SOLUTIONS TO HOMEWORK ASSIGNMENT # 5

1. Use Cauchy's Integral Theorem to evaluate the following integrals.

   (a) $\displaystyle\int_C \frac{z}{z^3+1}dz$, where C is the positively oriented circle $|z-2|=2$.

   (b) $\displaystyle\int_C \frac{z}{z^2+z-2}dz$, where C is the circle $|z|=3$, oriented in the clockwise direction.

   (c) $\displaystyle\int_C \frac{\cos\pi(z-1)}{z(z^2+16)(z^2-16)(z^2+25)^2(z^2-25)^2}dz$, where C is the circle $|z|=\pi$ oriented positively.

   Solution:

   (a) Suppose the partial fraction decomposition of $\dfrac{z}{z^3+1}$ is

$$\frac{z}{z^3+1}=\frac{A}{z+1}+\frac{B}{z-e^{\pi\imath/3}}+\frac{C}{z-e^{-\pi\imath/3}}$$

Notice that $\displaystyle\int_C \frac{A}{z+1}dz=0$ since $z=-1$ is exterior to C. The other 2 roots are interior to C. Next we compute B and C :

$$\begin{aligned} B &= \frac{z(z-e^{\pi\imath/3})}{z^3+1}\Big|_{z=e^{\pi\imath/3}}=\frac{e^{\pi\imath/3}}{3e^{2\pi\imath/3}}=\frac{e^{-\pi\imath/3}}{3} \\ C &= \frac{z(z-e^{-\pi\imath/3})}{z^3+1}\Big|_{z=e^{-\pi\imath/3}}=\frac{e^{-\pi\imath/3}}{3e^{-2\pi\imath/3}}=\frac{e^{\pi\imath/3}}{3} \end{aligned}$$

Therefore

$$\int_C \frac{z}{z^3+1}dz=2\pi\imath(B+C)=\frac{2\pi\imath}{3}$$

(b) The partial fraction decomposition is $\dfrac{z}{z^2+z-2}=\dfrac{2/3}{z+2}+\dfrac{1/3}{z-1}$. Since both roots are interior to C we have $\displaystyle\int_C \frac{z}{z^2+z-2}dz=2\pi\imath\left(\frac{2}{3}+\frac{1}{3}\right)=2\pi\imath.$

(c) The partial fraction decomposition of $\dfrac{1}{z(z^2+16)(z^2-16)(z^2+25)^2(z^2-25)^2}$ will have the form $\dfrac{1}{z(z^2+16)(z^2-16)(z^2+25)^2(z^2-25)^2}=\dfrac{A}{z}+$ other terms, where the other terms contribute 0 to the integral. **Why is this?**

Now we compute $A=\dfrac{z}{z(z^2+16)(z^2-16)(z^2+25)^2(z^2-25)^2}\Big|_{z=0}=-10^{-8}$. Therefore

$$\int_C \frac{\cos\pi(z-1)}{z(z^2+16)(z^2-16)(z^2+25)^2(z^2-25)^2}dz=A\int_C \frac{\cos\pi(z-1)}{z}dz=2\pi\imath\times 10^{-8}.$$

Question Where did the minus sign go?

2. Let C be a simple closed contour and let D be its interior. Suppose $f(z)$ and $g(z)$ are analytic in D and on C.

   (a) Show that $f(z) = g(z)\ \forall z \in D$ if $f(z) = g(z)\ \forall z \in C$.

   (b) Show that $\int_C \frac{f'(\zeta)}{\zeta - z} d\zeta = \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta\ \forall z \in D$.

   Solution:

   (a) By the Cauchy Integral Theorem:

$$\text{for any } z \in D,\ f(z) = \frac{1}{2\pi\imath}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi\imath}\int_C \frac{g(\zeta)}{\zeta - z} d\zeta = g(z)$$

   (b) Apply the Cauchy Integral Theorem twice:

$$\begin{aligned} f(z) &= \frac{1}{2\pi\imath}\int_C \frac{f(\zeta)}{\zeta - z} dz \Longrightarrow f'(z) = \frac{1}{2\pi\imath}\int_C \frac{f(\zeta)}{(\zeta - z)^2} dz \\ f'(z) &= \frac{1}{2\pi\imath}\int_C \frac{f'(\zeta)}{\zeta - z} dz \end{aligned}$$

