that the general solution now has four arbitrary constants (A_1, A_2, B_1, B_2), corresponding to each possible solution for m (but there are still only two eigenvectors).

We may also add inhomogeneous terms to the second-order system and then find the corresponding particular solutions following the usual strategy of educated guess work. Again, the best thing to do in Math 256 is to write a formal solution containing matrix inverses, and avoid the linear algebra required to compute any of them! See the worked problem below by way of an example.

A model of a water molecule

A model of the water molecule consists of two particles of mass m at positions $x_1(t)$ and $x_3(t)$, sandwiching a third with mass M at position $x_2(t)$. The particles (atoms) are connected to their neighbours by springs with constant k. Newton's law states that

$$\begin{aligned} m\ddot{x}_1 &= k(x_2 - x_1), \\ M\ddot{x}_2 &= k(x_3 - x_2) - k(x_2 - x_1), \\ m\ddot{x}_3 &= -k(x_3 - x_2). \end{aligned}$$



These three ODEs can be written as the system,

$$\ddot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \omega^2 \begin{pmatrix} -1 & 1 & 0 \\ \alpha & -2\alpha & \alpha \\ 0 & 1 & -1 \end{pmatrix},$$

where $\omega^2 = k/m$ and $\alpha = m/M$. The eigenvalues and eigenvectors of A are

$$\lambda = -\omega^2, \quad \mathbf{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}; \qquad \lambda = -(1+2\alpha)\omega^2, \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2\alpha \\ 1 \end{pmatrix}; \qquad \lambda = 0, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(you can confirm this by plugging them into the eignvalue/vector equation and doing the matrix multiplication).

The first of these eigenvalue/vector pairs corresponds to a pair of solutions with the form

$$[A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t)] \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

for two arbitrary constants A_1 and B_1. This is an oscillation of the molecule at frequency $\omega_1 = \omega$ in which the middle particle (*i.e.* the oxygen atom) does not participate in the motion, and the two outer particles (the hydrogen atoms) oscillate out of phase with one another.

The second eigenvalue/vector pair is an oscillation at frequency $\omega_2 = \omega\sqrt{1+2\alpha}$, with solution is

$$[A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)] \begin{pmatrix} 1 \\ -2\alpha \\ 1 \end{pmatrix}.$$

In this motion, the two outer particles oscillate in phase with one another, but are out of phase with the central particle, which also oscillates at a different amplitude.

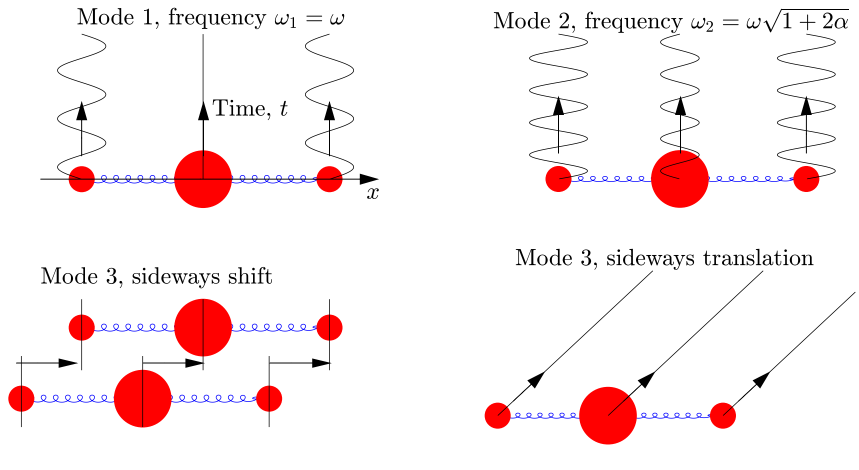
The last eigenvalue/vector pair is not an oscillation. If $\mathbf{x}(t) = T(t)\mathbf{v}$, where $\mathbf{v}$ is the relevant eigenvector, then substitution into the system of ODEs gives

$$\ddot{T}\mathbf{v} = \mathbf{0}.$$

i.e. $T(t)$ is the linear function of time, $A_3 + tB_3$. In other words, the eigenvalue/vector pair leads to a solution

$$(A_3 + tB_3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The A_3 part refers to a uniform sideways shift of the molecule; the B_3 part describes a launch of the molecule sideways at constant speed.



Overall the general solution to the system is

$$\mathbf{x}(t) = [A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t)] \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + [A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)] \begin{pmatrix} 1 \\ -2\alpha \\ 1 \end{pmatrix} + (A_3 + tB_3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (2)$$

The various pieces of the solution, or "modes", are illustrated in the figure,

If hydrogen bonding with other water molecules agitates the two outer atoms, their equations of motion become revised to

$$m\ddot{x}_1 = k(x_2 - x_1) + f(t), \quad m\ddot{x}_3 = -k(x_3 - x_2) + f(t).$$

for some prescribed function $f(t)$. In this case, our system is revised to

$$\ddot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}, \quad \mathbf{f}(t) = f(t) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The solution to this problem consists of the homogeneous solutions in (2) and a particular solution whose form is dictated by $f(t)$. For example, if $f(t) = \cos(\Omega t)$, we can pose the trial particular solution

$$\mathbf{x}_p(t) = \mathbf{d}\cos(\Omega t)$$

(there are no first derivatives in the system!). Plugging this trial into the system gives

$$-\Omega^2 \mathbf{d}\cos(\Omega t) = A\mathbf{d}\cos(\Omega t) + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cos(\Omega t).$$

That is, we may formally write

$$\mathbf{d} = -(A + \Omega^2 I)^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

where I is the 3×3 identity matrix, and so

$$\mathbf{x}(t) = \mathbf{x}_h(t) - (A + \Omega^2 I)^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cos(\Omega t),$$

where $\mathbf{x}_h(t)$ denotes the homogeneous solutions in (2). This formal solution of the problem boils down to some linear algebra as long as the matrix $(A + \Omega^2 I)$ is invertible. When it is not, we need to solve the problem differently. The matrix $(A + \Omega^2 I)$ is no longer invertible when $\det(A + \Omega^2 I) = 0$, or when $-\Omega^2$ is one of the eigenvalues of A; *i.e.* $\Omega = \omega$, $\Omega = \omega\sqrt{1+2\alpha}$ or $\Omega = 0$. The first two conditions reflect the occurrence of resonance for our system of ODEs. The last one corresponds to driving the molecule with a constant force (so it must accelerate).