Math 302, Assignment 4

1. Suppose that X has moment generating function

$$M_X(t) = \tfrac{1}{3}e^{-3t} + \tfrac{1}{6} + \tfrac{1}{2}e^t.$$

   (a) Find the mean and variance of X by differentiating the m.g.f. above.

   (b) Find the p.m.f. of X. Use your expression for the p.m.f. to check your answers from part (a).

Solutions: (a) We have

$$\mu = M_X'(0) = -1 + \frac{1}{2} = -\frac{1}{2}$$
$$\sigma^2 = M_X''(0) - \frac{1}{4} = 3 + \frac{1}{4} - \frac{1}{4} = \frac{13}{4}$$

(b) By looking at the m.g.f., we recognize that X takes values $-3, 0, 1$, with $\mathbb{P}(X = -3) = \frac{1}{3}$, $\mathbb{P}(X = 0) = \frac{1}{6}, \mathbb{P}(X = 1) = \frac{1}{2}$. Therefore, we can calculate again

$$\mu = \frac{1}{3}\cdot(-3) + \frac{1}{6}\cdot(0) + \frac{1}{2}\cdot 1 = -\frac{1}{2}$$
$$\sigma^2 = \Big(\frac{1}{3}\cdot(-3)^2 + \frac{1}{6}\cdot(0)^2 + \frac{1}{2}\cdot(1)^2\Big) - \frac{1}{4} = \frac{9}{4}$$

2. You have two dice, one with three sides labeled $0, 1, 2$ and one with 4 sides, labeled $0, 1, 2, 3$. Let X_1 be the outcome of rolling the first die, and X_2 the outcome of rolling the second. The rolls are independent.

   (a) What is the joint p.m.f. of (X_1, X_2)?

   (b) Let $Y_1 = X_1 \cdot X_2$ and $Y_2 = \max\{X_1, X_2\}$. Make a table for the joint p.m.f. of (Y_1, Y_2).

   (c) Are Y_1, Y_2 independent?

Solution: (a) By independence we have $p(x, y) = (1/3)(1/4) = 1/12$ for all $x \in \{0, 1, 2\}$ and $y \in \{0, 1, 2, 3\}$.

(b)

Table 1: The p.m.f. of (Y_1, Y_2) with the marginals.

$Y_1 \downarrow Y_2 \rightarrow$	0	1	2	3	p_{Y_1}
0	1/12	1/6	1/6	1/12	1/2
1	0	1/12	0	0	1/12
2	0	0	1/6	0	1/6
3	0	0	0	1/12	1/12
4	0	0	1/12	0	1/12
6	0	0	0	1/12	1/12
p_{Y_2}	1/12	1/4	5/12	1/4	

(c) For the marginal distributions see the margins of the above table. Since

$$\mathbb{P}(Y_1 = 1, Y_2 = 0) = 0 \neq \mathbb{P}(Y_1 = 1)\mathbb{P}(Y_2 = 0),$$

the variables Y_1 and Y_2 are not independent.

3. Let $X \sim \text{Exp}(2)$, $Y \sim \text{Unif}([1,3])$, and assume that X and Y are independent. Calculate $\mathbb{P}(Y - X \geq \frac{1}{2})$.

Solution: The joint density function is

$$f(x,y) = f_X(x) f_Y(y) = \begin{cases} e^{-2x} & \text{if } x > 0 \text{ and } 1 < y < 3, \\ 0 & \text{otherwise.} \end{cases}$$

Let T be defined by

$$T = \{(x,y) : x > 0,\ 1 < y < 3,\ x \leq y - 1/2\},$$

then we have

$$\begin{aligned} \mathbb{P}(Y - X \geq 1/2) &= \iint_T f(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_1^3 \int_0^{y-\frac{1}{2}} e^{-2x}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\int_1^3 1 - e^{-2y+1}\mathrm{d}y \\ &= 1 + e\left[\frac{1}{2}e^{-2y}\right]_1^3 = 1 + \frac{1}{2}e^{-5} - \frac{1}{2}e^{-1}. \end{aligned}$$

4. The random variables X, Y have joint probability density function

$$f(x,y) = \begin{cases} C\frac{e^{-x} - e^{-x-2y}}{e^y - 1} & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of C?

(b) Are X and Y independent?

(c) Find $\mathbb{P}(X < Y)$.

Solution: (a) We have

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = C\int_0^{\infty} e^{-x}\,\mathrm{d}x \int_0^{\infty} \frac{1 - e^{-2y}}{e^y - 1}\,\mathrm{d}y = \frac{3}{2}C,$$

thus $C = \frac{2}{3}$.

(b) The variables X and Y are independent since the joint p.d.f. factors into a function of x times a function of y.

