SOLUTIONS TO HOMEWORK ASSIGNMENT # 7

1. Determine the nature of all singularities of the following functions $f(z)$.

   (a) $f(z) = \cos 1/z$.

   (b) $f(z) = \dfrac{1}{z^2 \sin z}$.

   (c) $f(z) = \dfrac{z}{e^{z^2} - 1}$.

   Solution:

   (a) $z = 0$ is the only singularity. It is an essential singularity since the Laurent series expansion about $z = 0$,

   $$\cos 1/z = 1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} + \cdots,$$

   has infinitely many negative powers of z.

   (b) The singularities are $z = 0$ and $z = n\pi, n = \pm 1, \pm 2, \ldots$. The singularity at $z = 0$ is a pole of order 3 since $z = 0$ is a zero of order 3 of $z^2 \sin z$. This follows easily from the Maclaurin series about $z = 0$:

   $$z^2 \sin z = z^3 - \frac{1}{3!}z^5 + \frac{1}{5!}z^7 + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} z^{2n+3}.$$

   The singularities $z = n\pi, n = \pm 1, \pm 2, \ldots,$ are simple poles since they are simple zeros of $z^2 \sin z$.

   (c) $z = 0$ is a simple pole since

   $$\frac{z}{e^{z^2} - 1} = \frac{z}{z^2 + z^4/2! + z^6/3! + \cdots} = \frac{1}{z + z^3/2! + z^5/3! + \cdots} = \frac{1}{z} g(z)$$

   where $g(z)$ is analytic at $z = 0$ and $g(0) \neq 0$. In fact $g(0) = 1$, although what's important is just that $g(0) \neq 0$.

   The other singularities are the non-zero solutions of $e^{z^2} = 1$, that is $z = \sqrt{2n\pi\imath}$, where n is a non-zero integer. They are all simple poles since

   $$\frac{d}{dz}(e^{z^2} - 1) \mid_{z=\sqrt{2n\pi\imath}} = 2\sqrt{2n\pi\imath} e^{2n\pi\imath} = 2\sqrt{2n\pi\imath} \neq 0.$$

2. Evaluate the following integrals. In each case the contour is positively oriented.

   (a) $\displaystyle\int_{|z|=R} \bar{z}^n dz$, where n is an integer.

   (b) $\displaystyle\int_{|z|=3} \cot z dz$.

(c) $\displaystyle\int_{|z-1|=4} \frac{1}{z\sin z} dz.$

Solution:

(a) Make the substitution $z = Re^{\imath\theta}$. Then $dz = R\imath e^{\imath\theta} d\theta$ and so

$$\int_{|z|=R} \bar{z}^n dz = \int_{\theta=0}^{\theta=2\pi} \imath R^{n+1} e^{(-n+1)\imath\theta} d\theta = \imath R^{n+1} \int_{\theta=0}^{\theta=2\pi} e^{(-n+1)\imath\theta} d\theta = \begin{cases} 2\pi\imath R^2 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

It is obvious $\displaystyle\int_{\theta=0}^{\theta=2\pi} e^{(-n+1)\imath\theta} d\theta = 2\pi$ if $n = 1$. If $n \neq 1$ then the Fundamental Theorem of Calculus gives

$$\int_{\theta=0}^{\theta=2\pi} e^{(-n+1)\imath\theta} d\theta = \frac{e^{(-n+1)\imath\theta}}{-n+1} \Big|_{\theta=0}^{\theta=2\pi} = 0$$

The point to this question is that the function $f(z) = \bar{z}$ is not analytic, for if it were the Cauchy Integral Theorem would tell us that $\displaystyle\int_{|z|=R} \bar{z}^n dz = 0$ for $n \geq 0$.

(b) This is a straight forward application of the Cauchy Residue Theorem:

$$\int_{|z|=3} \cot z dz = 2\pi\imath Residue(\cot z, z = 0) = 2\pi\imath \frac{z\cos z}{\sin z} \Big|_{z=0} = 2\pi\imath.$$

The singularities of $\cot z = \dfrac{\cos z}{\sin z}$ are $z = n\pi, n = 0, \pm 1, \pm 2, \ldots$ They are all simple poles, but only the singularity at $z = 0$ is inside the cirlce $|z| = 3$.

