SOLUTIONS TO HOMEWORK ASSIGNMENT # 6

1. Evaluate the following series:

   (a) $\sum_{n=3}^{\infty}(-1)^n \frac{1}{3^n}$.

   (b) $\sum_{n=2}^{\infty} \frac{1}{n(n+1)}$.

   (c) $\sum_{n=1}^{\infty}(-1)^n \frac{z^{2n}}{2^{n+1/2}}$.

   Solution:

   (a) $\sum_{n=3}^{\infty}(-1)^n \frac{1}{3^n} = \sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)^n - (1 - 1/3 + 1/9) = -\frac{1}{36}$

   (b)

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n(n+1)} &= \lim_{N\to\infty} \sum_{n=2}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \lim_{N\to\infty} \left(\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right)\right) \\
&= \lim_{N\to\infty} \left(\frac{1}{2} - \frac{1}{N+1}\right) = \frac{1}{2}
\end{aligned}$$

   (c)

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^n \frac{z^{2n}}{2^{n+1/2}} &= \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \left(\frac{-z^2}{2}\right)^n - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\frac{1}{1+z^2/2}\right) - \frac{1}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}\left(\frac{2}{2+z^2} - 1\right) = -\frac{z^2}{\sqrt{2}(2+z^2)}
\end{aligned}$$

   Note: This is valid only for $|z| < \sqrt{2}$ since we have used the geometric series.

2. Determine the radius of convergence of following series.

   (a) $\sum_{n=5}^{\infty} n z^{2n}$.

   (b) $\sum_{n=0}^{\infty} 4^n z^{3n}$.

   (c) $\sum_{n=1}^{\infty} \sqrt{n} z^n$

Solution: Let R denote the radius of convergence.

(a) $\lim_{n\to\infty} \left| \frac{(n+1)z^{2n+2}}{nz^{2n}} \right| = |z^2| \Longrightarrow R = 1.$

(b) $\lim_{n\to\infty} \left| \frac{4^{n+1}z^{3n+3}}{4^n z^{3n}} \right| = 4|z|^3 \Longrightarrow R = \frac{1}{4^{1/3}}.$

(c) $\lim_{n\to\infty} \left| \frac{\sqrt{n+1}z^{n+1}}{\sqrt{n}z^n} \right| = |z| \Longrightarrow R = 1.$

3. Find closed form expressions for the following series.

(a) $\sum_{n=1}^{\infty} nz^n$.

(b) $\sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n+1)!}$. Hint: the sine function.

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} z^{2n}$.

Solution:

(a)

$$\begin{aligned}
\sum_{n=1}^{\infty} nz^n &= z\sum_{n=1}^{\infty} \frac{d}{dz}(z^n) = z\frac{d}{dz}\left(\sum_{n=1}^{\infty} z^n\right) \quad \text{(why, and when, is this valid?)} \\
&= z\frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right) \quad \text{(why is this valid?)} \\
&= z\frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{z}{(1-z)^2}
\end{aligned}$$

(b) $\sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n+1)!} = \frac{\sin\sqrt{z}}{\sqrt{z}}.$

Remarks: The function $\sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n+1)!}$ is analytic on the entire complex plane, but $\sqrt{z}$ and $\sin\sqrt{z}$ are not. On the other hand $\sqrt{z}$ and $\sin\sqrt{z}$ are both single valued and analytic on the cut plane $\Omega = \mathbb{C} - \{z = x + \imath y \mid x \le 0, y = 0\}$. Suppose C is a circle going once around the origin. If we were to start somewhere on this circle, say at z_0, with a given value of $\sqrt{z_0}$ (and the resulting value of $\sin\sqrt{z_0}$) and continuously compute the values of $\sqrt{z}$ and $\sin\sqrt{z}$ as we go once around the circle we would end up at $-\sqrt{z_0}$ and $\sin(-\sqrt{z_0}) = -\sin\sqrt{z_0}$. Thus $\frac{\sin\sqrt{z}}{\sqrt{z}}$ is single valued and analytic on $\mathbb{C}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} z^{2n} = e^{-z^2/2}$.

4. Use the comparison test to show that the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\sqrt{2}n\pi)}{2^n}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - n - 1}{n^{7/2}}$.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{\imath^n + (-1)^{n^2}}{n(\sqrt{n} - 1)}$.

