SOLUTIONS TO HOMEWORK ASSIGNMENT # 4

1. Suppose $f(z)$ is defined for $|z - z_0| < \epsilon$, where ϵ is some positive number. If $f'(z_0)\ \exists$ show that $f(z)$ is continuous at $z = z_0$.

   Solution: $f'(z_0)\ \exists$ means that $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}\ \exists$ and equals $f'(z_0)$. Write this in the form $\frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) + R(z)$, where $\lim_{z \to z_0} R(z) = 0$. Then

$$\begin{aligned} \lim_{z \to z_0} f(z) &= \lim_{z \to z_0} (f(z_0) + (z - z_0)f'(z_0) + (z - z_0)R(z)) \\ &= f(z_0) + \lim_{z \to z_0} (z - z_0)f'(z_0) + \lim_{z \to z_0} (z - z_0)R(z) \\ &= f(z_0) + 0 + 0 = f(z_0) \end{aligned}$$

   This proves that $f(z)$ is continuous at $z = z_0$.

2. Determine the domain Σ of analyticity of the function $f(z) = \text{Log}(4 + \imath - z)$. Note: $\text{Log} z$ is analytic on the domain $\Omega = \{z \mid -\pi < Arg z < \pi.\}$

   Solution: The domain of analyticity of any function $f(z) = \text{Log}(g(z))$, where $g(z)$ is analytic, will be the set of points z such that $g(z)$ is defined and $g(z)$ does not belong to the set $\{z = x + \imath y \mid -\infty < x \leq 0, y = 0\}$. Thus $f(z) = \text{Log}(4 + \imath - z)$ will be analytic on the domain

$$\Sigma = \mathbb{C} - \{z = x + \imath y \mid x \geq 4, y = 1\}$$

3. Show that the linear fractional transformation $L(z) = \frac{z - \imath}{z + \imath}$ maps the upper half plane $\mathbb{U} = \{z = x + \imath y \mid y > 0\}$ onto the interior of the unit circle. Hint: Show that the real axis is mapped to the unit circle and $z = \imath$ is mapped to 0.

   Solution: If x is real then $|L(x)| = \frac{|x - \imath|}{|x + \imath|} = 1$ since x is equidistant from $\imath$ and $-\imath$. Therefore L maps the real axis to the unit circle. The transformation L is $1 - 1$ and onto as a map from the extended complex plane $\mathbb{C} \cup \{\infty\}$ to itself. This follows from the fact that

$$w = L(z) = \frac{z - \imath}{z + \imath} \Longleftrightarrow z = \imath \frac{1 + w}{1 - w}$$

   The real axis (with a point at ∞) separates the extended complex plane $\mathbb{C} \cup \{\infty\}$ into disjoint pieces, namely the upper half plane $\mathbb{U}$ and the lower half plane. Likewise the unit circle separates the extended complex plane $\mathbb{C} \cup \{\infty\}$ into the interior of the unit circle and its exterior. From the properties of L mentioned above it follows that the $L(\mathbb{U})$ must be either the interior of the unit circle or the exterior. It is the interior since $L(\imath) = 0$.

4. Derive the identity $\sec^{-1} z = -\imath \log\left(\frac{1}{z} + \sqrt{\frac{1}{z^2} - 1}\right)$.

Solution:

$$\begin{aligned} w &= \sec^{-1} z \iff z = \sec w \iff z = \frac{2}{e^{\imath w} + e^{-\imath w}} \iff z e^{2\imath w} - 2e^{\imath w} + z = 0 \\ \iff e^{\imath w} &= \frac{2 + \sqrt{4 - 4z^2}}{2z} = \frac{1}{z} + \sqrt{\frac{1}{z^2} - 1} \iff w = -\imath \log\left(\frac{1}{z} + \sqrt{\frac{1}{z^2} - 1}\right) \end{aligned}$$

5. Show that $\int_C e^z dz = 0$, where C is the square with vertices $0, 1, 1+\imath, \imath$, traversed once in that order.

Solution: Since we do not yet have the Cauchy Integral Theorem we compute. The curve C can be broken up into the four sides of the square, each of which has a simple parametrization.

- $C_1 : z = x,\ x$ goes from 0 to 1.
- $C_2 : z = 1 + \imath y,\ y$ goes from 0 to 1.
- $C_3 : z = x + \imath,\ x$ goes from 1 to 0.
- $C_4 : z = \imath y,\ y$ goes from 1 to 0.

Therefore

$$\begin{aligned} \int_C e^z dz &= \int_{C_1} e^z dz + \int_{C_2} e^z dz + \int_{C_3} e^z dz + \int_{C_4} e^z dz \\ &= \int_{x=0}^{x=1} e^x dx + \int_{y=0}^{y=1} e^{1+\imath y} \imath dy + \int_{x=1}^{x=0} e^{x+\imath} dx + \int_{y=1}^{y=0} e^{\imath y} \imath dy \\ &= e^x \Big|_{x=0}^{x=1} + e^{1+\imath y} \Big|_{y=0}^{y=1} + e^{x+\imath} \Big|_{x=1}^{x=0} + e^{\imath y} \Big|_{y=1}^{y=0} \\ &= (e-1) + (e^{1+\imath} - e) + (e^{\imath} - e^{1+\imath}) + (1 - e^{\imath}) = 0 \end{aligned}$$

6. Suppose $f(z)$ is analytic on the domain $D = \{z \mid |z| < 1\}$ and satisfies $|f'(z)| \leq M$ in D. Prove that $|f(z_1) - f(z_2)| \leq M|z_1 - z_2|$ for all z_1, z_2 in D. See problem #12 on page 180.

