SOLUTIONS TO HOMEWORK ASSIGNMENT #2

1. Find all n^{th} roots of the following complex numbers z. Express your answers in the form $a+b\imath$.

(a) $n=3,\ z=-8\imath$.
(b) $n=4,\ z=-2+\sqrt{12}\imath$.

Solution:
(a) $-8\imath = 8e^{\imath(3\pi/2+2k\pi)} \Longrightarrow$ the 3^{rd} roots are: $2e^{\imath(3\pi/2+2k\pi)/3},\ k=0,1,2.$

$$\begin{aligned}
2e^{\imath(\pi/2)} &= 2\imath \text{ (for } k=0)\\
2e^{\imath(3\pi/2+2\pi)/3} &= 2\imath(-1/2+\imath\sqrt{3}/2) = -\sqrt{3}-\imath \text{ (for } k=1)\\
2e^{\imath(3\pi/2+4\pi)/3} &= 2\imath(-1/2-\imath\sqrt{3}/2) = \sqrt{3}-\imath \text{ (for } k=2)
\end{aligned}$$

(b) $-2+\sqrt{12}\imath = 4(-1/2+\imath\sqrt{3}/2) = 4e^{\imath\frac{2\pi}{3}} = 4e^{\imath(\frac{2\pi}{3}+2k\pi)}$. Thus the 4^{th} roots are

$$\begin{aligned}
\sqrt{2}e^{\imath(\frac{2\pi}{3}+2k\pi)/4} &= \sqrt{2}e^{\imath\frac{\pi}{6}}e^{\imath k\pi/2} = \sqrt{2}\left(\sqrt{3}/2+\imath/2\right)\imath^k\\
&= \pm\frac{1}{\sqrt{2}}(\sqrt{3}+\imath), \pm\frac{\imath}{\sqrt{2}}(\sqrt{3}+\imath)
\end{aligned}$$

2. Find all complex numbers z satisfying the following equations.
(a) $e^z=\imath$.
(b) $\cos z = 2$.
Solution:
(a) $e^z=\imath \Longleftrightarrow e^x\cos y = 0$ and $e^x \sin y = 1 \Longleftrightarrow y=\pi/2+2n\pi,\ x=0 \Longleftrightarrow z=\imath(\pi/2+2n\pi)$, where n is any integer.
Question: why are the numbers $z=\imath(\pi/2+(2n+1)\pi)$ not included?
(b)

$$\begin{aligned}
\cos z &= \frac{e^{\imath z}+e^{-\imath z}}{2} = 2 \Longleftrightarrow e^{2\imath z}-4e^{\imath z}+1=0 \Longleftrightarrow e^{\imath z} = 2\pm\sqrt{3}\\
&\Longleftrightarrow e^{-y}\cos x = 2\pm\sqrt{3} \text{ and } e^{-y}\sin x = 0\\
&\Longleftrightarrow x=2n\pi \text{ and } -y=\ln(2\pm\sqrt{3})\\
&\Longleftrightarrow x=2n\pi \text{ and } y=-\ln(2\pm\sqrt{3}) = \ln(2\mp\sqrt{3})\\
&\Longleftrightarrow z=2n\pi+\ln(2\mp\sqrt{3})\imath,\ n=0,\pm1,\pm2,\ldots
\end{aligned}$$

Question: Why only even multiples of π?
3. Use De Moivre's formula to prove that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1\ \forall\ \theta$.
Solution:

$$\begin{aligned}
(\cos\theta+\imath\sin\theta)^4 &= \cos^4\theta + 4\cos^3\theta(\imath\sin\theta)^3 + 6\cos^2\theta\,(\imath\sin\theta)^2 + 4\cos\theta(\imath\sin\theta)^3 + (\imath\sin\theta)^4\\
&= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 + \imath(\text{someting real})\\
&\Longrightarrow \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1 \text{ by De Moivre's Theorem}
\end{aligned}$$

4. For any 2 complex numbers z_1, z_2 show that $|z_2| - |z_1| \leq |z_2 - z_1|$.
Solution: This follows from the triangle inequality:

$$|z_2| = |z_2 - z_1 + z_1| \leq |z_2 - z_1| + |z_1| \Longrightarrow |z_2| - |z_1| \leq |z_2 - z_1|$$

5. Solve for $z : z^3 = \dfrac{2\imath}{1+\imath}$.

Solution: $\dfrac{2\imath}{1+\imath} = 1+\imath = \sqrt{2}e^{\imath\pi/4+2k\pi} \Longrightarrow z = 2^{1/6}e^{\pi/12+2k\pi/3}$. Now $\cos(\pi/12) = \dfrac{1+\sqrt{3}}{2\sqrt{2}}$ and $\sin(\pi/12) = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$. To see this note that

$$\begin{aligned}\cos(\pi/12) + \imath\sin(\pi/12) &= e^{\imath\pi/12} = \frac{e^{\imath\pi/3}}{e^{\imath\pi/4}} = \frac{1/2+\imath\sqrt{3}/2}{\frac{1}{\sqrt{2}}(1+\imath)} \\ &= \frac{1}{\sqrt{2}}\frac{1+\imath\sqrt{3}}{1+\imath} \times \frac{1-\imath}{1-\imath} \\ &= \frac{1}{2\sqrt{2}}(1+\sqrt{3}+\imath(\sqrt{3}-1))\end{aligned}$$

After some computation we see that the solutions are

$$\frac{-1+\imath}{2^{1/3}}, \frac{(1-\sqrt{3}-(\sqrt{3}+1)\imath}{2^{4/3}}, \frac{(1+\sqrt{3}-(-\sqrt{3}+1)\imath}{2^{4/3}}$$

6. Sketch each of the following sets Ω. If Ω is open (resp. closed or connected) put the words open (resp. closed or connected) in the boxes; otherwise leave blank.
(a) $\Omega = \{z = x + \imath y \mid \; x \geq 1 \text{ or } x \leq 0\}$
(b) $\Omega = \mathbb{C} - \{z \mid 0 \leq x \leq 1, y = 0\}$
(c) $\Omega = \{z \mid 1 < |z| < 2\}$
(d) $\Omega = \{z \mid x^2 - xy + y^2 \leq 1\}$
(e) $\Omega = \{z \mid -1/2 \leq x \leq 1/2 \text{ and } |z| \geq 1\}$
(f) $\Omega = \{z = re^{\imath\theta} \mid r > 0 \text{ and } \pi/4 \leq \theta \leq \pi/2\}$
(g) $\Omega = \mathbb{C} - \{z \mid 0 \leq x, y = 0\}$
Solution:
(a) Closed.
(b) Open and connected.
(c) Open and connected.
(d) Closed and connected.
(e) Closed.
(f) Connected.
(g) Open and connected.