SOLUTIONS TO HOMEWORK ASSIGNMENT #1

1. Express the following complex numbers in the form $a + b\imath$.

(a) $(1+\imath)^8$
(b) $(\sqrt{3}+\imath)^6$
(c) $\dfrac{1+\imath}{1-\imath} - \dfrac{1}{1+\imath}$.

Solution:
(a) $(1+\imath)^8 = (2\imath)^4 = 16$.
(b) $(\sqrt{3}+\imath)^6 = \left(2e^{\pi\imath/6}\right)^6 = -64$.
(c) $\dfrac{1+\imath}{1-\imath} - \dfrac{1}{1+\imath} = \dfrac{(1+\imath)^2-(1-\imath)}{2} = \dfrac{-1+3\imath}{2} = -\dfrac{1}{2} + \dfrac{3}{2}\imath$.

2. Show that $\mathbb{C}$ can not be ordered (see #30 on page 7).
Solution: Suppose it is possible to order $\mathbb{C}$. Then there would be a set P of positive elements. It would have the following properties

1. Given any $z \in \mathbb{C}$ one, and only one of the following, would hold: $z = 0, z \in P, -z \in P$.
2. $z_1, z_2 \in P \Longrightarrow z_1 + z_2, z_1 z_2 \in P$. That is P is closed under sums and products.

The number $\imath$ would be either positive or negative (that is, either $\imath \in P$ or $-\imath \in P$). Assume $\imath \in P$. Then $-\imath = \imath^3 \in P$, a contradiction. If we assume $-\imath \in P$ we also get a contradiction since $(-\imath)^3 = \imath$. Therefore there can be no such set P, and thus $\mathbb{C}$ can not be ordered.

3. Find all solutions of the following equations. Express your answers in the form $a + b\imath$.
(a) $z^2 + \imath z + 2 = 0$.
(b) $z^4 - 16 = 0$.
(c) $z^3 + 1 = 0$.

Solution:
(a) Applying the quadratic formula we get $z = \dfrac{-\imath \pm \sqrt{-1-8}}{2} = -2\imath, \imath$.
(b) $z^4 = 16 \Longleftrightarrow z = \pm 2, \pm 2\imath$.
(c) $z^3 + 1 = 0 \Longleftrightarrow z = -1, e^{\pm\pi\imath/3} = -1, \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}\imath$.

4. Show that $\dfrac{\pi}{4} = 4\arctan(1/5) - \arctan(1/239)$. Hint: $(1+\imath)(5-\imath)^4$. This is question #11 on page 23.
Solution: By computation

$$\begin{aligned}
(1+\imath)(5-\imath)^4 &= 956 - 4\imath = 4(239-\imath) = 676\sqrt{2}e^{\imath\theta},\ \theta = \arctan\frac{-1}{239} = -\arctan\frac{1}{239} \\
(1+\imath)(5-\imath)^4 &= \sqrt{2}e^{\pi\imath/4}(\sqrt{26})^4 e^{4\imath\phi} = 676\sqrt{2}e^{\imath(\pi/4+4\phi)},\ \phi = \arctan\frac{-1}{5} = -\arctan\frac{1}{5}.
\end{aligned}$$

Comparing arguments gives $\dfrac{\pi}{4} + 4\phi = \theta$, that is $\dfrac{\pi}{4} = 4\arctan\dfrac{1}{5} - \arctan\dfrac{1}{239}$.