(c) Let $E = \{(x, y) : x < y\}$, then

$$\mathbb{P}(X < Y) = \iint_E f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \frac{2}{3}\int_0^\infty \frac{1 - e^{-2y}}{e^y - 1}\int_0^y e^{-x}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \frac{2}{3}\int_0^\infty \frac{1 - e^{-2y}}{e^y - 1}(1 - e^{-y})$$
$$= \frac{2}{3}\int_0^\infty (e^{-y} - e^{-3y}) = \frac{2}{3}(1 - 1/3) = \frac{4}{9}.$$

5. Let X_1 and X_2 be two discrete random variables with joint p.m.f. $\mathbb{P}(X_1 = k_1, X_2 = k_2)$. Prove the following claims from the lecture:

   (a) If $g : \mathbb{R}^2 \to \mathbb{R}$ is a function, then

   $$\mathbb{E}\, g(X_1, X_2) = \sum_{k_1, k_2} g(k_1, k_2) \cdot \mathbb{P}(X_1 = k_1, X_2 = k_2).$$

   Hint: Remember that the left hand side is by definition $\mathbb{E}\, g(X_1, X_2) = \sum_l l \cdot \mathbb{P}(g(X_1, X_2) = l)$, where the sum is over all values of $g(X_1, X_2)$, i.e. over all l such that $l = g(k_1, k_2)$ for some value k_1 of X_1 and some value k_2 of X_2.

   (b) $\mathbb{E}[X_1 + X_2] = \mathbb{E}\, X_1 + \mathbb{E}\, X_2$. Hint: Use part (a).

Solution: (a) Consider the set V_1 of possible values of X_1, and V_2 of X_2. In other words, V_1 consists of all k_1 such that $\mathbb{P}(X_1 = k_1) > 0$, and similarly for V_2. Consider now $L = g(V_1, V_2) = \bigcup_{(k_1, k_2)} \{g(k_1, k_2)\}$. By definition, we have

$$\mathbb{E}\, g(X_1, X_2) = \sum_{\ell \in L} \ell \cdot \mathbb{P}(g(X_1, X_2) = \ell).$$

Now, observe that $\{g(X_1, X_2) = \ell\}$ is the event consisting of the union of all events $\{X_1 = k_1, X_2 = k_2\}$ where the pair (k_1, k_2) satisfies $g(k_1, k_2) = \ell$. Thus, by additivity of the probability

$$\mathbb{P}(g(X_1, X_2) = \ell) = \sum_{g(k_1, k_2) = \ell} \mathbb{P}(X_1 = k_1, X_2 = k_2).$$

Then,

$$\mathbb{E}\, g(X_1, X_2) = \sum_{\ell \in L} \sum_{g(k_1, k_2) = \ell} \ell \cdot \mathbb{P}(X_1 = k_1, X_2 = k_2)$$
$$= \sum_{\substack{\ell \in L \\ g(k_1, k_2) = \ell}} g(k_1, k_2) \cdot \mathbb{P}(X_1 = k_1, X_2 = k_2).$$

Next, observe that $\bigcup_{\ell \in L} \bigcup_{g(k_1, k_2) = \ell} \{(k_1, k_2)\}$ is the set of possible values of (X_1, X_2) (this is a tautology). Therefore,

$$\mathbb{E} g(X_1, X_2) = \sum_{(k_1, k_2)} g(k_1, k_2) \cdot \mathbb{P}(X_1 = k_1, X_2 = k_2).$$

This proves the claim.

(b) By part (a) and the axioms of probability, we have

$$\begin{aligned}\mathbb{E}(X_1+X_2) &= \sum_{k_1,k_2}(k_1+k_2)\mathbb{P}(X_1=k_1,X_2=k_2)\\ &= \sum_{k_1,k_2} k_1\mathbb{P}(X_1=k_1,X_2=k_2)+\sum_{k_1,k_2} k_2\mathbb{P}(X_1=k_1,X_2=k_2)\\ &= \sum_{k_1} k_1\mathbb{P}(X_1=k_1)+\sum_{k_2} k_2\mathbb{P}(X_2=k_2)=\mathbb{E}X_1+\mathbb{E}X_2.\end{aligned}$$

6. Let X and Y be either two independent Poisson RV's, or two independent Exponential RV's, with parameters μ, λ. Compute the p.m.f. / p.d.f. of $X+Y$.

Solution:

Case Poisson: Remember that a Poisson RV takes values $k = 0, 1, 2, \ldots$ with probability $\mathbb{P}(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}$. If X, Y are independent Poisson, we therefore have that $X+Y$ also takes values $0, 1, 2, \ldots$, and we have to find $\mathbb{P}(X+Y=k)$.