(c) The singularities of $\dfrac{1}{z\sin z}$ inside the circle $|z-1| = 4$ are $z = 0$ and $z = \pi$. The singularity at $z = 0$ is a pole of order 2 since the Laurent series at $z = 0$ is

$$\frac{1}{z\sin z} = \frac{1}{z^2(1 - z^2/3! + z^4/5! - + \cdots)} = \frac{1}{z^2} + \frac{1}{6} + \cdots$$

Here we have used the geometric series:

$$\begin{aligned}
\frac{1}{z\sin z} &= \frac{1}{z(z - z^3/3! + z^5/5! - + \cdots)} = \frac{1}{z^2(1 - z^2/3! + z^4/5! - + \cdots)} \\
&= \frac{1}{z^2(1 - (z^2/3! - z^4/5! + \cdots))} \\
&= \frac{1}{z^2}\left(1 + (z^2/3! - z^4/5! + \cdots) + (z^2/3! - z^4/5! + \cdots)^2 + \cdots\right) \\
&= \frac{1}{z^2} + 1/3! + \text{ higher powers of } z
\end{aligned}$$

Therefore the residue at $z = 0$ is 0.

Another way to see this is that

$$\frac{1}{z\sin z} = \frac{1}{z^2}g(z) \text{ where } g(z) = \frac{z}{\sin z}$$

Now we could expand $g(z) = z/\sin z$ as a Taylor series about $z = 0$. But since $g(z)$ is an even function it follows that the Taylor series will have the form $a_0 + a_1 z^2 + a_4 z^4 + \cdots$, and therefore the residue at $z = 0$ is 0. We don't actually have to compute the Taylor series.

The singularity at $z = \pi$ is a simple pole and therefore the residue at $z = \pi$ is $\left.\frac{z-\pi}{z\sin z}\right|_{z=\pi} = -1/\pi$. Therefore $\int_{|z-1|=4} \frac{1}{z\sin z}dz = -2\imath$.

3. Let $f(z)$ be the power series $\sum_{n=0}^{\infty} n^2 z^n$.

   (a) Find all z such that the power series converges.

   (b) Find a closed form expression for $f(z)$.

   Solution:

   (a) By the ratio test the series converges for $|z| < 1$ and diverges for $|z| > 1$. The series diverges for $|z| = 1$ since the terms $n^2 z^n$ do not go to 0 as $n \to \infty$ if $|z| = 1$.

   (b) Consider the geometric series $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$ Then

$$z\frac{d}{dz}(1-z)^{-1} = z + 2z^2 + 3z^3 + \cdots$$

   Do it one more time:

$$z + 2^2z^2 + 3^2z^3 + \cdots = z\frac{d}{dz}\left(z\frac{d}{dz}(1-z)^{-1}\right) = z\frac{d}{dz}(z(1-z)^{-2}) = \frac{z(1+z)}{(1-z)^3}$$

4. Find all z such that the power series $\sum_{n=1}^{\infty} \frac{1}{n^2} z^n$ converges.

   Solution: By the ratio test we see that $\sum_{n=1}^{\infty} \frac{1}{n^2} z^n$ converges for $|z| < 1$ and diverges for $|z| > 1$. It also converges for $|z| = 1$ by comparison with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

5. Suppose $f(z)$ is analytic for $|z| \le 1$ and $|f(z)| \le M$ for $|z| = 1$, where M is some constant. Show that $|f(0)| \le M$ and $|f'(0)| \le M$.

Solution: This follows from a Cauchy Integral Formula and the ML inequality:

$$\begin{aligned} f(0) &= \frac{1}{2\pi\imath}\int_{|z|=1}\frac{f(z)}{z}dz \Longrightarrow |f(0)| \le \frac{1}{2\pi}M2\pi = M \\ f'(0) &= \frac{1}{2\pi\imath}\int_{|z|=1}\frac{f(z)}{z^2}dz \Longrightarrow |f'(0)| \le \frac{1}{2\pi}M2\pi = M \end{aligned}$$

Exercise: What inequalities do you get for $|f^{(n)}(0)|$?

6. Determine if there is a function $f(z)$ which is analytic in some open neighbourhood of the origin and which satisfies the following. If there is such a function find a closed form for it and state where $f(z)$ is analytic.

   (a) $f^{(k)}(0) = k$ for $k \ge 0$.

   (b) $f^{(k)}(0) = (k!)^2$ for $k \ge 0$.

   (c) $f(0) = \pi$ and $f^{(k)}(0) = (-1)^{k+1}2^k(k-1)!$ for $k \ge 1$.

   Solution: In all cases we consider the Maclaurin series $f(z) = \displaystyle\sum_{k=0}^{\infty}\frac{f^k(0)}{k!}z^k$.

   (a) $f(z) = \displaystyle\sum_{k=0}^{\infty}\frac{f^k(0)}{k!}z^k = \sum_{k=1}^{\infty}\frac{1}{(k-1)!}z^k = ze^z$. Thus $f(z)$ is entire.

   (b) In this case we would have $f(z) = \displaystyle\sum_{k=0}^{\infty}k!z^k$, which diverges for all $z \ne 0$. Thus there is no such function.

   (c) $f(z) = \pi + \displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^k(k-1)!}{k!}z^k = \pi + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}(2z)^k = \pi + Log(1+2z)$.
   This converges for $|z| < 1/2$. .