Solution: Let z_1, z_2 be 2 points in D, and let C be the straight line from z_2 to z_1. This lies entirely within D. Therefore

$$|f(z_1) - f(z_2)| = \left|\int_C f'(z) dz\right| \leq M|z_1 - z_2|$$

by the ML inequality.

7. Use Theorem 5 on page 170 to establish the following estimates:

(a) $\left|\int_C \frac{dz}{z^2+\imath}\right| \le \frac{3\pi}{4}$, where C is the circle $|z|=3$ traversed once.

(b) $\left|\int_C \mathrm{Log}(z)\ dz\right| \le \frac{\pi^2}{4}$, where C is the first quadrant portion of the circle $|z|=1$.

Solution:

(a) As z traverses the circle $|z|=3$ once in the positive direction, $w=z^2$ will traverse the circle $|w|=9$ twice in the positive direction. The point on this circle that is closest to $-\imath$ is $w=-9\imath$, and therefore

$$\mathrm{Max}_{|z|=3}\frac{1}{|z^2+\imath|} = \frac{1}{\mathrm{Min}_{|z|=3}|z^2+\imath|} = \frac{1}{8} \Longrightarrow \left|\int_C \frac{dz}{z^2+\imath}\right| \le \frac{6\pi}{8} = \frac{3\pi}{4}$$

(b) On the contour C we have $z=e^{\imath\theta}$, where $0\le\theta\le\pi/2$. Therefore on C

$$M = \mathrm{Max}|\mathrm{Log}(z)| = \mathrm{Max}|\mathrm{Log}(e^{\imath\theta})| = \mathrm{Max}|\imath\theta| = \pi/2.$$

Since the length of C is $L=\pi/2$ we have

$$\left|\int_C \mathrm{Log}(z)\ dz\right| \le ML = \frac{\pi^2}{4}$$

8. Use the Cauchy Integral Theorem (see page 194) to prove that

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta+\theta)d\theta = 0,\quad \int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta+\theta)d\theta = 0$$

Hint: $\int_C e^z dz = 0$, where C is the unit circle parametrized by $z=e^{\imath\theta}, 0\le\theta\le 2\pi$.

Solution: If $z=e^{\imath\theta}=\cos\theta+\imath\sin\theta$ then

$$e^z = e^{\cos\theta+\imath\sin\theta} = e^{\cos\theta}(\cos(\sin\theta)+\imath\sin(\sin\theta)) \text{ and } dz = (-\sin\theta+\imath\cos\theta)d\theta$$

Therefore

$$\begin{aligned}
\int_C e^z dz &= \int_0^{2\pi} e^{\cos\theta}\left(\cos(\sin\theta)+\imath\sin(\sin\theta)\right)(-\sin\theta+\imath\cos\theta)d\theta \\
&= -\int_0^{2\pi} e^{\cos\theta}\left(\cos(\sin\theta)\sin\theta+\sin(\sin\theta)\cos\theta\right)d\theta \\
&\quad +\imath\int_0^{2\pi} e^{\cos\theta}\left(\cos(\sin\theta)\cos\theta-\sin(\sin\theta)\sin\theta\right)d\theta \\
&= -\int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta+\theta)d\theta + \imath\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta+\theta)d\theta
\end{aligned}$$

But $\int_C e^z dz = 0$ and therefore

$$\int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta+\theta)d\theta = 0 \text{ and } \int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta+\theta)d\theta = 0$$

9. Evaluate $\int_C \frac{dz}{(z^2+1)^2}$, where C is the circle of radius 2 about the origin, oriented in the counterclockwise direction. Hint: $\frac{1}{(z^2+1)^2} = \frac{A_1}{z+\imath} + \frac{A_2}{(z+\imath)^2} + \frac{A_3}{z-\imath} + \frac{A_4}{(z-\imath)^2}$.

   Solution: We only need to compute A_1 and A_3 since $\int_C \frac{dz}{(z\pm\imath)^2} = 0$. To compute A_1 multiply by $(z+\imath)^2$, differentiate, and then set $z=-\imath$:

$$A_1 = \frac{d}{dz}\frac{1}{(z-\imath)^2}\Big|_{z=-\imath} = -2(z-\imath)^{-3}\Big|_{z=-\imath} = \frac{-2}{(-2\imath)^3} = \frac{\imath}{4}$$

   Similarly

$$A_3 = \frac{d}{dz}\frac{1}{(z+\imath)^2}\Big|_{z=\imath} = -2(z+\imath)^{-3}\Big|_{z=\imath} = \frac{-2}{(2\imath)^3} = -\frac{\imath}{4}$$

   Therefore $\int_C \frac{dz}{(z^2+1)^2} = 0$.

10. Find a branch $f(z)$ of $\log(2z-1)$ that is analytic on $\mathbb{C} - \{x+\imath y \mid x \le 1/2, y=0\}$ and satisfies $f(1) = 2\pi\imath$.

   Solution: Let $\log z$ be that branch of the logarithm which satisfies $\log 1 = 2\pi\imath$, that is $\log z = \ln|z| + \imath arg(z)$, where $arg(z)$ is chosen to satisfy $\pi < arg(z) < 3\pi$. Then $f(z) = \log(2z-1)$.