First, we need to find all solutions to $x+y=k$, where $x, y \in \{0, 1, 2, \ldots\}$. Clearly, these are just the pairs $(0,k), (1,k-1), (2,k-2), \ldots, (k-1,1), (k,0)$. Now, we have to sum the joint p.m.f. of X and Y over these pairs (the joint p.m.f. is just the product of the individual p.m.f.'s by independence). We get

$$\begin{aligned}\mathbb{P}(X+Y=k) &= \sum_{x=0}^{k}\mathbb{P}(X=x,Y=k-x)\\ &= \sum_{x=0}^{k}\frac{\lambda^x}{x!}e^{-\lambda}\cdot\frac{\mu^y}{y!}e^{-\mu}\\ &= e^{-\lambda-\mu}\frac{1}{k!}\sum_{x=0}^{k}\frac{k!}{x!y!}\lambda^x\mu^y = e^{-\lambda-\mu}\frac{(\lambda+\mu)^k}{k!}\end{aligned}$$

where in the last step, we used the binomial theorem. Thus we have determined the p.m.f. of $X+Y$. We note that it is just the p.m.f. of Poisson RV with parameter $\lambda+\mu$. Thus, a sum of two independent Poisson RV's is again Poisson, with the sum of the parameters.

Case Exponential: Remember that the p.d.f. of Exponential is

$$f_X(x) = \begin{cases}\lambda e^{-\lambda x} & x > 0\\ 0 & x \le 0\end{cases}$$

We now use the convolution formula

$$f_{X+Y}(y) = \int_{-\infty}^{\infty} f_X(x) f_Y(y-x)\,dx.$$

Note that the integrand vanishes when $x < 0$ since $f_X(x)$ is zero there. It also vanishes if $x > y$, since $f_Y(y-x)$ has a negative argument for such x, and so vanishes. Thus, if $y < 0$, the integral vanishes, and if $y > 0$, it equals

$$f_{X+Y}(y) = \int_0^y \lambda e^{-\lambda x} \cdot \mu e^{-\mu(y-x)}\ dx = \lambda \cdot \mu \cdot e^{-\mu y} \cdot \int_0^y e^{(\mu-\lambda)x}\ dx = \frac{\lambda\mu}{\mu-\lambda}\left[e^{-\lambda y} - e^{-\mu y}\right].$$

(This holds for $\mu \neq \lambda$. For $\mu = \lambda$, we get $\lambda^2 \cdot y \cdot e^{-\lambda y}$).

7. Compute the moment generating functions of the Geom(p) and the Exp(λ) random variables.

Solution:

Case Geometric: Using the definition of the m.g.f. and the geometric series, we get

$$\begin{aligned}
\mathbb{E}e^{t\cdot \mathrm{Geom}(p)} &= \sum_{k\geq 1} e^{tk} \cdot \mathbb{P}(\mathrm{Geom}(p) = k) \\
&\sum_{k\geq 1} e^{tk} \cdot p \cdot (1-p)^{k-1} \\
&= p \cdot e^t \cdot \sum_{k\geq 1} \left(e^t\right)^{k-1} \cdot (1-p)^{k-1} = \frac{p \cdot e^t}{1-(1-p)e^t}
\end{aligned}$$

Note: We could now compute the mean / variance from this function by taking derivatives at 0. Compare this to the tricks we needed in the lecture to compute the mean of Geometric!

Case Exponential: Using the definition, we have

$$\begin{aligned}
\mathbb{E}e^{t\cdot \mathrm{Exp}(\lambda)} &= \int_{-\infty}^{\infty} e^{t\cdot x} f(x)\ dx \\
&= \int_0^{\infty} e^{t\cdot x} \cdot \lambda e^{-\lambda x}\ dx \\
&= \begin{cases} \frac{\lambda}{\lambda - t} & t < \lambda \\ \infty & \text{else} \end{cases}
\end{aligned}$$

8.* Let X be a continuous random variable with p.d.f. $f(x)$ and $g : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Show that the p.d.f. of $g(X)$ equals

$$f_{g(X)}(y) = \frac{f(g^{-1}(y))}{g'(g^{-1}(y))}$$

Solution:

We compute the c.d.f., using that g is strictly increasing.

$$F_{g(X)}(b) = \mathbb{P}(g(X) \leq b) = \mathbb{P}(X \leq g^{-1}(b)) = F_X(g^{-1}(b)).$$

Here, $g^{-1}(b)$ is the inverse function of g (e.g. $g^{-1}(b) = \sqrt{b}$ if $g(x) = x^2$, or $g^{-1}(b) = \arctan b$ if $g(x) = \tan x$). Using the chain rule

$$\begin{aligned}
\frac{d}{dx} F_{g(X)}(x) &= F_X'(g^{-1}(x)) \cdot \frac{d}{dx} g^{-1}(x) \\
&= f_X(g^{-1}(x)) \cdot \frac{1}{g'(g^{-1}(x))},
\end{aligned}$$

where we used a theorem about the derivative of the inverse function.