7. Evaluate the following integrals. In each case the contour is positively oriented.

   (a) $\displaystyle\int_{C_R}\frac{1}{z^2+z+1}dz$, where $R > 1$ and C_R is the real axis from $-R$ to R together with the upper half of the circle $|z| = R$.

   (b) $\displaystyle\int_{|z|=1} z^2e^{1/z}\sin(1/z)dz$.

   Solution:

   (a) The singularities of $f(z) = \dfrac{1}{z^2+z+1}$ occur at the roots of z^2+z+1. The only root inside the contour C_R is $z = e^{2\pi\imath/3}$, and it is a simple pole. Thus

$$\begin{aligned}\int_{C_R} \frac{1}{z^2+z+1}dz &= 2\pi\imath Residue\left(\frac{1}{z^2+z+1}, z=e^{2\pi\imath/3}\right)\\ &= 2\pi\imath\frac{z-e^{2\pi\imath/3}}{z^2+z+1}\Big|_{z=e^{2\pi\imath/3}}\\ &= 2\pi\imath\frac{1}{2z+1}\Big|_{z=e^{2\pi\imath/3}} = \frac{2\pi}{\sqrt{3}}\end{aligned}$$

(b) The only singularity of $z^2e^{1/z}\sin(1/z)$ occurs at $z = 0$, and it is an essential singularity. Therefore the formula for computing the residue at a pole will not work, but we can still compute some of the coefficients in the Laurent series expansion about $z = 0$:

$$\begin{aligned} z^2e^{1/z}\sin(1/z) &= z^2\left(1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\cdots\right)\left(\frac{1}{z}-\frac{1}{3!z^3}+\frac{1}{5!z^5}-+\cdots\right)\\ &= z^2\left(\frac{1}{z}+\frac{1}{z^2}+\left(\frac{1}{2}-\frac{1}{6}\right)\frac{1}{z^3}+\cdots\right) = z+1+\frac{1}{3z}+\cdots\\ &\Longrightarrow \ Residue(z^2e^{1/z}\sin(1/z), z=0) = \frac{1}{3}\end{aligned}$$

Therefore $\displaystyle\int_{|z|=1} z^2e^{1/z}\sin(1/z)dz = \frac{2\pi\imath}{3}$.

Exercise: Read about the Cauchy product in the text.

8. Evaluate $\displaystyle\int_0^\infty \frac{x^2+1}{x^4+1}dx$.

Solution:

Consider the integral $\displaystyle\int_{C_R} \frac{1+z^2}{1+z^4}dz$, where $R > 0$ and C_R is the positively oriented contour comprised of the segment of the real axis from $-R$ to R and then the upper half of the circle $|z| = R$. Let C'_R, C''_R denote the real axis portion and the circular portion resp. Then $\displaystyle\lim_{R\to\infty}\int_{C''_R}\frac{1+z^2}{1+z^4}dz = 0$ since the degree of z^4+1 is 2 more than the degree of z^2+1. The singularities are at the solutions of the equation $z^4+1=0$, that is

$$z = e^{\pi\imath/4}, z = e^{3\pi\imath/4}, z = e^{5\pi\imath/4}, z = e^{7\pi\imath/4}.$$

The only singularities in the upper half plane are $z = e^{\pi\imath/4}, z = e^{3\pi\imath/4}$, and they are simple poles. It follows that

$$\begin{aligned}\int_{-\infty}^\infty \frac{x^2+1}{x^4+1}dx &= 2\pi\imath\left(Residue\left(\frac{1+z^2}{1+z^4}, e^{\pi\imath/4}\right) + Residue\left(\frac{1+z^2}{1+z^4}, e^{3\pi\imath/4}\right)\right)\\ &= 2\pi\imath\left(\frac{(z-e^{\pi\imath/4})(1+z^2)}{1+z^4}\Big|_{z=e^{\pi\imath/4}} + \frac{(z-e^{3\pi\imath/4})(1+z^2)}{1+z^4}\Big|_{z=e^{3\pi\imath/4}}\right)\end{aligned}$$

$$
\begin{aligned}
&= 2\pi\imath\left(\frac{1+\imath}{4e^{3\pi\imath/4}}+\frac{1-\imath}{4e^{\pi\imath/4}}\right) = -\frac{\pi\imath}{2}\left((1-\imath)e^{3\pi\imath/4}+(1+\imath)e^{\pi\imath/4}\right)\\
&= -\frac{\pi\imath}{2}\left((1-\imath)\left(\frac{-1+\imath}{\sqrt{2}}\right)+(1+\imath)\left(\frac{1+\imath}{\sqrt{2}}\right)\right)\\
&= -\frac{\pi\imath}{2\sqrt{2}}\left(-(1-\imath)^2+(1+\imath)^2\right) = \pi\sqrt{2}
\end{aligned}
$$

Therefore $\int_0^\infty \frac{x^2+1}{x^4+1}dx = \frac{\pi}{\sqrt{2}}$.