Solution:

(a) $\displaystyle\left|\frac{\sin(\sqrt{2}n\pi)}{2^n}\right| \le \frac{1}{2^n}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges so does $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\sqrt{2}n\pi)}{2^n}$.

(b) $\displaystyle\left|\frac{n^2 - n - 1}{n^{7/2}}\right| \le \frac{n^2}{n^{7/2}} = \frac{1}{n^{3/2}}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges so does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - n - 1}{n^{7/2}}$.

(c) $\displaystyle\left|\frac{\imath^n + (-1)^{n^2}}{n(\sqrt{n} - 1)}\right| \le \frac{2}{n(\sqrt{n}/2)}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges so does $\displaystyle\sum_{n=2}^{\infty} \frac{\imath^n + (-1)^{n^2}}{n(\sqrt{n} - 1)}$.

5. Show that the sequence of functions $F_n(z) = \dfrac{z^n}{z^n - \imath}$, $n = 1, 2, \ldots$ converges to 0 if $|z| < 1$, and to 1 if $|z| > 1$.

Solution: First assume $|z| < 1$. Then

$$\lim_{n\to\infty} F_n(z) = \lim_{n\to\infty} \frac{z^n}{z^n - \imath} = \frac{\lim_{n\to\infty} z^n}{\lim_{n\to\infty} z^n - \imath} = \frac{0}{0 - \imath} = 0$$

If $|z| > 1$ then

$$\lim_{n\to\infty} F_n(z) = \lim_{n\to\infty} \frac{z^n}{z^n - \imath} = \lim_{n\to\infty} \frac{1}{1 - \imath/z^n} = 1$$

Note: The behaviour of the sequence $F_n(z)$ for $|z| = 1$ is quite chaotic. For example if $z = e^{2\pi\imath/k}$, where k is not divisible by 4, then the values of $F_n(z)$ are all finite and repeat in groups of k since $F_n(z) = F_{k+n}(z)$ for all n. On the other hand if $z = e^{\imath\theta}$, where θ is irrational, then the values of $F_n(z)$ are all finite, but can be arbitrarily large. This follows from the fact that the values of z^n are never equal to $\imath$, but they can get arbitrarily close to $\imath$.

6. The Bernoulli numbers B_n are defined by the power series $\displaystyle\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$.

(a) Show that $\dfrac{z}{e^z-1}+\dfrac{z}{2}=\dfrac{z}{2}\coth\dfrac{z}{2}$.

(b) Show that $B_1=-1/2$ and $B_3=B_5=B_7=\cdots=0$.

(c) Show that $z\cot(z)=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}B_{2n}}{(2n)!}(z)^{2n}$.

Solution:

(a) $\dfrac{z}{e^z-1}+\dfrac{z}{2}=\dfrac{z+ze^z}{2(e^z-1)}=\dfrac{z}{2}\dfrac{e^z-1}{e^z+1}=\dfrac{z}{2}\dfrac{e^{z/2}-e^{-z/2}}{e^{z/2}-e^{-z/2}}=\dfrac{z}{2}\coth(z/2)$.

(b) Since $\dfrac{z}{2}\coth(z/2)$ is an even function it follows that its Maclaurin series representation must have only even powers of z, and therefore

$$\frac{z}{2}\coth(z/2)=\frac{z}{e^z-1}+\frac{z}{2}=\frac{z}{2}+\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n=\sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}z^{2n}$$

That is $B_1=-1/2$ and $B_3=B_5=B_7=\cdots=0$.

(c) By definition $\cot z=\dfrac{\cos z}{\sin z}=\dfrac{(e^{\imath z}+e^{-\imath z})/2}{(e^{\imath z}-e^{-\imath z})/2\imath}=\imath\coth(\imath z)$ and therefore

$$z\cot z=\imath z\coth(\imath z)=\sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}(2\imath z)^{2n}=\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}B_{2n}}{(2n)!}z^{2n}$$

This follows by replacing z by $2\imath z$ in the formula for $\dfrac{z}{2}\coth(z/2)$.

Note: The function $f(z)=\dfrac{z}{e^z-1}$ is analytic for $|z|<2\pi$ since the singularity nearest the origin is $z=2\pi\imath$. The singularity at $z=0$ is **removable**. It follows that the series reprsentation for $z\cot z$ is valid for $|z|<\pi$.