SOLUTIONS TO MIDTERM #1, MATH 300

1. (9 marks) Answer true or false to the following questions by putting either true or false in the boxes. If the answer is true give a proof, and if the answer is false give a counter-example.

   (a) $Log\ e^z = z\ \forall$ complex numbers z.

   (b) $\left(\sqrt{2}\cos\frac{\pi}{6} + \imath\sqrt{2}\sin\frac{\pi}{6}\right)^4 = -2 + 2\sqrt{3}\imath$.

   (c) $Arg(z_1 z_2) = Arg(z_1) + Arg(z_2)\ \forall$ complex numbers z.

   Solution:

   (a) FALSE. For example $\text{Log}\ e^{2\pi\imath} = 0 \neq 2\pi\imath$. In class we proved that

   $$\text{Log}\ e^z = z \iff -\pi < y \leq \pi.$$

   (b) TRUE. By De Moivre's theorem

   $$\left(\sqrt{2}\cos\frac{\pi}{6} + \imath\sqrt{2}\sin\frac{\pi}{6}\right)^4 = (\sqrt{2})^4\left(\cos\frac{4\pi}{6} + \imath\sin\frac{4\pi}{6}\right) = -2 + 2\sqrt{3}\imath.$$

   (c) FALSE. This would be true if all 3 values $Arg(z_1 z_2), Arg(z_1), Arg(z_2)$ were in the interval $(-\pi, \pi]$. An example where it is not true is $z_1 = z_2 = e^{2\pi\imath/3}$.

2. (9 marks) The following questions require little or no computation.

   (a) Let $f(z) = u(x,y) + \imath v(x,y)$ be an entire function. What are the Cauchy-Riemann equations?

   (b) Express $\text{Log}(\sqrt{3} + \imath)$ in the form $a + b\imath$.

   (c) Find the principal value of $(1+\imath)^{1+\imath}$.

   Solution:

   (a) The Cauchy-Riemann equations for an analytic function $f(z) = u(x,y) + \imath v(x,y)$ are

   $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

   (b) $\text{Log}\ (\sqrt{3} + \imath) = \text{Log}\ \left(2e^{\pi\imath/6}\right) = \ln 2 + \pi\imath/6$.

   (c) The principal branch of $(1+\imath)^{(1+\imath)}$ is by definition

   $$\begin{aligned}(1+\imath)^{(1+\imath)} &= e^{(1+\imath)\text{Log}\ (1+\imath)} = e^{(1+\imath)\text{Log}\ (\sqrt{2}e^{\pi\imath/4})} \\ &= e^{(1+\imath)\left(\ln\sqrt{2} + \pi\imath/4\right)} = e^{(\ln\sqrt{2} - \pi/4)}e^{\imath(\ln\sqrt{2} + \pi/4)}\end{aligned}$$

3. (9 marks) Find all solutions of the following equations. Express your answers in the form $a + b\imath$.

   (a) $\dfrac{1+z^2}{1-z^2} = \imath$.

   (b) $z^3 + 1 = 0$.

   (c) $\cos z = 2\imath \sin z$.

   Solution:

   (a) $\dfrac{1+z^2}{1-z^2} = \imath \iff z^2 = \dfrac{\imath - 1}{\imath + 1} = \imath \iff z = \sqrt{\imath} = \pm\dfrac{\imath + 1}{\sqrt{2}}$.

   (b)

$$\begin{aligned} z^3 + 1 = 0 \quad &\iff \quad z = -1, -\omega, -\omega^2 \quad \left(\text{where } \omega = e^{2\pi\imath/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\imath\right) \\ &\iff \quad z = -1, \frac{1}{2} - \frac{\sqrt{3}}{2}\imath, \frac{1}{2} + \frac{\sqrt{3}}{2}\imath \end{aligned}$$

   (c)

$$\begin{aligned} \cos z = 2\imath \sin z \quad &\iff \quad \frac{e^{\imath z} + e^{-\imath z}}{2} = 2\imath \frac{e^{\imath z} - e^{-\imath z}}{2\imath} = e^{\imath z} - e^{-\imath z} \iff e^{\imath z} = 3e^{-\imath z} \\ &\iff \quad e^{2\imath z} = 3 \iff z = \frac{1}{2\imath}\log 3 = \frac{1}{2\imath}(\ln 3 + 2k\pi\imath) \\ &\iff \quad z = k\pi - \frac{\ln 3}{2}\imath, \ k = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

4. (3 marks) Suppose $u(x, y)$ is harmonic $\forall\ (x, y)$ and $v(x, y)$ is a harmonic conjugate of $u(x, y)$. Show that $u^2(x, y) - v^2(x, y)$ is harmonic $\forall\ (x, y)$.

   Solution: $f(z) = u + \imath v$ is entire $\Longrightarrow f^2(z) = u^2 - v^2 + 2uv\imath$ is entire $\Longrightarrow u^2 - v^2$ is harmonic $\forall (x, y)$.

5. (6 marks) Describe the image of the rectangle $\{z \mid 0 \le x \le 1, 0 \le y \le \pi\}$ under the mapping $f(z) = e^z$. Hint: plot the images of the curves $x =$constant, y=constant.

   Solution: $f(z) = e^z = e^x e^{\imath y} = u + \imath v$, where $u = e^x \cos y$ and $v = e^x \sin y$. Therefore the image is the half annulus $\{u + \imath v \mid 1 \le \sqrt{u^2 + v^2} \le e, v \ge 0\}$.