Remarks: In this calculation we have used the fact that $\lim_{R\to\infty}\int_{C''_R}\frac{P(z)}{Q(z)}dz = 0$, where $P(z), Q(z)$ are polynomials such that $degree(Q) \geq degree(P)+2$. **See page 322**. The basic reason for this is that $\frac{P(z)}{Q(z)}$ behaves like $1/R^d$ on the arc, where $d = degree(Q) - degree(P)$; whereas the arc only has length πR. Therefore the ML inequality guarantees that the integral goes to 0.

9. Evaluate $\int_{-\pi}^{\pi}\frac{1}{1+\sin^2\theta}d\theta$.

Solution:

We make the substitution $z = e^{\imath\theta}$. Then $dz = \imath e^{\imath\theta}d\theta = \imath z d\theta$, or $d\theta = \frac{dz}{\imath z}$. Moreover $\sin\theta = \frac{e^{\imath\theta}-e^{-\imath\theta}}{2\imath} = \frac{z-1/z}{2\imath}$. Therefore

$$
\begin{aligned}
\int_{-\pi}^{\pi}\frac{1}{1+\sin^2\theta}d\theta &= \int_{|z|=1}\frac{1}{\imath z\left(1+\left(\frac{z-1/z}{2\imath}\right)^2\right)}dz = \int_{|z|=1}\frac{1}{\imath z\left(1-\frac{z^2-2+1/z^2}{4}\right)}dz\\
&= \frac{4}{\imath}\int_{|z|=1}\frac{1}{z\left(6-z^2-1/z^2\right)}dz = \frac{4}{\imath}\int_{|z|=1}\frac{z}{6z^2-z^4-1}dz
\end{aligned}
$$

The singularities occur at solutions of $z^4-6z^2+1=0$, that is $z=\pm\sqrt{3\pm2\sqrt{2}}$. All of them are simple poles, but only $z=\pm\sqrt{3-2\sqrt{2}}$ are inside the circle $|z|=1$. Next we compute the residues at these singularities:

$$
Residue\left(\frac{z}{6z^2-z^4-1}, z=\sqrt{3-2\sqrt{2}}\right) = \frac{z(z-\sqrt{3-2\sqrt{2}})}{6z^2-z^4-1}\Big|_{z=\sqrt{3-2\sqrt{2}}}
$$

$$
= \frac{\sqrt{3-2\sqrt{2}}}{-4(3-2\sqrt{2})^{3/2}+12\sqrt{3-2\sqrt{2}}} = \frac{1}{-4(3-2\sqrt{2})+12} = \frac{1}{8\sqrt{2}}
$$

In a similar manner we calculate that

$$
Residue\left(\frac{z}{6z^2-z^4-1}, z=-\sqrt{3-2\sqrt{2}}\right) = \frac{1}{8\sqrt{2}}.
$$

Therefore

$$\int_{-\pi}^{\pi} \frac{1}{1+\sin^2\theta} d\theta = \frac{4}{\imath} \times 2\pi\imath \times (\text{the sum of the residues}) = \frac{4}{\imath} \times 2\pi\imath \times \frac{1}{4\sqrt{2}} = \pi\sqrt{2}$$

10. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}} dx = \frac{\pi(2n)!}{2^{2n}(n!)^2}$ for $n = 0, 1, 2, \ldots$

Solution: We use an argument similar to that used in question 8. In particular see the remark at the end of that question. The only singularity of $\dfrac{1}{(1+z^2)^{n+1}}$ in the upper half plane is at $z = \imath$, and it is a pole of order $n+1$. Therefore

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}} dx &= 2\pi\imath Residue\left(\frac{1}{(1+x^2)^{n+1}}, z=\imath\right) = \frac{2\pi\imath}{n!} \frac{d^n}{dz^n} \frac{(z-\imath)^{n+1}}{(1+z^2)^{n+1}} \Big|_{z=\imath} \\
&= \frac{2\pi\imath}{n!} \frac{d^n}{dz^n} (z+\imath)^{-n-1} \Big|_{z=\imath} \\
&= \frac{2\pi\imath}{n!} (-n-1)(-n-2)\cdots(-n-n)(2\imath)^{-2n-1} \\
&= \frac{2\pi\imath}{n!} (-1)^n \frac{(n+1)(n+2)\cdots(2n)}{(2\imath)^{2n+1}} \\
&= \frac{\pi}{2^{2n}} \frac{(n+1)(n+2)\cdots(2n)}{n!} = \frac{\pi(2n)!}{2^{2n}(n!)^2}
\end{aligned}$$