3. Suppose $f(z)$ is an entire function such that $Im(f(z))$ is bounded. Show that $f(z)$ is a constant.

   Solution: Let $f(z) = u(x, y) + \imath v(x, y)$. Then apply Liouville's theorem to the entire function $g(z) = e^{\imath f(z)} = e^{-v+\imath u}$:

$$|g(z)| = \left|= e^{-v+\imath u}\right| = e^{-v} = e^{-Im(f(z))} \text{ is bounded } \Longrightarrow f(z) \text{ is a constant.}$$

4. Suppose $P(z) = \prod_{j=1}^{j=k} (z - r_j)^{s_j}$ is a polynomial in factored form and C is a positively oriented simple closed contour such that $r_1, \dots, r_n$ are in the interior of C and the rest of the roots are exterior to C. Show that $\int_C \frac{P'(z)}{P(z)} dz = 2\pi\imath(s_1 + \cdots + s_n)$.

   Solution: The partial fraction decomposition of $\frac{P'(z)}{P(z)}$ is $\frac{P'(z)}{P(z)} = \sum_{j=1}^{j=k} \frac{s_j}{z - r_j}$, and therefore

$$\int_C \frac{P'(z)}{P(z)} dz = \sum_{j=1}^{j=k} \int_C \frac{s_j}{z - r_j} dz = 2\pi\imath \sum_{j=1}^{j=n} s_j \text{ since the other roots are exterior to } C$$

5. Evaluate $\int_C \frac{e^{\imath z}}{(z^2 + 1)^3} dz$, where C is the circle $x^2 + (y - 1)^2 = 1$, oriented positively.

Solution: The integrand $\dfrac{e^{\imath z}}{(z^2+1)^3}$ fails to be analytic only at $z = \pm\imath$. First we determine the partial fraction decomposition of $\dfrac{1}{(z^2+1)^3}$.

$$
\begin{aligned}
\frac{1}{(z^2+1)^3} &= \frac{A_1}{z-\imath} + \frac{A_2}{(z-\imath)^2} + \frac{A_3}{(z-\imath)^3} + \frac{B_1}{z+\imath} + \frac{B_2}{(z+\imath)^2} + \frac{B_3}{(z+\imath)^3} \\
A_1 &= \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{(z-\imath)^3}{(z^2+1)^3}\right)\Big|_{z=\imath} = \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{(z+\imath)^3}\right)\Big|_{z=\imath} = -\frac{3}{16}\imath \\
A_2 &= \frac{d}{dz}\left(\frac{(z-\imath)^3}{(z^2+1)^3}\right)\Big|_{z=\imath} = \frac{d}{dz}\left(\frac{1}{(z+\imath)^3}\right)\Big|_{z=\imath} = -\frac{3}{16} \\
A_3 &= \frac{(z-\imath)^3}{(z^2+1)^3}\Big|_{z=\imath} = \frac{1}{(z+\imath)^3}\Big|_{z=\imath} = \frac{\imath}{8}
\end{aligned}
$$

We don't need to compute the other numerators since $\dfrac{e^{\imath z}}{(z+\imath)^n}$ is analytic on and within C for any integer, and therefore $\displaystyle\int_C \frac{e^{\imath z}}{(z+\imath)^n}dz = 0$. Finally

$$
\begin{aligned}
\int_C \frac{e^{\imath z}}{(z^2+1)^3}dz &= A_1\int_C \frac{e^{\imath z}}{z-\imath}dz + A_2\int_C \frac{e^{\imath z}}{(z-\imath)^2}dz + A_3\int_C \frac{e^{\imath z}}{(z-\imath)^3}dz \\
&= 2\pi\imath A_1 e^{\imath z}\,|_{z=\imath} + 2\pi\imath A_2\frac{d}{dz}(e^{\imath z})\,|_{z=\imath} + \pi\imath A_3\frac{d^2}{dz^2}(e^{\imath z})\,|_{z=\imath} \\
&= \frac{3\pi}{8e} + \frac{3\pi}{8e} + \frac{\pi}{8e} = \frac{7\pi}{8e}
\end{aligned}
$$