SOLUTIONS TOHOMEWORK ASSIGNMENT # 3

1. Find the harmonic function $u(x, y)$ on the region $\Omega = \{z \mid y > 0,\ 2 \le xy \le 4\}$ that satisfies the boundary conditions $u(x, y) = \begin{cases} \alpha & \text{if } xy = 2 \\ \beta & \text{if } xy = 4 \end{cases}$

   Solution: The solution is $u = Axy + B$, where the constants A, B are chosen to satisfy the boundary conditions. Thus $u(x, y) = \dfrac{\beta - \alpha}{2} xy + 2\alpha - \beta$.

2. Let $f(z) = u(x, y) + \imath v(x, y)$ be analytic on some domain Ω. Show that the Jacobian of the mapping $\Omega \to \mathbb{R}^2$, $(x, y) \to (u(x, y), v(x, y))$, satisfies $J(x, y) = |f'(z)|^2$.

   Solution: $f'(z) = \dfrac{\partial u}{\partial x} + \imath \dfrac{\partial v}{\partial x} \Longrightarrow |f'(z)| = \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2$. The Jacobian is

$$J(x, y) = det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

   Using the Cauchy-Riemann equations gives

$$J(x, y) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2.$$

3. Suppose $f(z)$ is an entire function satisfying $f(z)$ is real $\forall z$ and $f(0) = 2$. Show that $f(z) = 2\ \forall z \in \mathbb{C}$.

   Solution: $f(z) = u(x, y) + \imath v(x, y)$, where $v(x, y) = 0\ \forall (x, y)$ (since $f(z)$ is real $\forall z$). The Cauchy-Riemann equations then imply that $\dfrac{\partial u}{\partial x} = 0$, $\dfrac{\partial u}{\partial y} = 0$ and therefore $f(z)$ is a constant, which must be 2.

4. (a) Show that the function $u(x, y) = \sin x \cosh y$ is harmonic on $\mathbb{R}^2$.

   (b) Find the harmonic conjugate $v(x, y)$ of $u(x, y)$ that satisfies $v(0, 0) = 1$.

   Solution:

   (a) $\dfrac{\partial^2 u}{\partial x^2} = -\sin x \cosh y$ and $\dfrac{\partial^2 u}{\partial y^2} = \sin x \cosh y$. Therefore $\nabla^2 u = 0$, that is $u(x, y)$ is harmonic.

   (b) We have to find a function $v(x, y)$ such that $f(z) = u(x, y) + \imath v(x, y)$ is entire and $v(0, 0) = 1$. The Cauchy-Riemann equations give:

$$\begin{aligned} \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} = -\sin x \sinh y \Longrightarrow v(x, y) = \cos x \sinh y + C(y) \\ \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} = \cos x \cosh y = \cos x \cosh y + C'(y) \Longrightarrow C(y) = \text{a constant.} \end{aligned}$$

   Therefore $v(x, y) = \cos x \sinh y + 1$.

5. Let C be the circle which is the intersection of the plane $ax_1 + bx_2 + cx_3 = d$ with the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$ (assume a, b, c, d are such that there is a non-trivial intersection). Prove that the stereographic projection of C onto the complex plane $\mathbb{C}$ is either a straight line or a circle.

   Solution: The formulas for stereographic projection, relating coordinates (x_1, x_2, x_3) on the unit sphere and points $z = x + \imath y$ in the complex plane $\mathbb{C}$, are:

$$x = \frac{x_1}{1 - x_3},\ y = \frac{x_2}{1 - x_3},\ x_1 = \frac{2x}{x^2 + y^2 + 1},\ x_2 = \frac{2y}{x^2 + y^2 + 1},\ x_3 = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$$

   Substituting the formulas for x_1, x_2, x_3 into the equation $ax_1 + bx_2 + cx_3 = d$ gives $(c - d)x^2 + (c - d)y^2 + 2ax + 2by = c + d$. If $c \neq d$ this is a circle, and if $c = d$ this is a straight line.

6. Let $f(z) = 1/z$ be the reciprocal map. Prove that this map corresponds to rotation by π about the x_1-axis under stereographic projection.

   Solution: The mapping $z \to 1/z$ is just $x + \imath y \to X + \imath Y$ where

$$X = \frac{x}{x^2 + y^2},\ Y = -\frac{y}{x^2 + y^2}.$$

   Therefore the mapping $z \to 1/z$ on the unit sphere becomes

$$(x_1, x_2, x_3) \to x + \imath y \to X + \imath Y \to \left(\frac{2X}{X^2 + Y^2 + 1}, \frac{2Y}{X^2 + Y^2 + 1}, \frac{X^2 + Y^2 - 1}{X^2 + Y^2 + 1}\right)$$

   A little algebra shows that

$$\frac{2X}{X^2 + Y^2 + 1} = x_1, \frac{2Y}{X^2 + Y^2 + 1} = -x_2, \frac{X^2 + Y^2 - 1}{X^2 + Y^2 + 1} = -x_3.$$

   Thus the mapping on the unit sphere is $(x_1, x_2, x_3) \to (x_1, -x_2, -x_3)$. This is just rotation by π about the x_1-axis.

7. For each of the following subsets Ω of the unit sphere describe the stereographic projection. Let the positive x_1-axis correspond to longitude 0^o and the positive x_2-axis correspond to longitude 90^o.

   (a) Ω is everything "north of 60", including the 60^{th} parallel.

   (b) Ω is everything "south of 60", not including the 60^{th} parallel.

   (c) Ω is everything between the tropic of cancer ($23^o 27'$ north) and the tropic of capricorn ($23^o 27'$ south) , but not including these parallels.

   (d) $\Omega = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$.

(e) Ω is the closed portion of the southern hemisphere from longitude 30^o to longitude 60^o.

Solution:

(a) On the 60^{th} parallel north we have $x_3 = \dfrac{\sqrt{3}}{2}$, and therefore the image is given by

$$|z| \geq \sqrt{\frac{1+x_3}{1-x_3}} = \sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}}.$$

(b) On the 60^{th} parallel south we have $x_3 = -\sqrt{3}/2$. Therefore the image is

$$|z| < \sqrt{\frac{1+x_3}{1-x_3}} = \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}.$$

(c) Let θ be the angle in radians corresponding to $23^o27'$. For a point on the tropic of cancer we have $x_3 = \sin\theta$ and so $|z| = \sqrt{\dfrac{1+x_3}{1-x_3}} = \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}$. For a point on the tropic of capricorn we have $x_3 = -\sin\theta$ and $|z| = \sqrt{\dfrac{1+x_3}{1-x_3}} = \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$. Therefore the image is $\left\{ z \;\middle|\; 0.656 \approx \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} < |z| < \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} \approx 1.524 \right\}$

(d) The image is $\{z = x + \imath y \mid x \geq 0, y \geq 0, |z| \geq 1\}$.

(e) The image is $\{z = re^{\imath\theta} \mid r \leq 1, \pi/6 \leq \theta \leq \pi/3\}$.