SOLUTIONS TO MIDTERM #2, MATH 300

1. (12 marks) Answer true or false to the following statements. Give valid reasons for all your answers.

   (a) If $f(z)$ is analytic on a simple closed smooth curve C then $\oint_C f(z)dz = 0$.

   (b) The function $f(z) = ze^{1/z}$ has a pole at $z = 0$.

   (c) The power series $\sum_{n=0}^{\infty}(-1)^n \frac{z^n}{(2n)!}$ converges to the function $\cos\sqrt{z}$ for all z.

   (d) If the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $z = 2 + \imath$ then it converges for $z = \imath$.

   Solution:

   (a) False. For example $\int_C \frac{1}{z}dz = 2\pi\imath$ where C is the positively oriented unit circle. The function $f(z) = 1/z$ is analytic in the punctured plane $\{z \mid z \neq 0\}$ but not at the origin. The Cauchy Integral Theorem states that $\oint f(z)dz = 0$ if $f(z)$ is analytic on the simple closed contour C and **analytic inside** C.

   (b) False. It has an essential singularity at $z = 0$ since the Laurent series about $z = 0$ is $ze^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n+1}$. There are infinitely many negative powers of z.

   Remark: Suppose $f(z)$ has an isolated singularity at $z = z_0$, that is, $f(z)$ is analytic in some open annulus $\{z \mid 0 < |z - z_0| < R\}$. Then the singularity at z_0 is:

   - removable if $f(z)$ can be defined at $z = z_0$ so that it becomes analytic there. A good example of this is the function $f(z) = \frac{\sin z}{z}$. The formula for the function doesn't make sense at $z = 0$, but if we define $f(0) = 1$ then it becomes analytic at $z = 0$.
   - a pole of order n if the Laurent series expansion at z_0 has the form

     $$f(z) = a_{-n}(z - z_0)^{-n} + a_{-n+1}(z - z_0)^{-n+1} + \cdots + + a_0 + \cdots, \text{ where } a_{-n} \neq 0, n > 0.$$

     In other words $f(z)$ has a pole at $z = z_0$ if the Laurent series expansion has negative powers of $z - z_0$, but only finitely many. The order of the pole is n if there are no negative powers $(z - z_0)^k$ for $k < -n$.
   - essential if there are infinitely many negative powers of $z - z_0$ in the Laurent series expansion of $f(z)$ at $z = z_0$. Typical examples are $e^{1/z}$ and $\sin(1/z)$.

(c) True. This follows from replacing z by $\sqrt{z} = z^{1/2}$ in the Maclaurin expansion of the cosine function: $\cos z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$. Remark: The function $\sqrt{z}$ is multivalued on the complex plane $\mathbb{C}$, but $\cos\sqrt{z}$ is single valued because the cosine function is even.

EXERCISE: Determine the Maclaurin series expansions of the functions $\sqrt{z}\sin\sqrt{z}$ and $\frac{\sin\sqrt{z}}{\sqrt{z}}$.

(d) True since $|2+\imath| > |\imath|$. Lemma 2 on page 253 of the text states that if a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for some z with $|z| = R$ then it converges for all z with $|z| < R$.

2. (12 marks) The following questions require little or no computation.

   (a) Suppose $f(z)$ and $g(z)$ are analytic for $|z| \le 1$ and $f(z) + g(z) = 0$ for all z such that $|z| = 1$. Show that $f(z) + g(z) = 0$ for all z such that $|z| \le 1$.

   (b) Find the Laurent series for $f(z) = \frac{1}{z^2(z-1)}$ valid for $|z| > 1$.

   (c) Find the radius of convergence R of the power series $\sum_{j=0}^{\infty} \frac{z^{2j}}{3^j}$.

Solution:

(a) Let C be the unit circle positively oriented. We need only show that $f(z)+g(z) = 0$ for all z inside C. Since $f(z)+g(z)$ is analytic on C and inside C we can apply a Cauchy Integral Formula:

$$f(z) + g(z) = \frac{1}{2\pi\imath}\int_C \frac{f(\zeta) + g(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi\imath}\int_C \frac{0}{\zeta - z} d\zeta = 0$$

(b) $\frac{1}{z^2(z-1)} = \frac{1}{z^3(1-1/z)} = \sum_{n=0}^{\infty} \frac{1}{z^{n+3}}$ by the geometric series: $\frac{1}{1-1/z} = \sum_{n=0}^{\infty} \frac{1}{z^n}$, which converges since $|z| > 1$.

(c) The radius of convergence is $R = \sqrt{3}$ since

$$\lim_{j\to\infty} \left| \frac{z^{2j+2}}{3^{j+1}} / \frac{z^{2j}}{3^j} \right| = \frac{|z^2|}{3} < 1 \iff |z| < \sqrt{3}$$

3. (12 marks) Compute $\int_C \frac{\sin \pi z}{z^2(z-2)} dz$, where C is the circle $|z| = 1$ with the positive orientation.

First we compute the partial fraction decomposition of $f(z) = \dfrac{1}{z^2(z-2)}$:

$$\begin{aligned} f(z) &= \frac{1}{z^2(z-2)} = \frac{A_1}{z} + \frac{A_2}{z^2} + \frac{A_3}{z-2} \\ A_1 &= \frac{d}{dz}\left(z^2 f(z)\right)\Big|_{z=0} = \frac{d}{dz}(z-2)^{-1}\Big|_{z=0} = -\frac{1}{4} \\ A_2 &= z^2 f(z)|_{z=0} = \frac{1}{z-2}\Big|_{z=0} = -\frac{1}{2} \end{aligned}$$

The value of A_3 is immaterial since $\int_C \frac{\sin \pi z}{z-2} dz = 0$ by the Cauchy Integral Theorem. Thus

$$\begin{aligned} \int_C \frac{\sin \pi z}{z^2(z-2)} dz &= -\frac{1}{4}\int_C \frac{\sin \pi z}{z} dz - \frac{1}{2}\int_C \frac{\sin \pi z}{z^2} dz \\ &= -\frac{1}{4} \times 2\pi\imath \times \sin \pi z|_{z=0} - \frac{1}{2} \times 2\pi\imath \times \frac{d}{dz}(\sin \pi z)|_{z=0} \\ &= -\pi^2 \imath \end{aligned}$$

4. (12 marks) Suppose $P(z) = (z-r_1)^{s_1}(z-r_2)^{s_2}$ is a polynomial with distinct roots ($r_1 \neq r_2$). Show that $\oint_{C_R} \frac{zP'(z)}{P(z)} dz = 2\pi\imath(r_1 s_1 + r_2 s_2)$ for all R sufficiently large, where C_R is the positively oriented circle $|z| = R$.

Solution: $\dfrac{zP'(z)}{P(z)} = \dfrac{s_1 z}{z-r_1} + \dfrac{s_2 z}{z-r_2}$ and therefore

$$\begin{aligned} \oint_{C_R} \frac{zP'(z)}{P(z)} dz &= \oint_{C_R} \left(\frac{s_1 z}{z-r_1} + \frac{s_2 z}{z-r_2}\right) dz \\ &= 2\pi\imath\,(s_1 r_1 + s_2 r_2) \end{aligned}$$

so long as R is large enough that the roots r_1, r_2 are inside the circle $|z| = R$.